Count of oscillatory modes of quarks in baryons for 3 quark flavors u , d , s

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Abstract

The present notes prepare the counting of 'oscillatory modes of $N_{fl} = 3$ light quarks' , – u , d , s – , using the SU $\left(2 N_{fl} = 6 \right) \times$ SO3 $\left(\vec{L} \right)$ broken symmetry classification , extended to the harmonic oscillator symmetry of 3 paired oscillator modes . $\vec{L} = \sum_{n=1}^{N_{fl}} \vec{L}_n$ stands for the space rotation group generated by the sum of the 3 individual angular momenta of quarks in their c.m. system . The motivation arises from modeling the yields of hadrons in heavy ion collisions at RHIC and LHC , necessitating at the respective highest c.m. energies per nucleon pairs an increase of heavy hadron resonances relative to e.g. SPS energies , whence the included hadrons are treated as a noninteracting gas .

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List of contents

$\longrightarrow$

List of contents II

Figures 1

List of figures

$\longrightarrow$

1 - Introduction

This outline serves to initiate the actual counting of osillatory modes of light flavored u, d , s quarks inside baryons , based on material worked out between December 2012 and June 2013 , contained in ref. 1 – [1-2013] – which in turn constitutes a note-file to ref. 2 – [2-2013] .

The introduction continues establishing the main ingredients resulting from ref. 1 – [1-2013] – in subsequent subsections 1-1 , ...

1-1 – Counting oscillatory modes using the circular oscillatory wave function basis enforcing overall Bose symmetry under the combined permutations of $SU6 \ (\ fl \ \times \ spin \ ) \ \times$ barycentric coordinates

We begin recalling the definition of barycentric coordinates $(\ \beta\alpha\rho\upsilon\kappa\epsilon\nu\tau\rho\iota\kappa\acute{\epsilon}s \ \sigma\upsilon\nu\tau\epsilon\tau\alpha\gamma\mu\acute{\epsilon}\nu\epsilon s \ )$ in the c.m. system of three valence quarks among the light 3 quark flavors u, d, s . They appear first in ref. [1-2013] in section 2-4-1 , eq. (18) on page 2-18 [PDF page # 26] through eq. (28) on page 2-22 [PDF page # 30] . The definitions correspond to the usage in (classical) Hamiltonian mechanics and applied in my original paper [3-1980] . For $N \ \rightarrow \ N_{\ c} \ = \ 3$ they take the form

(1)
$$\begin{array}{c} z_{\ 1} \ = \ \frac{1}{\sqrt{2}} \ (\ x_{\ 1} \ - \ x_{\ 2} \ ) \quad , \quad z_{\ 2} \ = \ \frac{1}{\sqrt{6}} \ (\ x_{\ 1} \ + \ x_{\ 2} \ - \ 2 \ x_{\ 3} \ ) \\ z_{\ 3} \ = \ \frac{1}{\sqrt{3}} \ (\ x_{\ 1} \ + \ x_{\ 2} \ + \ x_{\ 3} \ ) \ = \ \sqrt{3} \ X_{\ c.m.} \quad \rightarrow \ 0 \end{array}$$

Here we replace the symbol N by $N_{\ c}$, reserving N for the main oscillatory quantum number as defined in the following (sub-)sections. In eq. 1 $x_{\ k} \ ; \ k = 1, 2, 3$ denote configuration space 3-vectors $\longrightarrow$

all this in the overall c.m. frame, wherein time is defined and denoted t . We recall from ref. [3-1980] that only 2 out of the three barycentric three-vector variables as well as their conjugate momenta, i.e.

(2) $$z_1 , z_2 \longleftrightarrow \pi_1 , \pi_2$$

exhibit oscillatory (confined) motion , wheres the c.m. related position and conjugate momentum are left in free motion . Thus the universal , i.e. N_c independent oscillatory frequency appears in the form of the induced $\mathcal{M}^2$ operator in the N_c $(= 3$ here $)$ dependent combination

$$\mathcal{M}^2 = \sum_{\nu=1}^{N_c - 1} \left[K_{N_c} \pi_\nu^2 + (K_{N_c})^{-1} \Lambda^2 z_\nu^2 \right]$$

(3) $$K_{N_c} = N_c / (N_c - 1) \quad \left(= \tfrac{3}{2} \text{ here} \right)$$

$$\pi_\mu = \frac{1}{i} \partial_{z_\mu} \quad ; \quad \mu = 1, 2$$

(4) $$z_\mu = \lambda \overline{z}_\mu \longleftrightarrow \pi_\mu = \lambda^{-1} \overline{\pi}_\mu \quad ; \quad \mu = 1, 2$$

under which $\mathcal{M}^2$ in eq. 3 becomes

(5) $$\mathcal{M}^2 = \sum_{\nu=1}^{N_c - 1} \left[\frac{K_{N_c}}{\lambda^2} \overline{\pi}_\nu^2 + \frac{\lambda^2 \Lambda^2}{K_{N_c}} \overline{z}_\nu^2 \right]$$

$\longrightarrow$

The quantity λ of dimension length , introduced in eqs. 4 and 5 , is to be chosen such that

(6) $$\frac{K_{N_c}}{\lambda^2} = \frac{\lambda^2 \Lambda^2}{K_{N_c}} \rightarrow \lambda^4 = \left(\frac{K_{N_c}}{\Lambda} \right)^2 \rightarrow \lambda^2 = \frac{K_{N_c}}{\Lambda}$$

whereupon $\mathcal{M}^2$ in eqs. 3 , 5 assumes the reduced universal form

$$\mathcal{M}^2 = \Lambda \sum_{\nu=1}^{N_c - 1} \left[\overline{\pi}_\nu^2 + \overline{z}_\nu^2 \right]$$

(7) $$z_\mu = \lambda \overline{z}_\mu \longleftrightarrow \pi_\mu = \lambda^{-1} \overline{\pi}_\mu \ ; \ \mu = 1,2 \ ; \ \lambda^2 = \frac{K_{N_c}}{\Lambda}$$

$$i \overline{\pi}_\mu = \partial_{\overline{z}_\mu} \longleftrightarrow \overline{z}_\mu \ ; \ \mu = 1,2 \ :$$ **4 three vectors with dimensionless components**

The universal oscillator vector-variables derive from the relation on the last line in eq. 7 $\longrightarrow$

(8)
$$a_{\mu\, k} \;=\; \frac{1}{\sqrt{2}}\;\left(i\, \overline{\pi}_{\mu\, k} \;+\; \overline{z}_{\mu\, k} \right) \quad ; \quad a^{\dagger}_{\mu\, k} \;=\; \frac{1}{\sqrt{2}}\;\left(-\, i\, \overline{\pi}_{\mu\, k} \;+\; \overline{z}_{\mu\, k} \right)$$
$$\mu \;=\; 1, 2 \quad ; \quad k \;=\; 1, 2, 3$$

which in turn yields the relativistic structure of the oscillatory $\mathcal{M}^{\,2}$ operator , substituting in eqs. 3 , 5 and 7

(9)
$$\mathcal{M}^{\,2} \;=\; (\, 2\, \Lambda\,)\, \sum_{\nu=1,\; k=1}^{\nu=2\;,k=3} \left[\, \overline{a}^{\,\dagger}_{\nu\;k}\, \overline{a}_{\,\nu\;k} \;+\; \frac{1}{2}\,\right] \quad ; \quad 2\, \Lambda \;=\; 1\, /\, \alpha^{\,\prime}$$

The contribution of the zero mode oscillations , $\frac{1}{2}$ for each dimension of configuration space (= 3) is inherent to the classical limiting form of oscillatory motion , and is accompanied in the sense of a long range approximation (especially given finite quark masses) by a constant correction .The latter remains non zero also in the limit of vanishing quark masses , as discussed in ref. [3-1980] .
In eq. 9 $1\, /\alpha^{\,\prime}$ denotes the inverse of the Regge slope . $\longrightarrow$

If we determine it from the positive parity Λ trajectory from the present PDG tables [4-2012]

(10)
$$\begin{array}{lcccc} \Lambda\,,\ J^{P} & : & \frac{1}{2}^{+} & \frac{5}{2}^{+} & \frac{9}{2}^{+} \\ M_{j} & : & 1.115683 & 1.820 & 2.350 \\ M_{j}^{2} & : & 1.2447485 & 3.3124 & 5.5225 \\ \frac{1}{2}\,\Delta M^{2} & : & & 1.034 & 1.105 \end{array}$$

and average the two half mass square difference entries in the last line of eq. 10 with weights two to one we obtain

(11)
$$1/\alpha' = \frac{1}{3}\left(M_{2}^{2} - M_{1}^{2}\right) + \frac{1}{6}\left(M_{3}^{2} - M_{2}^{2}\right) \sim 1.06 \text{ GeV}^{2}$$

I remark that in ref. [4-2012] $\Lambda\,\frac{9}{2}^{+}$ has only three stars , and furthermore the trajectory contains only three entries , whereas I think to remember that it contained four sometimes back [a].
$\Lambda\,\frac{13}{2}^{+}$ would extrapolate to 2.755 GeV using eq. 11 . $\longrightarrow$

---

[a] ”Tempora mutantur nos et mutamur in illis .”

From eqs. 6 and 11 we find

$$\begin{aligned} \lambda &= \sqrt{\frac{3}{1.06}} \ \text{GeV}^{-1} \ \sqrt{\alpha' \, 1.06 \ \text{GeV}^{2}} \\ &= 1.682 \ \text{GeV}^{-1} \ \sqrt{\alpha' \, 1.06 \ \text{GeV}^{2}} \\ &= 0.322 \ \text{fm} \ \sqrt{\alpha' \, 1.06 \ \text{GeV}^{2}} \\ \lambda^{-1} &= 0.594 \ \text{GeV} \left(\alpha' \, 1.06 \ \text{GeV}^{2} \right)^{-\frac{1}{2}} \\ \sqrt{\alpha' \, 1.06 \ \text{GeV}^{2}} &= 1 \pm 5 \, \% \end{aligned} \tag{12}$$

The circular oscillatory wave function basis arises from the complex coordinates , derived from eq. 6 . We repeat first the linear oscillatory wave function basis , defined in subsection 3-1-3b in ref. 1 – [1-2013], page 3-11-7 [PDF page # 52] . ⟶

The scalar product in general barycentric coordinates with dimension length becomes

$$\langle\, \Psi^{(2)} \mid \Psi^{(1)} \,\rangle = \lambda^6 \int \Pi_{i=1}^{3} d^3 \bar{\xi}_i \, \delta^3 \left(\bar{\xi}_3 \right) \times$$
$$\times\, \Psi^{*\,(2)} (\bar{x}_1 , , \bar{x}_2 , \bar{x}_3) \, \Psi^{(1)} (\bar{x}_1 , \bar{x}_2 , \bar{x}_3)$$

---

(13) $$\bar{\xi}_1 = \frac{1}{\sqrt{2}} (\bar{x}_1 - \bar{x}_2) \quad , \quad \bar{\xi}_2 = \frac{1}{\sqrt{6}} (\bar{x}_1 + \bar{x}_2 - 2\, \bar{x}_3)$$

$$\bar{\xi}_3 = \frac{1}{\sqrt{3}} (\bar{x}_1 + \bar{x}_2 + \bar{x}_3) = \sqrt{3}\, \overline{X}_{c.m.} \;\; \rightarrow 0$$

$$z_j = \lambda\, \bar{z}_j \quad , \quad x_j = \lambda\, \bar{x}_j \;\; ; \; j = 1,2,3 \;\; ; \; X_{c.m.} = \lambda\, \overline{X}_{c.m.} = \frac{1}{3} \sum_{i=1}^{3} \bar{x}_i$$

In order to eliminate the scale factor λ we redefine the wave functions $\Psi^{(1),(2)}$ in eq. 13

(14) $$\psi^{(1),(2)} = \lambda^3\, \Psi^{(1),(2)}$$

Using the dimensionless wave functions

(15) $$\psi^{(1),(2)} (\bar{x}_1 , \bar{x}_2 , \bar{x}_3)$$

defined in eq. 15 the scalar product (eq. 13) becomes $\longrightarrow$

(16) $$\langle \psi^{(2)} \mid \psi^{(1)} \rangle = \int \Pi_{i=1}^{3} d^3 \bar{\xi}_i \, \delta^3 \left(\bar{\xi}_3 \right) \times \times \psi^{*(2)} (\bar{x}_1 , , \bar{x}_2 , \bar{x}_3) \, \psi^{(1)} (\bar{x}_1 , \bar{x}_2 , \bar{x}_3)$$

1-1-1 - The barycentric 6 spatial oscillatory variables and their symmetries with respect to the 3 quark positions in dimensionless universal variables

The central properies under the 3 quark position permutation group S_3 can perfectly be discussed according to the dimensionless variables $\bar{x}_i \, ; \, i = 1, 2, 3$

(17) $$\pi \begin{pmatrix} 1 & 2 & 3 \\ i_1 & i_2 & i_3 \end{pmatrix} \longrightarrow \begin{pmatrix} \bar{x}_1 & \bar{x}_2 & \bar{x}_3 \\ \bar{x}_{i_1} & \bar{x}_{i_2} & \bar{x}_{i_3} \end{pmatrix}$$

To this end we invert the linear relations in eq. 13

(18) $$\begin{aligned} \bar{x}_1 &= \frac{1}{\sqrt{2}} \bar{\xi}_1 + \frac{1}{\sqrt{6}} \bar{\xi}_2 \\ \bar{x}_2 &= -\frac{1}{\sqrt{2}} \bar{\xi}_1 + \frac{1}{\sqrt{6}} \bar{\xi}_2 \\ \bar{x}_3 &= -\frac{2}{\sqrt{6}} \bar{\xi}_2 = \frac{1}{3} (2 \bar{x}_3 - \bar{x}_1 - \bar{x}_2) \quad (\surd) \end{aligned}$$

The relation on the last line of eq. 18 takes into account the vanishing of $\sum_{i=1}^{3} \bar{x}_i$. $\longrightarrow$

For completeness we give both barycentric variable transformations alongside (eqs. 13 and 18)

$$\overline{\xi}_{1} = \frac{1}{\sqrt{2}} \left(\overline{x}_{1} - \overline{x}_{2} \right) \quad , \quad \overline{\xi}_{2} = \frac{1}{\sqrt{6}} \left(\overline{x}_{1} + \overline{x}_{2} - 2 \, \overline{x}_{3} \right)$$

$$\overline{\xi}_{3} = \frac{1}{\sqrt{3}} \left(\overline{x}_{1} + \overline{x}_{2} + \overline{x}_{3} \right) = \sqrt{3} \; \overline{X}_{c.m.} \; \rightarrow \; 0$$

$$z_{j} = \lambda \, \overline{z}_{j} \quad , \quad x_{j} = \lambda \, \overline{x}_{j} \quad ; \; j = 1, 2, 3 \quad ; \; X_{c.m.} = \lambda \, \overline{X}_{c.m.} = \frac{1}{3} \sum_{i=1}^{3} \overline{x}_{i}$$

(19)

$$\begin{aligned} \overline{x}_{1} &= \frac{1}{\sqrt{2}} \overline{\xi}_{1} + \frac{1}{\sqrt{6}} \overline{\xi}_{2} \\ \overline{x}_{2} &= - \frac{1}{\sqrt{2}} \overline{\xi}_{1} + \frac{1}{\sqrt{6}} \overline{\xi}_{2} \\ \overline{x}_{3} &= \phantom{- \frac{1}{\sqrt{2}} \overline{\xi}_{1}} - \frac{2}{\sqrt{6}} \overline{\xi}_{2} \end{aligned}$$

Eqs. 1 - 9 characterize what is denoted in ref. 1 [1-2013] *and here* **: oscillator mode wave functions in the linear basis , comprising 6 linear oscillators , as defined in eq. 8 .**
The oscillator mode wave functions in the circular basis , to which we turn in subsection (1-3) , decompose the induced representation of the permutation group of the configuration space quark-modes $\longleftrightarrow$ S_{3} into 3 pair-modes , of which one individual pair is described in ref. 1 in subsection 'Modes of a pair of onedimensional oscillators pairmodes and the complex plane' on pages 3-pmodes-1 [PDF page # 57] – 3-pmodes-5 (PDF page # 61) and ref. 5 [5-2010] . $\longrightarrow$

1-2 – Aligning statistics between the u, d , s $SU6\ (\ fl\ \times\ spin\ )$ group and oscillator modes in 6 barycentric configuration space variables

Alfred Young + 16 April 1873 in Widnes , Lancashire , England
† 15 December 1940 in Birdbrook , Essex , England
, cited from ref. [6-2012] .

Ferdinand Frobenius + 26 October 1849 in Berlin-Charlottenburg , Prussia , Germany
† 3 August 1917 in Berlin , Prussia , Germany

cited from ref. [7-2013] .

Issai Schur + 10 January , 1875 in Mogilev , Russian Empire
† 10 January , 1941 in Tel Aviv , Mandatory Palestine

shall be cited here as eminent pioneers of preparing and using the association of finite groups with representations of unitary symmetries associated with oscillatory modes and their statistics .

1-2-1 – Modes of a pair of onedimensional oscillators – pair-modes and the complex plane

We recapitulate the properties of pair-modes first considering the the two independent barycentric dimensionless vector coordinates

(20) $$\overline{\xi}_{\,1} = \frac{1}{\sqrt{2}}\left(\overline{x}_{\,1} - \overline{x}_{\,2}\right) \quad , \quad \overline{\xi}_{\,2} = \frac{1}{\sqrt{6}}\left(\overline{x}_{\,1} + \overline{x}_{\,2} - 2\,\overline{x}_{\,3}\right)$$

in eq. 19 as if pertaining to only 1 space dimension . $\longrightarrow$

The even dimension already for 1 space dimension : 2 , of the barycentric *relative* configuration coordinates with vanishing values for the c.m. coordinate(s) , derive from the 3 base positions of the quarks bound in baryons , discussed in ref. 1 [1-2013] in subsections 2-4 , and all subsections of section 3 , and we follow this outline here .

The pairing mode allows to reveal explicity the hidden SU2 symmetry inherent to a pair of oscillators . For N_c odd , i.e. whence baryons are *fermions* the number of oscillator modes for 3 space dimensions is $3 \left(N_c - 1 \right)$, i.e. all modes (6 for $N_c = 3$) can be paired – yielding for $N_c = 3$ three pair-modes in baryons . One pair-mode was extensively described in ref. 5 [5-2010] besides ref. 1 [1-2013] . Here we repeat eq. 82 in ref. 1 , op.cit. , on page 3-pmodes-1 [PDF page # 57] adapting the variables $\xi_{1,2} \rightarrow \overline{\xi}_{1,2}$

$$
\left(a , b \right) \equiv \left(a_1 , a_2 \right) \quad
\left|
\begin{array}{l}
\zeta = \frac{1}{\sqrt{2}} \left(x + i y \right) \quad ; \quad x = \overline{\xi}_2 , y = \overline{\xi}_1 \\[2ex]
a = \frac{1}{\sqrt{2}} \left(\partial_{\zeta} + \overline{\zeta} \right) \quad ; \quad b = \frac{1}{\sqrt{2}} \left(\partial_{\overline{\zeta}} + \zeta \right) \\[1ex]
\hline
\left[a , a^{\dagger} \right] = \P \quad , \quad \left[b , b^{\dagger} \right] = \P \\[2ex]
\left[a , b \right] = \left[a , b^{\dagger} \right] = \left[b , a^{\dagger} \right] = \left[a^{\dagger} , b^{\dagger} \right] = 0
\end{array}
\right.
\tag{21}
$$

$\longrightarrow$

We obtain the $(x\,,\,y) \equiv \left(\bar{\xi}_2\,,\,\bar{\xi}_1\right)$ **representation of the paired oscillators** $(a\,,\,b) \equiv (a_1\,,\,a_2)$ **, defined in eq. 21 , in the table-equations below**

$\frac{1}{\sqrt{2}}\,\zeta = \frac{1}{2}\,(x + i\,y)$ $\frac{1}{\sqrt{2}}\,\partial_{\bar{\zeta}} = \frac{1}{2}\,(\partial_x + i\,\partial_y)$	$\frac{1}{\sqrt{2}}\,\bar{\zeta} = \frac{1}{2}\,(x - i\,y)$ $\frac{1}{\sqrt{2}}\,\partial_{\zeta} = \frac{1}{2}\,(\partial x - i\,\partial_y)$
$\frac{1}{\sqrt{2}}\left(\zeta + \partial_{\bar{\zeta}}\right) = \frac{1}{2}\left[\begin{array}{c} x + \partial_x + \\ + i\,(y + \partial_y) \end{array}\right]$	$\frac{1}{\sqrt{2}}\left(\bar{\zeta} + \partial_{\zeta}\right) = \frac{1}{2}\left[\begin{array}{c} x + \partial_x - \\ - i\,(y + \partial_y) \end{array}\right]$

(22)

Further it follows for the adjoint operators from eq. 22

$$a = \frac{1}{\sqrt{2}}\left(\zeta + \partial_{\bar{\zeta}}\right) \;\; ; \;\; a^{\dagger} = \frac{1}{2}\left[\begin{array}{c} x - \partial_x - \\ - i\,(y - \partial_y) \end{array}\right] = \frac{1}{\sqrt{2}}\left(\bar{\zeta} - \partial_{\zeta}\right)$$

(23)

$$b = \frac{1}{\sqrt{2}}\left(\bar{\zeta} + \partial_{\zeta}\right) \;\; ; \;\; b^{\dagger} = \frac{1}{2}\left[\begin{array}{c} x - \partial_x + \\ + i\,(y - \partial_y) \end{array}\right] = \frac{1}{\sqrt{2}}\left(\zeta - \partial_{\bar{\zeta}}\right)$$

$\longrightarrow$

The polynomial basis of normalized wave function associated with the two paired absorption oscillators (a_1 , a_2) follows from the associated construction of the creation oscillators $a^{\dagger}_{1,2}$ in eq. 23 . It lays the ground for the assignment of 'oscillator mode wave functions in the circular pair-mode basis', *in particular* $a^{\dagger}_{1,2} \rightarrow n_{1,2}$ **, in eq. 24 below , to be completed in subsection 1-3 .**

$$\psi_{n_1, n_2} \left(\zeta , \overline{\zeta} \right) = \mathcal{N} \, 2^{-\frac{1}{2} (n_1 + n_2)} \left(\overline{\zeta} - \partial_{\zeta} \right)^{n_1} \left(\zeta - \partial_{\overline{\zeta}} \right)^{n_2} \exp \left(- \zeta \overline{\zeta} \right)$$

$$\zeta \overline{\zeta} = \tfrac{1}{2} (x^2 + y^2)$$

(24)

In eq. 24 $\mathcal{N}$ **denotes the normalization constant of the ground state with** $N = 0$

$$\mathcal{N}^{-2} = (n_1 !) (n_2 !) \int \tfrac{1}{2} \left| d \zeta \wedge d\overline{\zeta} \right| \exp \left(- 2 \zeta \overline{\zeta} \right)$$

$$\tfrac{1}{2} d \zeta \wedge d \overline{\zeta} == \tfrac{1}{4} (d x + i d y) \wedge (d x - i d y) = \tfrac{1}{2 i} (d x \wedge d y)$$

(25) $$\tfrac{1}{2} \left| d \zeta \wedge d \overline{\zeta} \right| = d\zeta \, d \overline{\zeta} = d x \, d y$$

$$\mathcal{N}^{-2} = (n_1 !) (n_2 !) \int d x \, d y \exp (- x^2 - y^2) =$$

$$= (n_1 !) (n_2 !) \, \pi \int_0^{\infty} d \varrho \, e^{- \varrho} = (n_1 !) (n_2 !) \, \pi$$

$\longrightarrow$

Finally we turn to the paired oscillator mode orthogonal polynomials . These are not Hermite polynomials, which prevail for unpaired modes, but simple monomials. This structure is derived from substituting the two expressions for the creation operators $a^{\dagger}_{1,2}$: $\left(\overline{\zeta} - \partial_{\zeta} \right)^{n_1}$ and $\left(\zeta - \partial_{\overline{\zeta}} \right)^{n_2}$ in eq. 24 , as combined operators inside the powers , acting on the left on the given paired mode ground state

$$
\begin{array}{ll}
 & \sqrt{2}\, a^{\dagger}_{1} = \overline{\zeta} - \partial_{\zeta} = \exp\left(\zeta \overline{\zeta} \right) \left(- \partial_{\zeta} \right) \exp\left(- \zeta \overline{\zeta} \right) \\
(26) & \\
 & \sqrt{2}\, a^{\dagger}_{2} = \zeta - \partial_{\overline{\zeta}} = \exp\left(\zeta \overline{\zeta} \right) \left(- \partial_{\overline{\zeta}} \right) \exp\left(- \zeta \overline{\zeta} \right)
\end{array}
$$

The expression for the paired wave function $\psi_{n_1, n_2}\left(\zeta , \overline{\zeta} \right)$ in eq. 24 then takes the form

$$
\begin{array}{lll}
 & \psi_{n_1, n_2}\left(\zeta , \overline{\zeta} \right) = & \\
(27) & = & \mathcal{N}\, 2^{-\frac{1}{2}(n_1 + n_2)} \exp\left(\zeta \overline{\zeta} \right) \left(- \partial_{\zeta} \right)^{n_1} \left(- \partial_{\overline{\zeta}} \right)^{n_2} \exp\left(- 2\, \zeta \overline{\zeta} \right) \\
 & = & \mathcal{N}\, 2^{\frac{1}{2}(n_1 + n_2)}\, \overline{\zeta}^{\,n_1} \zeta^{n_2} \exp\left(- \zeta \overline{\zeta} \right)
\end{array}
$$

We use polar coordinates , as they are representing finite rotations of the complex ζ - plane $\longrightarrow$

leading to the wave function representation

$$\psi_{n_1, n_2}\left(\zeta, \overline{\zeta}\right) =$$

(28) $$= \left(\frac{2^{(n_1+n_2)}}{\pi\,(n_1!)\,(n_2!)}\right)^{\frac{1}{2}} \exp\left(i\,(n_2 - n_1)\,\varphi\right)\left[\varrho^{(n_1+n_2)} \exp\left(-\varrho^2\right)\right]$$

$$\varrho = |\zeta| \quad ; \quad \varphi = arg\,(\zeta)$$

The functions ψ_{n_1, n_2} in eq. 28 form a complete basis in the space $L_2(\zeta, \overline{\zeta})$. They are combined with the restrictions from overall Fermi statistics – including an overall color antisymmetric selection rule in conjunction with the three Young tableaux as displayed in eq. (to be dfined) .
Thus we study the action of the symmetric group S_3 on these base functions , as defined in eq. 17 .

(29) $$\left(U\left[\pi\begin{pmatrix} 1 & 2 & 3 \\ i_1 & i_2 & i_3 \end{pmatrix}\right]\psi_{n_1, n_2}\right)\left(\zeta, \overline{\zeta}\right) = \\ = \psi_{n_1, n_2}\left(\left[\pi\begin{pmatrix} 1 & 2 & 3 \\ i_1 & i_2 & i_3 \end{pmatrix}\right]^{-1}\left(\zeta, \overline{\zeta}\right)\right)$$

$\longrightarrow$

Eq. 29 needs to be elaborated, as follows

(30)
$$\left[\pi \begin{pmatrix} 1 & 2 & 3 \\ i_1 & i_2 & i_3 \end{pmatrix} \right]^{-1} = \pi \begin{pmatrix} i_1 & i_2 & i_3 \\ 1 & 2 & 3 \end{pmatrix}$$

Next we identify the subgroup of even permutations – $A_3 = Z_3$ **– of** S_3

(31)
$$Z = \pi \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \text{ or } \pi \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$
$$Z^2 = \pi \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \text{ or } \pi \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \; ; \; Z^3 = \P$$
$$\rightarrow \; Z^{-1} = Z^2 \; ; \; Z^{-2} = Z$$

Here is the place to emphasize that the discussion within subsection 1-2-1 is for the time beeing restricted to oscillatory modes in *one* space dimension, to be generalized to three subsequently, but after finishing the selection rules staying with 1 space dimension for the time beeing.
We proceed to identify the permutation Z as defined in eq. 31 with a rotation of the ζ **plane by 120 degrees , completing the action of** S_3 **displayed in eq. 29** $\longrightarrow$

(32)
$$\left(U \left[\pi \begin{pmatrix} 1 & 2 & 3 \\ i_1 & i_2 & i_3 \end{pmatrix} \right] \psi_{n_1, n_2} \right) \left(\zeta, \overline{\zeta} \right) =$$
$$= D_{m_1 m_2 n_1 n_2} (\pi(.)) \ \psi_{m_1, m_2} \left(\zeta, \overline{\zeta} \right)$$

and for the abelian cyclic subgroup A_3 of even permutations eq. 32 becomes

(33)
$$\left(U \left[\pi \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \right] \psi_{n_1, n_2} \right) \left(\zeta, \overline{\zeta} \right) =$$
$$= D_{n_1 n_2}(Z) \ \psi_{n_1, n_2} \left(\zeta, \overline{\zeta} \right) = \psi_{n_1, n_2} \left(Z^{-1} \zeta, Z \overline{\zeta} \right)$$

Eqs. 28 and 33 imply

(34)
$$D_{n_1 n_2}(Z) \ \psi_{n_1, n_2} \left(\zeta, \overline{\zeta} \right) =$$
$$= \exp\left(i \left(n_1 - n_2 \right) \left(2 \pi / 3 \right) \right) \psi_{n_1, n_2} \left(\zeta, \overline{\zeta} \right) =$$
$$= \psi_{n_1, n_2} \left(Z^{-1} \zeta, Z \overline{\zeta} \right)$$

It follows from eq. 34

(35)
$$D_{n_1 n_2}(Z) = Z^{n_1 - n_2} \ , \quad Z = \exp\left(i \left(2 \pi / 3 \right) \right)$$

$\longrightarrow$

Next we decompose the action of Z on ζ into real and imaginary parts

$$\zeta \longrightarrow Z\,\zeta = \zeta' \;:$$

(36)
$$\zeta = \begin{pmatrix} \bar{\xi}_2 \\ \bar{\xi}_1 \end{pmatrix} \;,\; \zeta' = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \bar{\xi}_2 \\ \bar{\xi}_1 \end{pmatrix} \longrightarrow$$

$$\bar{\xi}'_2 = -\frac{1}{2}\,\bar{\xi}_2 - \frac{\sqrt{3}}{2}\,\bar{\xi}_1$$

$$\bar{\xi}'_1 = \frac{\sqrt{3}}{2}\,\bar{\xi}_2 - \frac{1}{2}\,\bar{\xi}_1$$

We recall eq. 19 , repeating it below

(37)
$$\bar{\xi}_1 = \frac{1}{\sqrt{2}}\,(\bar{x}_1 - \bar{x}_2) \;,\; \bar{\xi}_2 = \frac{1}{\sqrt{6}}\,(\bar{x}_1 + \bar{x}_2 - 2\,\bar{x}_3)$$

$$\bar{\xi}_3 = \frac{1}{\sqrt{3}}\,(\bar{x}_1 + \bar{x}_2 + \bar{x}_3) = \sqrt{3}\;\overline{X}_{c.m.} \;\rightarrow\; 0$$

Substituting the expressions on the first line in eq. 37 in eq. 36 we obtain

(38)
$$\bar{\xi}'_2 = -\frac{1}{2\sqrt{6}}\,(\bar{x}_1 + \bar{x}_2 - 2\,\bar{x}_3) - \frac{\sqrt{3}}{2\sqrt{2}}\,(\bar{x}_1 - \bar{x}_2)$$

$$\bar{\xi}'_1 = \frac{\sqrt{3}}{2\sqrt{6}}\,(\bar{x}_1 + \bar{x}_2 - 2\,\bar{x}_3) - \frac{1}{2\sqrt{2}}\,(\bar{x}_1 - \bar{x}_2)$$

$\longrightarrow$

and arranging the factors yielding the result

$$\overline{\xi}\,'_2 = -\frac{1}{2\sqrt{6}}\,[\,(\,\overline{x}_1 + \overline{x}_2 - 2\,\overline{x}_3\,) + 3\,(\,\overline{x}_1 - \overline{x}_2\,)\,]$$

(39)

$$\overline{\xi}\,'_1 = \frac{1}{2\sqrt{2}}\,[\,(\,\overline{x}_1 + \overline{x}_2 - 2\,\overline{x}_3\,) - (\,\overline{x}_1 - \overline{x}_2\,)\,]$$

The final result is compared with the initial choice of barycentric variables in eq. 37

$$\overline{\xi}\,'_2 = \frac{1}{\sqrt{6}}\,(\,\overline{x}_2 + \overline{x}_3 - 2\,\overline{x}_1\,) \quad \leftarrow \quad \overline{\xi}_2 = \frac{1}{\sqrt{6}}\,(\,\overline{x}_1 + \overline{x}_2 - 2\,\overline{x}_3\,)$$

(40)

$$\overline{\xi}\,'_1 = \frac{1}{\sqrt{2}}\,(\,\overline{x}_2 - \overline{x}_3\,) \quad \leftarrow \quad \overline{\xi}_1 = \frac{1}{\sqrt{2}}\,(\,\overline{x}_1 - \overline{x}_2\,)$$

The choice , marked by ’or’ in eq. 31 , is revealed inspecting the substitution of the $\overline{\xi}_j$ indices from the right hand - to the left hand side of eq. 40 , corresponding to the cyclic permutation associated with the actions of Z and $Z^{-1} \equiv Z^2$

$$Z \quad : \quad \zeta \longrightarrow Z\,\zeta \quad \simeq \quad \pi \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

(41)

$$Z^2 \quad : \quad \zeta \longrightarrow Z^2\,\zeta \quad \simeq \quad \pi \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}$$

$\longrightarrow$

The following remarks are pertinent to *this outline* .

1) The complex structure defining the 'oscillator mode wave functions in the circular pair-mode basis', resides – for 1 spatial dimension – in the complex plane $\mathbb{C}$, characterized by the variable $\zeta \in \mathbb{C}$, which contains the quantity Z

(42) $$Z = \exp \left(i \left(2 \pi / 3 \right) \right)$$

on its counterclockwiese oriented unit circle , *in contrast* **to the complex conjugate plane $\overline{\mathbb{C}}$. The latter is naturally associated with $\overline{\zeta} \in \overline{\mathbb{C}}$, wherein the complex conjugate value relative to Z**

(43) $$\overline{Z} = Z^{-1} = \exp \left(- i \left(2 \pi / 3 \right) \right)$$

is situated .

2) While $Z \longleftrightarrow \overline{Z}$, $\zeta \longleftrightarrow \overline{\zeta}$ – together – can freely be exchanged , the ceation operators of oscillatory modes of quarks in baryons

(44) $$a_1^{\dagger} , a_2^{\dagger}$$

defined in eq. 21 , are locked in this order to the quantities Z , ζ , i.e. to the complex structure chosen and they have to be exchanged , together with the operation of complex conjugation , i.e.

(45) $$\zeta \rightarrow \overline{\zeta} \quad \text{has to go together with} \quad a_1^{\dagger} \rightarrow a_2^{\dagger} , a_2^{\dagger} \rightarrow a_1^{\dagger}$$

in order to represent an intrisically immaaterial reparametrization .

$\longrightarrow$

1-3 – Associating symmetric and antisymmetric representations of $SU6\ (\ fl\ \times\ spin\ )$ to oscillator mode wave functions in the circular pair-mode basis

1-3-1 - Choosing a complex basis for transforming the basis derived in ref. 1 , op.cit. [eq. 124] for the two-dimensional irreducible unitary representation of S_3 from 1 spacelike dimension

Here we invert the decomposition of the complex numbers $\zeta\ ,\ \overline{\zeta}$ into real and imaginary parts , as displayed (e.g.) in eq. 36

(46)

$$\zeta\ \longrightarrow\ Z\,\zeta\ =\ \zeta\,'\ \ :$$

$$\zeta\ =\ \begin{pmatrix} \overline{\xi}_{\,2} \\ \overline{\xi}_{\,1} \end{pmatrix}\ \ ,\ \ \zeta\,'\ =\ \begin{pmatrix} -\ \frac{1}{2} & -\ \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} \overline{\xi}_{\,2} \\ \overline{\xi}_{\,1} \end{pmatrix}\ \longrightarrow$$

$$\overline{\xi}_{\,2}{}' \ =\ -\ \frac{1}{2}\ \overline{\xi}_{\,2}\ -\ \frac{\sqrt{3}}{2}\ \overline{\xi}_{\,1}$$

$$\overline{\xi}_{\,1}{}' \ =\ \frac{\sqrt{3}}{2}\ \overline{\xi}_{\,2}\ -\ \ \frac{1}{2}\ \ \overline{\xi}_{\,1}$$

repeated above and on next page below

$\longrightarrow$

$$\zeta \rightarrow \overline{\xi}_2 + i\,\overline{\xi}_1 \rightarrow \begin{pmatrix} \overline{\xi}_2 \\ \overline{\xi}_1 \end{pmatrix} \;\; ; \;\; 2) \circ 4) \; : \; \zeta'' \rightarrow \begin{pmatrix} \overline{\xi}_2'' \\ \overline{\xi}_1'' \end{pmatrix}$$

(47)

$$\begin{pmatrix} \overline{\xi}_2'' \\ \overline{\xi}_1'' \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \overline{\xi}_2 \\ \overline{\xi}_1 \end{pmatrix}$$

back to the original complex- and complex conjugate variables ζ , $\overline{\zeta}$, as defined on the left hand side of the first relation in eq. 47

$$\begin{pmatrix} \overline{\xi}_2 \\ \overline{\xi}_1 \end{pmatrix} \rightarrow \begin{pmatrix} \zeta = \overline{\xi}_2 + i\,\overline{\xi}_1 \\ \overline{\zeta} = \overline{\xi}_2 - i\,\overline{\xi}_1 \end{pmatrix} = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} \overline{\xi}_2 \\ \overline{\xi}_1 \end{pmatrix} = \mathcal{M} \begin{pmatrix} \overline{\xi}_2 \\ \overline{\xi}_1 \end{pmatrix}$$

$$\mathcal{M}^{\dagger} = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \longrightarrow \mathcal{M}\,\mathcal{M}^{\dagger} = \mathcal{M}^{\dagger}\,\mathcal{M} = 2\,(\,\P\,)_{2\times 2}$$

(48)

Thus the unitary 2 x 2 matrix

(49)

$$u = \frac{1}{\sqrt{2}}\,\mathcal{M}$$

is a unitary 2 x 2 matrix which generates the similarity transformation $\longrightarrow$

through the following steps , denoting by $D_\pi \; ; \; \pi \in S_3$ the six 2 x 2 unitary matrices in the basis given in ref 1. , op.cit. , [eq. 124] and likewise by $d_\pi \; ; \; \pi \in S_3$ the six transformed 2 x 2 unitary matrices , associated with the basis as described in eq. 48

$$\pi \to d_\pi \; : \begin{pmatrix} \zeta \\ \bar{\zeta} \end{pmatrix} \longrightarrow d_\pi \begin{pmatrix} \zeta \\ \bar{\zeta} \end{pmatrix} = d_\pi \mathcal{M} \begin{pmatrix} \bar{\xi}_2 \\ \bar{\xi}_1 \end{pmatrix}$$

(50)
$$\begin{pmatrix} \zeta \\ \bar{\zeta} \end{pmatrix} = \mathcal{M} \begin{pmatrix} \bar{\xi}_2 \\ \bar{\xi}_1 \end{pmatrix}$$

$$\pi \to D_\pi \; : \begin{pmatrix} \bar{\xi}_2 \\ \bar{\xi}_1 \end{pmatrix} \longrightarrow D_\pi \begin{pmatrix} \bar{\xi}_2 \\ \bar{\xi}_1 \end{pmatrix}$$

Next we multiply the last relation in eq. 50 by $\mathcal{M}$ from the left

(51)
$$\pi \to D_\pi \; : \begin{pmatrix} \zeta \\ \bar{\zeta} \end{pmatrix} \longrightarrow \mathcal{M} D_\pi \begin{pmatrix} \bar{\xi}_2 \\ \bar{\xi}_1 \end{pmatrix} = \mathcal{M} D_\pi \mathcal{M}^{-1} \begin{pmatrix} \zeta \\ \bar{\zeta} \end{pmatrix}$$

$\longrightarrow$

Comparing eq. 51 with the first relation in eq. 50 we obtain the sought similarity transformation

$$d_{\pi} = \mathcal{M} D_{\pi} \mathcal{M}^{-1} = u D_{\pi} u^{-1}$$

(52)

$$u = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} , \quad u^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$$

The detailed calculations of the matrix product associated with the sixth permutation representation matrices d_{π} from the basis formed by D_{π} given in ref. 1 , op.cit. , [eq. 124] are performed in ref. 1 , op.cit. , in Appendix 3 . The collection of 2 x 2 representation matrices $d_{\pi} ; \pi = 1, \cdots, 6$ is displayed in eq. 53 below .

It is the $d_{\pi} ; \pi = 1, \cdots, 6$ representation , which is aligned with the circular pair-mode basis borne out by the one dimensional irreducible representation of the abelian subgroup of cyclic permutaions or $A_3 \subset S_3$ in this basis (of 2 x 2 matrices) $\longrightarrow$

$$1)\ d_{\pi=1} \quad \pi \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad + \quad \to ¶$$

$$2)\ d_{\pi=2} \quad \pi \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} e^{+i(2\pi/3)} & 0 \\ 0 & e^{-i(2\pi/3)} \end{pmatrix} \quad + \quad \to Z$$

$$3)\ d_{\pi=3} \quad \pi \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} e^{-i(2\pi/3)} & 0 \\ 0 & e^{+i(2\pi/3)} \end{pmatrix} \quad + \to Z^2 \equiv Z^{-1}$$

$$4)\ d_{\pi=4} \quad \pi \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad - \quad \to T_{12}$$

$$5)\ d_{\pi=5} \quad \pi \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & e^{+i(2\pi/3)} \\ e^{-i(2\pi/3)} & 0 \end{pmatrix} \quad - \quad \to T_{23}$$

$$6)\ d_{\pi=6} \quad \pi \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & e^{-i(2\pi/3)} \\ e^{+i(2\pi/3)} & 0 \end{pmatrix} \quad - \quad \to T_{13}$$

(53)

1-3-2 - Extending 1 spatial dimension to 3

We go back to subsection 1-2-1 .

The 1 complex variable ζ associated with the 1 pairmode as appropriate for 1 space dimension and defined in eq. 21 , for 3 space-time dimensions with axes (X) , (Y) , (Z) , thus becomes a complex three vector

(54) $$\vec{\zeta} = (\, \zeta^{\,(X)} \, , \, \zeta^{\,(Y)} \, , \, \zeta^{\,(Z)} \,)$$

The notation (X) , (Y) , (Z) for the three orthogonal axes of the 3-dimensional configuration space in the c.m. system is chosen in order to prevent confusing these with the 1-dimensional quantities denoted $x \, \ldots$, $y \, \ldots$, $z \, \ldots$, as introduced for 1 spatial dimension and defined in eqs. 21 and 19 , which become 3-vectors for 3 space dimensions .

The extension of the various space variables from 1 to 3 dimensions we shall do in segmented steps :

$\longrightarrow$

1-1) The center of mass position variables in 1 spatial dimension

These variables appear (last) in eq. 13 repeated below

$$\langle \Psi^{(2)} \mid \Psi^{(1)} \rangle = \lambda^6 \int \Pi_{i=1}^{3} d^3 \overline{\xi}_i \, \delta^3 \left(\overline{\xi}_3 \right) \times$$
$$\times \Psi^{*(2)} (\overline{x}_1 , , \overline{x}_2 , \overline{x}_3) \Psi^{(1)} (\overline{x}_1 , \overline{x}_2 , \overline{x}_3)$$

$$\overline{\xi}_1 = \frac{1}{\sqrt{2}} (\overline{x}_1 - \overline{x}_2) \quad , \quad \overline{\xi}_2 = \frac{1}{\sqrt{6}} (\overline{x}_1 + \overline{x}_2 - 2 \overline{x}_3)$$
$$\overline{\xi}_3 = \frac{1}{\sqrt{3}} (\overline{x}_1 + \overline{x}_2 + \overline{x}_3) = \sqrt{3} \, \overline{X}_{c.m.} \rightarrow 0$$

$$z_j = \lambda \overline{z}_j \quad , \quad x_j = \lambda \overline{x}_j \quad ; \; j = 1, 2, 3 \quad ; \; X_{c.m.} = \lambda \overline{X}_{c.m.} = \frac{1}{3} \sum_{i=1}^{3} \overline{x}_i$$

(55)

1-3) Extension of the last three relations in eq. 55 to 3 spatial dimensions

The extension takes the form

$$x_j \rightarrow \vec{x}_j \quad \textbf{with} \quad \vec{x}_j = \left(x_j^{(X)} , x_j^{(Y)} , x_j^{(Z)} \right) ; \; j = 1, 2, 3$$

$$X_{c.m.} \rightarrow \vec{X}_{c.m.} \quad \textbf{with} \quad \vec{X}_{c.m.} = \left(X_{c.m.}^{(X)} , X_{c.m.}^{(Y)} , X_{c.m.}^{(Z)} \right) = 0$$

(56)

Again note that $\vec{X}_{c.m.}$ **and the axis superfix** (X) **denote very different objects .** $\longrightarrow$

1-3) (continued) The configuration space 3-vectors $\vec{x}_{1,2,3}$, $\vec{X}_{c.m.}$ in eq. 56 have dimension $\left[\right.$ mass $^{-1}\left.\right]$ in rational units. They can be reduced to dimensionless configuration space variables , as given in eqs. 3 - 9 for 1 spatial dimesnsion and in the case of 3 spatial dimensions follows straightforwardly from eqs. 3 , 7 and 11

(57) $$\vec{\overline{x}}_{j} = \lambda^{-1} \vec{x}_{j} \ ; \ j = 1,2,3 \ ; \ \lambda^{-1} = \left(\frac{\Lambda}{K_{N_c}} \right)^{1/2}$$

$$K_{N_c} = N_c / (N_c - 1) \ \left(= \tfrac{3}{2} \text{ here} \right) \ ; \ 2\Lambda = 1 / \alpha' \sim 1.06 \text{ GeV}^2$$

2-1) Dimensionless barycentric coordinates in 1 space dimension

We recall the definition of the dimensionless barycentric coordinates asociated with the dimensionless quantities $\overline{x}_{1,2,3}$, $\overline{X}_{c.m.} = 0$ for 1 spatial dimension in eqs. 19 and 55 in point 1-1)

$\longrightarrow$

2-1) (continued)

$$\overline{\xi}_1 = \frac{1}{\sqrt{2}} (\overline{x}_1 - \overline{x}_2) \quad , \quad \overline{\xi}_2 = \frac{1}{\sqrt{6}} (\overline{x}_1 + \overline{x}_2 - 2\,\overline{x}_3)$$

$$\overline{\xi}_3 = \frac{1}{\sqrt{3}} (\overline{x}_1 + \overline{x}_2 + \overline{x}_3) = \sqrt{3}\ \overline{X}_{c.m.} \ \rightarrow 0$$

$$x_j = \lambda\,\overline{x}_j \ ; \ j = 1,2,3 \ ; \ X_{c.m.} = \lambda\,\overline{X}_{c.m.} = \frac{1}{3}\sum_{i=1}^{3} \overline{x}_i = 0 \tag{58}$$

$$\overline{x}_1 = \frac{1}{\sqrt{2}}\,\overline{\xi}_1 + \frac{1}{\sqrt{6}}\,\overline{\xi}_2$$

$$\overline{x}_2 = -\frac{1}{\sqrt{2}}\,\overline{\xi}_1 + \frac{1}{\sqrt{6}}\,\overline{\xi}_2$$

$$\overline{x}_3 = -\frac{2}{\sqrt{6}}\,\overline{\xi}_2$$

2-3) Extension of the dimensionless variables in eq. 58 to 3 spatial coordinates

The extension of the configuration variables $\overline{x}_{1,2,3} \rightarrow \overline{\vec{x}}_{1,2,3}$ from d = 1 to d=3 dimensions is defined in point 1-3) . Similarly the 2 barycentric coordinates for d = 1 become three vectors for d = 3 .

2-3) (continued)

$$\vec{\overline{\xi}}_{\mathbf{1}} = \frac{1}{\sqrt{2}} \left(\vec{\overline{x}}_1 - \vec{\overline{x}}_2 \right) \quad , \quad \vec{\overline{\xi}}_{\mathbf{2}} = \frac{1}{\sqrt{6}} \left(\vec{\overline{x}}_1 + \vec{\overline{x}}_2 - 2\,\vec{\overline{x}}_3 \right)$$

(59)
$$\vec{\overline{\xi}}_{\mathbf{3}} = \frac{1}{\sqrt{3}} \left(\vec{\overline{x}}_1 + \vec{\overline{x}}_2 + \vec{\overline{x}}_3 \right) = \sqrt{3}\ \vec{\overline{X}}_{c.m.} \ \rightarrow \ 0$$

$$\vec{\overline{\xi}}_{\mathbf{j}} = \left(\overline{\xi}_{\mathbf{j}}^{\,(X)} , \overline{\xi}_{\mathbf{j}}^{\,(Y)} , \overline{\xi}_{\mathbf{j}}^{\,(Z)} \right) \quad ; \quad \mathbf{j = 1.2.3}$$

The suffix numbering the different barycentric 3-vectors (in 3 spacelike dimension) are displayed in boldface style in order to emphasize that the numerals labelling $\vec{\overline{\xi}}_{\mathbf{j}}$; j = 1.2.3 are logically quite distinct from those numerals labelling $\overline{x}_{1,2,3}$; $j = 1.2.3$, displayed in cursive mode . The inverse relations expressing $\overline{x}_{1,2,3}$; $j = 1.2.3$ as a function of the 2 independent, baricentric dimensionless 3-vectors become

(60)
$$\begin{aligned} \vec{\overline{x}}_1 &= \frac{1}{\sqrt{2}} \vec{\overline{\xi}}_{\mathbf{1}} + \frac{1}{\sqrt{6}} \vec{\overline{\xi}}_{\mathbf{2}} \\ \vec{\overline{x}}_2 &= -\frac{1}{\sqrt{2}} \vec{\overline{\xi}}_{\mathbf{1}} + \frac{1}{\sqrt{6}} \vec{\overline{\xi}}_{\mathbf{2}} \\ \vec{\overline{x}}_3 &= -\frac{2}{\sqrt{6}} \vec{\overline{\xi}}_{\mathbf{2}} \end{aligned}$$

The relations in eq. 60 elucidate the different meaning of the suffix labels in cursive mode and boldface mode . $\longrightarrow$

3-1) The *linear* oscillator basis and mode excitation numbers $n_1, \; n_2$ for 1 and 3 spatial dimensions

In order to recall the meaning of the originally adopted labels 1,2 we refer back to subsection 1-1 comprising eqs. 1 - 4 . First we adapt from these equations eq. 2 from the barycentric coordinates $z_1 \; , \; z_2$, defined in ref. [3-1980] , to the dimensionless ones $\overline{\xi}_1 \; , \; \overline{\xi}_2$ as used here for 1 spatial dimension , with the identification as in eq. 58 , together with their relative canonical momenta , denoted $\overline{\pi}_1 \; , \; \overline{\pi}_2$ (as displayed in eqs. 3 - 6)

(61)
$$\overline{\xi}_1 \, , \, \overline{\xi}_2 \longleftrightarrow \overline{\pi}_1 \, , \, \overline{\pi}_2$$
$$\overline{\pi}_\mu = \frac{1}{i} \, \partial_{\overline{\xi}_\mu} \quad ; \quad \mu = 1, 2$$

$$\overline{\xi}_1 = \frac{1}{\sqrt{2}} \left(\overline{x}_1 - \overline{x}_2 \right) \quad , \quad \overline{\xi}_2 = \frac{1}{\sqrt{6}} \left(\overline{x}_1 + \overline{x}_2 - 2 \, \overline{x}_3 \right)$$

The associated (universal) oscillator vector-variables derive from the relation on the last line in eq. 7

(62)
$$a_{\mu \, k} = \frac{1}{\sqrt{2}} \left(i \, \overline{\pi}_{\mu \, k} + \overline{\xi}_{\mu \, k} \right) \quad ; \quad a^{\dagger}_{\mu \, k} = \frac{1}{\sqrt{2}} \left(- i \, \overline{\pi}_{\mu \, k} + \overline{\xi}_{\mu \, k} \right)$$
$$\mu = 1, 2 \quad ; \quad k = 1, 2, 3 \quad ; \quad i \, \overline{\pi}_{\mu \, k} = \partial_{\overline{\xi}_{\mu \, k}}$$

which in turn yields the relativistic structure of the oscillatory $\mathcal{M}^2$ operator $\longrightarrow$

3-1) (continued)

substituting in eqs. 3 , 5 and 7

(63) $$\mathcal{M}^{2} = (2\Lambda) \sum_{\nu=1,\,k=1}^{\nu=2\,,k=3} \left[\overline{a}^{\dagger}_{\nu\,k}\,\overline{a}_{\nu\,k} + \frac{1}{2} \right] \; ; \; 2\Lambda = 1/\alpha^{\prime}$$

The contribution of the zero mode oscillations , $\frac{1}{2}$ for each dimension of configuration space (= 3) is inherent to the classical limiting form of oscillatory motion , and is accompanied in the sense of a long range approximation (especially given finite quark masses) by a constant correction .The latter remains non zero also in the limit of vanishing quark masses , as discussed in ref. [3-1980] . In eq. 63 $1/\alpha^{\prime}$ denotes the inverse of the Regge slope.

The last relation in eq. 62 determines the oscillator basis corresponding to the decomposition into *linear* **oscillatory modes , which straightforwardly extend from 1 to 3 spatial dimensions**

(64) $$a_{\mu\,k} = \frac{1}{\sqrt{2}} \left(\partial_{\,\overline{\xi}_{\mu\,k}} + \overline{\xi}_{\mu\,k} \right) \; ; \; a^{\dagger}_{\mu\,k} = \frac{1}{\sqrt{2}} \left(-\partial_{\,\overline{\xi}_{\mu\,k}} + \overline{\xi}_{\mu\,k} \right)$$
$$\mu = 1,2 \; ; \; k = 1,2,3$$

Eq. 63 refers to 3 spatial dimensions while the case d = 1 would correspond to limit the suffix k in eqs. 61 - 63 to k = 1 .

$\longrightarrow$

3-3) Extending to 3 spatial dimensions and circular oscillatory pair-modes and wave function basis

Pair-modes have been discussed for 1 spatial dimension in subsection

”Modes of a pair of onedimensional oscillators – pairmodes and the complex plane”

which comprizes eqs. 21 - 41 .

For 1 spatial dimension we have 2 independent oscillators , which exhibit , whence the flavor & spin degrees of freedom are decoupled according to the decomposition in iref. 1 , op.cit. , [eq. (63)] , an intrinsic SU2 symmetry of the 2 barycentric oscillatory modes . This symmetry allows to choose – always for 1 spacial dimension – a basis , henceforth called circular mode basis , in which the two oscillators take the form repeated in adapted notation below

(65)

$$\begin{array}{lcl} \zeta = \frac{1}{\sqrt{2}} \left(\overline{\xi}_{\,2} + i \, \overline{\xi}_{\,1} \right) & ; & x = \overline{\xi}_{\,2} \, , \, y = \overline{\xi}_{\,1} \\ a_{\,1} = \frac{1}{\sqrt{2}} \left(\partial_{\,\zeta} + \overline{\zeta} \right) & ; & a_{\,2} = \frac{1}{\sqrt{2}} \left(\partial_{\,\overline{\zeta}} + \zeta \right) \\ a_{\,1}^{\,\dagger} = \frac{1}{\sqrt{2}} \left(- \partial_{\,\overline{\zeta}} + \zeta \right) & ; & a_{\,2}^{\,\dagger} = \frac{1}{\sqrt{2}} \left(- \partial_{\,\zeta} + \overline{\zeta} \right) \end{array}$$

$$\left[\, a_{\,1} \, , \, a_{\,2} \, \right] = \left[\, a_{\,1} \, , \, a_{\,2}^{\,\dagger} \, \right] = \left[\, a_{\,2} \, , \, a_{\,1}^{\,\dagger} \, \right] = 0$$

$\longrightarrow$

3-3) (continued)

From here (eq. 65) extension from 1 to 3 space dimensions follows 'stroke by stroke' .

To demonstrate this we repeat eq. 54 in adapted form below

(66) $$\zeta \to \zeta \longrightarrow \vec{\zeta} = \left(\zeta^{(X)} , \zeta^{(Y)} , \zeta^{(Z)} \right)$$

which entails the extension of the circular mode oscillators in eq. 65

(67)
$$a_{\mu} \to \vec{a}_{\mu} \;\; ; \;\; \mu = 1,2 \to \mu = \mathbf{1,2}$$
$$\vec{a}_{\mu} = \left(a_{\mu}^{(X)} , a_{\mu}^{(Y)} , a_{\mu}^{(Z)} \right) \;\; ; \;\; \mu = \mathbf{1,2}$$

Eq. 65 extended to 3 space dimensions becomes

(68)
$$\zeta_{\mathbf{j}} = \frac{1}{\sqrt{2}} \left(\overline{\xi}_{2\,\mathbf{j}} + i \, \overline{\xi}_{\mathbf{1}\;\mathbf{j}} \right) \;\; ; \;\; x_{\mathbf{j}} = \overline{\xi}_{\mathbf{2}\;\mathbf{j}} , \; y_{\mathbf{j}} = \overline{\xi}_{\mathbf{1}\;\mathbf{j}}$$
$$a_{\mathbf{1}\;\mathbf{j}} = \frac{1}{\sqrt{2}} \left(\partial_{\zeta_{\mathbf{j}}} + \overline{\zeta}_{\mathbf{j}} \right) \;\; ; \;\; a_{\mathbf{2}\;\mathbf{j}} = \frac{1}{\sqrt{2}} \left(\partial_{\overline{\zeta}_{\mathbf{j}}} + \zeta_{\mathbf{j}} \right)$$
$$a^{\dagger}_{\mathbf{1}\;\mathbf{j}} = \frac{1}{\sqrt{2}} \left(- \partial_{\overline{\zeta}_{\mathbf{j}}} + \zeta_{\mathbf{j}} \right) \;\; ; \;\; a^{\dagger}_{\mathbf{2}\;\mathbf{j}} = \frac{1}{\sqrt{2}} \left(- \partial_{\zeta_{\mathbf{j}}} + \overline{\zeta}_{\mathbf{j}} \right)$$

$$\left[a_{\mathbf{1}\;\mathbf{j}} , a_{\mathbf{2}\;\mathbf{k}} \right] = \left[a_{\mathbf{1}\;\mathbf{j}} , a^{\dagger}_{\mathbf{2}\;\mathbf{k}} \right] = \left[a_{\mathbf{2}\;\mathbf{k}} , a^{\dagger}_{\mathbf{1}\;\mathbf{j}} \right] = 0 \;\; ; \;\; \mathbf{j} , \mathbf{k} = {}^{(X),(Y),(Z)}$$

⟶

3-3) (continued)

It is appropriate here – now for 3 space dimensions , and using the circular pair mode oscillator basis – to restate the vector nature of oscillator absorption and creation operators (as derived in eqs. 67 and 68) in vector- and component notation

(69)
$$\begin{array}{l} \vec{a}_{\mu} \leftrightarrow a_{\mu \, \mathbf{k}} = \left(\vec{a}_{\mu} \right)_{\mathbf{k}} \\ \vec{a}^{\,\dagger}_{\mu} \leftrightarrow a^{\dagger}_{\mu \, \mathbf{k}} = \left(\vec{a}^{\,\dagger}_{\mu} \right)_{\mathbf{k}} \end{array} \quad ; \quad \mu = \mathbf{1,2} \, , \, \mathbf{k} = (X) \, , \, (Y) \, , \, (Z)$$

The components – 6 each for creation- and annihilation operators – $a_{\mu \, \mathbf{j}} \, , \, a^{\dagger}_{\nu \, \mathbf{k}}$ **, interpreted in the circular pair mode oscillator basis , satisfy the nontrivial** ***commutation*** **relations**

(70)
$$\left[a_{\mu \, \mathbf{j}} \, , \, a^{\dagger}_{\nu \, \mathbf{k}} \right] = \delta_{\mu \nu} \, \delta_{\mathbf{jk}} \, \P \quad ; \quad \left\{ \begin{array}{c} \mu \, , \nu \\ \mathbf{j} \, , \mathbf{k} \end{array} \right\} = \left\{ \begin{array}{l} \mathbf{1,2} \\ (X), (Y), (Z) \end{array} \right\}$$

all other commutators vanish

The commutation relations in eq. 70 as such do not depend on the oscillator basis (allowing an SU6 invariant structure) , but the operator realization is particularly adapted , whence circular oscillator basis is chosen . This is done in the next step

⟶

3-3) (continued)

Extending the mode structure in the circular oscillator basis from 1 to 3 space dimensions

From here on new material , elaborated in 2013 , shapes the discussinon of counting oscillatory modes of quarks in baryons .

We repeat the form of the wave function in the circular oscillatry mode basis corresponding for 1 pair of oscillators , resulting from the associated structure of the pair of creation operators (eqs. 65 and 26 below

(71)

$$\begin{array}{lll} \zeta = \frac{1}{\sqrt{2}} \left(\overline{\xi}_{\,2} + i \, \overline{\xi}_{\,1} \right) & ; & x = \overline{\xi}_{\,2} \, , \, y = \overline{\xi}_{\,1} \\ a_{\,1} = \frac{1}{\sqrt{2}} \left(\partial_{\,\zeta} + \overline{\zeta} \right) & ; & a_{\,2} = \frac{1}{\sqrt{2}} \left(\partial_{\,\overline{\zeta}} + \zeta \right) \\ a^{\,\dagger}_{\,1} = \frac{1}{\sqrt{2}} \left(- \partial_{\,\overline{\zeta}} + \zeta \right) & ; & a^{\,\dagger}_{\,2} = \frac{1}{\sqrt{2}} \left(- \partial_{\,\zeta} + \overline{\zeta} \right) \end{array}$$

(72)

$$\left[\, a_{\,1} \, , \, a_{\,2} \, \right] = \left[\, a_{\,1} \, , \, a^{\,\dagger}_{\,2} \, \right] = \left[\, a_{\,2} \, , \, a^{\,\dagger}_{\,1} \, \right] = 0$$

$$\sqrt{2} \, a^{\,\dagger}_{\,1} = \overline{\zeta} - \partial_{\,\zeta} = \exp \left(\zeta \, \overline{\zeta} \right) \left(- \partial_{\,\zeta} \right) \exp \left(- \zeta \, \overline{\zeta} \right)$$

$$\sqrt{2} \, a^{\,\dagger}_{\,2} = \zeta - \partial_{\,\overline{\zeta}} = \exp \left(\zeta \, \overline{\zeta} \right) \left(- \partial_{\,\overline{\zeta}} \right) \exp \left(- \zeta \, \overline{\zeta} \right)$$

$\longrightarrow$

3-3) (continued)

Thus as shown in eqs. 27 and 28 for 1 space dimensions the wave function in the circular oscillator basis – also corresponding to 1 space dimension takes the form repeated below

$$\psi_{n_1, n_2}\left(\zeta, \overline{\zeta}\right) =$$
$$= \mathcal{N}\, 2^{-\frac{1}{2}(n_1 + n_2)} \exp\left(\zeta\, \overline{\zeta}\right) \left(-\partial_{\zeta}\right)^{n_1} \left(-\partial_{\overline{\zeta}}\right)^{n_2} \exp\left(-2\, \zeta\, \overline{\zeta}\right)$$
$$= \mathcal{N}\, 2^{\frac{1}{2}(n_1 + n_2)}\, \overline{\zeta}^{\, n_1}\, \zeta^{\, n_2} \exp\left(-\zeta\, \overline{\zeta}\right)$$

(73)

and

$$\psi_{n_1, n_2}\left(\zeta, \overline{\zeta}\right) =$$
$$= \left(\frac{2^{(n_1 + n_2)}}{\pi\,(n_1!)\,(n_2!)}\right)^{\frac{1}{2}} \exp\left(i\,(n_2 - n_1)\,\varphi\right) \left[\varrho^{(n_1 + n_2)} \exp\left(-\varrho^2\right)\right]$$

$$\varrho = |\zeta| \quad ; \quad \varphi = arg\,(\zeta)$$

(74)

→

3-3) (continued)

The further extension from 1 to 3 space dimensions consists in assigning to the single integers $n_{1,2}$ determining the wave function in eqs. 73 and 74 vectors , denoted $\mathbf{n}_{1,2}$, which in components become

(75) $$\mathbf{n}_{\mu} = \left(\mathbf{n}_{\mu}^{(X)} , \mathbf{n}_{\mu}^{(Y)} , \mathbf{n}_{\mu}^{(Z)} \right) \quad ; \quad \mu = \mathbf{1,2}$$

The wave function for 1 space dimension , displayed in eq. 74 becomes for 3 such

$$\psi_{\mathbf{n}_1 , \mathbf{n}_2} \left(\vec{\zeta} , \vec{\vec{\zeta}} \right) =$$

(76) $$= \prod_{\mathbf{k}} \left(\left(\frac{2^{\left(n_{\mathbf{1}}^{\mathbf{k}} + n_{\mathbf{2}}^{\mathbf{k}} \right)}}{\pi \left(n_{\mathbf{1}}^{\mathbf{k}} ! \right) \left(n_{\mathbf{2}}^{\mathbf{k}} ! \right)} \right)^{\frac{1}{2}} \exp \left(i \left(n_{\mathbf{2}}^{\mathbf{k}} - n_{\mathbf{1}}^{\mathbf{k}} \right) \varphi_{\mathbf{k}} \right) \times \right.$$
$$\left. \times \left[\varrho_{\mathbf{k}}^{\left(\mathbf{n}_{\mathbf{1}}^{\mathbf{k}} + \mathbf{n}_{\mathbf{2}}^{\mathbf{k}} \right)} \exp \left(- \varrho_{\mathbf{k}}^{2} \right) \right] \right)$$

$$\varrho_{\mathbf{j}} = \left| \zeta_{\mathbf{j}} \right| \quad ; \quad \varphi_{\mathbf{j}} = arg \left(\zeta_{\mathbf{j}} \right) \quad ; \quad \mathbf{j} = (X), (Y), (Z)$$

$\longrightarrow$

3-3) (continued)

The imeasure in the scalar product in eq. 55 repeated below

$$\langle \Psi^{(2)} \mid \Psi^{(1)} \rangle = \lambda^6 \int \Pi_{i=1}^{3} d^3 \overline{\xi}_i \, \delta^3 \left(\overline{\xi}_3 \right) \times \times \Psi^{*(2)} (\overline{x}_1 , , \overline{x}_2 , \overline{x}_3) \Psi^{(1)} (\overline{x}_1 , \overline{x}_2 , \overline{x}_3)$$

$$\overline{\xi}_1 = \frac{1}{\sqrt{2}} (\overline{x}_1 - \overline{x}_2) \quad , \quad \overline{\xi}_2 = \frac{1}{\sqrt{6}} (\overline{x}_1 + \overline{x}_2 - 2 \overline{x}_3)$$

$$\overline{\xi}_3 = \frac{1}{\sqrt{3}} (\overline{x}_1 + \overline{x}_2 + \overline{x}_3) = \sqrt{3} \; \overline{X}_{c.m.} \;\; \rightarrow 0$$

$$z_j = \lambda \overline{z}_j \quad , \quad x_j = \lambda \overline{x}_j \;\; ; \; j = 1,2,3 \;\; ; \; X_{c.m.} = \lambda \overline{X}_{c.m.} = \frac{1}{3} \sum_{i=1}^{3} \overline{x}_i$$

(77)

is already adapted to 3 space dimensions.

This concludes all extensions from 1 to 3 space dimensions.

→

2 - Counting

From the SU $\left(2\, N_{fl} \;=\; 6 \right)$, spin $\times$ flavor symmetry assumed here as limiting , i.e. neglecting definite breakings in a systematic approximation , we separate off this SU6 symmetry classidication, discussed in section 2 of ref. 1 , op.cit. . We branch to the classification of irreducible SU6 representations , guiding the counting .

2-1 - The reduction of an SUN group R-fold product representation through the symmetric group S $_R$ $\longleftrightarrow$ Young tableaux Y ^N_R

The three irreducible Young tableaux in figures 1 - 3 below , arise as tensors of rank 3 within a symmetry group of SU $\left(6 \;=\; \text{SU2}_{\,spin} \;\times \text{SUN}_{\,fl} \;=3 \right)$. Its broken character is discussed later .

Rank three corresponds to the wave function of a baryon formed from three valence quarks, confined with respect to color, exclusively . In the construction of this wave function the width of the clearly involved resonance shall be set approximately to zero .

We turn towards the functions asociated with these Young tableaux , depending on 6 integer arguments

$$D^{\,6}_{\,3} \left(m_{\,6} \,,\, m_{\,5} \,, \cdots \,,\, m_{\,1} \right) \;=\; D \left(m_{\,6} \,,\, m_{\,5} \,, \cdots \,,\, m_{\,1} \right)$$

(78) $$m_{\,6} \;>\; m_{\,5} \;\cdots\; >\; m_{\,1} \;>\; 0 \quad \text{: integers}$$

$$D \left(m_{\,6} \,,\, m_{\,5} \,, \cdots \,,\, m_{\,1} \right) \;=\; \Pi^{\,6}_{\,j=2}\, \Pi^{\,k\,=\,j-1}_{\,k=1} \left(m_{\,j} \;-\; m_{\,j-k} \right)$$

The D - functions for symmetric , mixed and antisymmetric representations of SU6 will be discussed after the 3 figures below . $\longrightarrow$

2-2

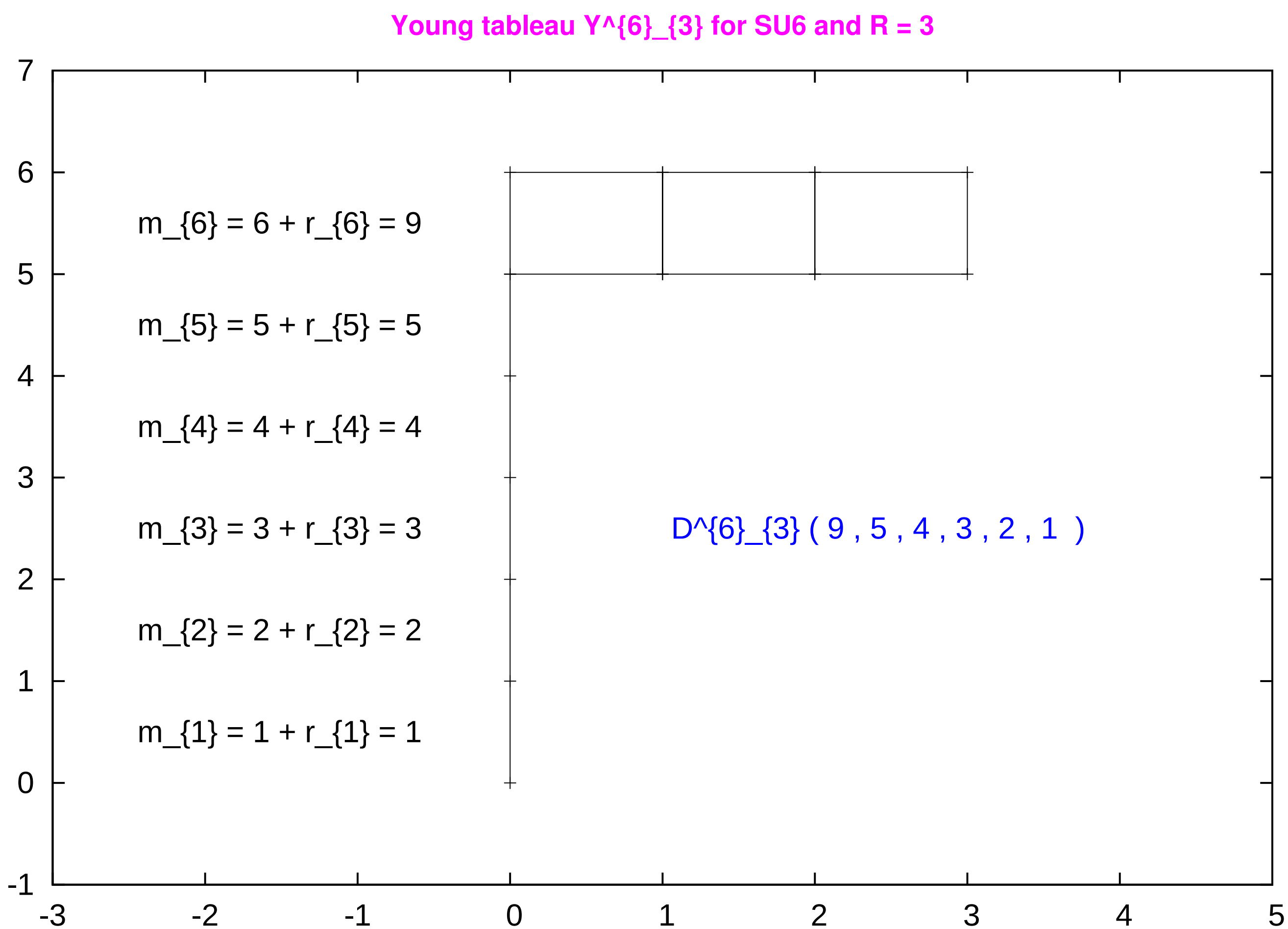


Fig 1 : The symmetric Young tableau for SU6 and rank $R = 3 \longleftrightarrow$

2-3

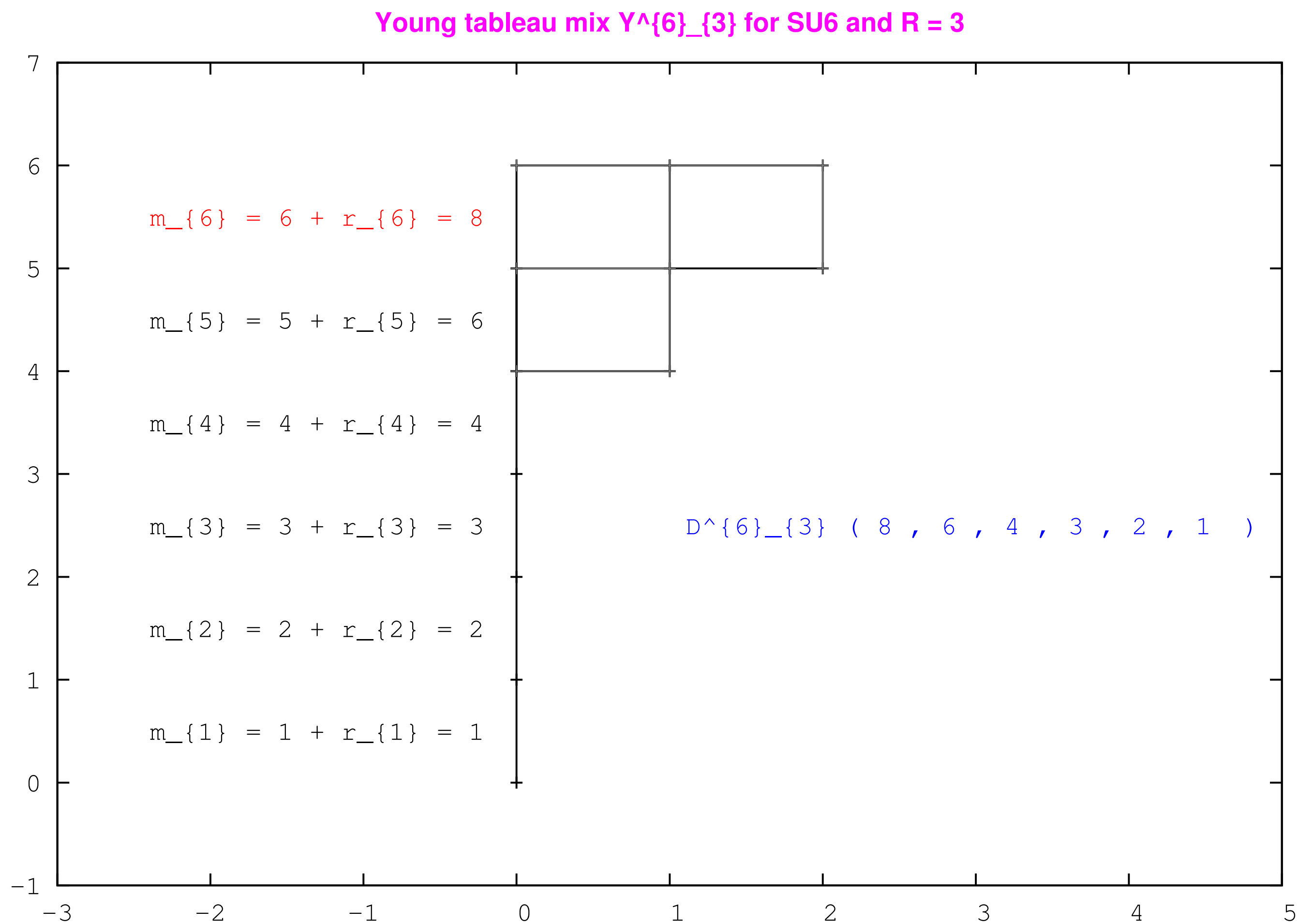


Fig 2 : The mixed Young tableau for SU6 and rank $R = 3 \longleftrightarrow$

2-4

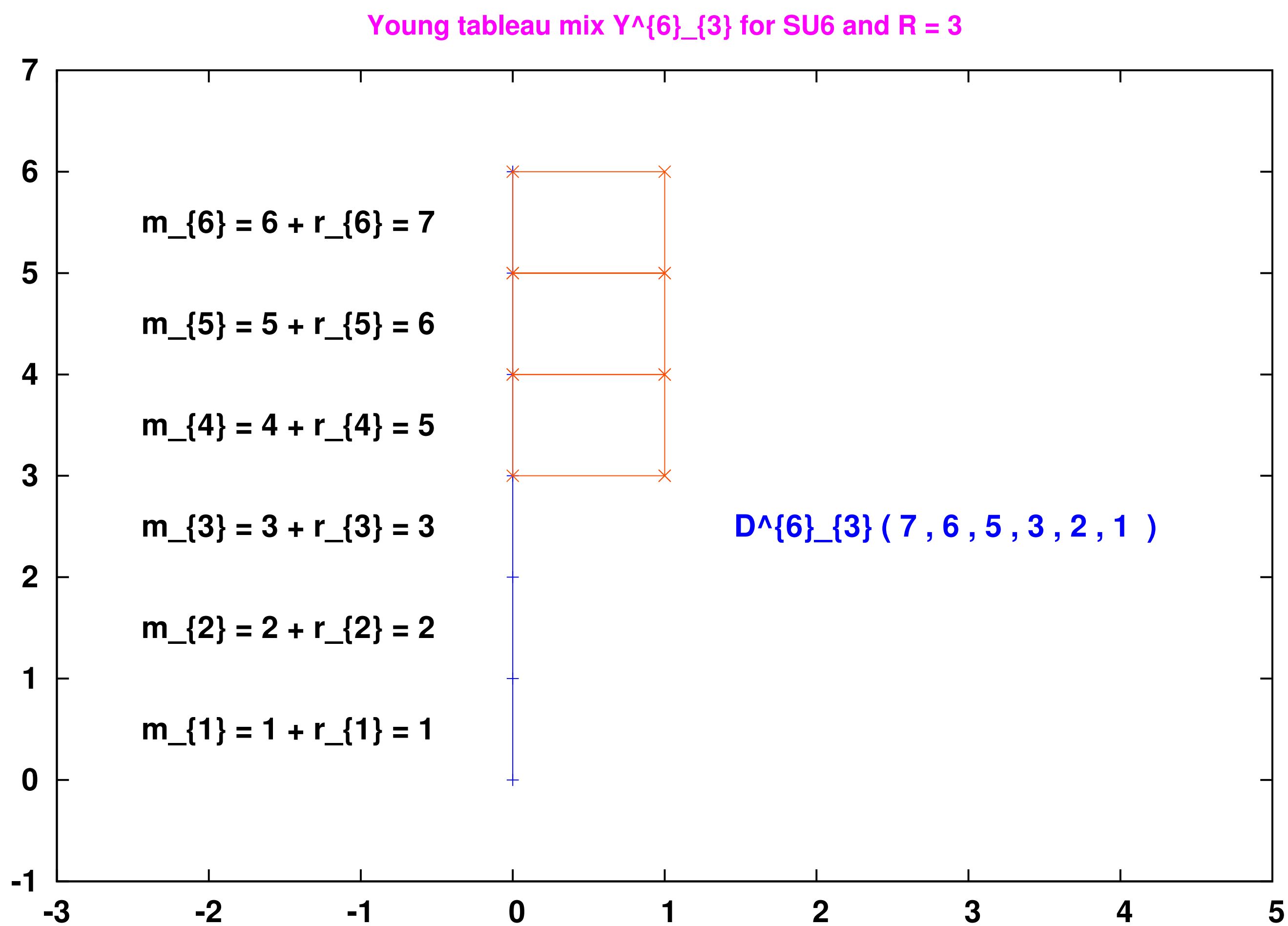
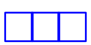


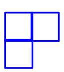
Fig 3 : The antisymmetric Young tableau for SU6 and rank $R = 3$ $\longleftrightarrow$

2-5

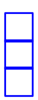
The significance of Young tableaux with respect to an SUN transformation group lies in the one to one correspondence of irreducible representations of this group formed by the R-fold tensor product of the defining one , symmetrized first along the rows and antisymmetrized thereafter along the columns associated with any given Young tableau , forming a representaion of the symmetric group S $_R$,
The three Young tableaux in figures 1 - 3 each determine such an irreducible representation of SU6 . The D functions , defined in eq. 78 , determine the respective dimensions of these representations .
We list the three D functions in eq. 79 below



$$: \quad D\,(9,5,4,3,2,1) = 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot d_5$$



$$: \quad D\,(8,6,4,3,2,1) = 2\cdot 4\cdot 5\cdot 6\cdot 7\cdot 2\cdot 3\cdot 4\cdot 5\cdot d_4$$



$$: \quad D\,(7,6,5,3,2,1) = 2\cdot 4\cdot 5\cdot 6\cdot 3\cdot 4\cdot 5\cdot 2\cdot 3\cdot 4\cdot d_3$$

$$d_j = D\,(j, j-1, \cdots, 1) = (j-1)!\,d_{j-1} = \Pi_{k=1}^{j-1} k!$$

$$d_1 = d_2 = 1,\ d_3 = 2,\ d_4 = 12,\ d_5 = 24\cdot 12,\ d_6 = 120\cdot 24\cdot 12$$

(79)

This concludes the discussion of the main premises contained in Young tableaux . ⟶

2-6

The dimensions of irreducible SUN representations belonging to a specific Young tableau are given by

$$dim\,(\textbf{Y-t}) = \frac{D\,(m_R\,,\, m_{R-1}\,,\,\cdots\,,\, m_1)}{D\,(R\,,\; R-1\,,\,\cdots\,,\,1)} \tag{80}$$

Substituting eq. 79 in eq. 80 it follows , always for SU6

$$1:\quad dim\,(\ydiagram{3}) = \frac{4\cdot5\cdot6\cdot7\cdot8}{5\,!} = 56$$

$$2:\quad dim\left(\ydiagram{2,1}\right) = \frac{2\cdot4\cdot5\cdot6\cdot7\cdot2\cdot3\cdot4\cdot5}{4\,!\,5\,!} = 70 \tag{81}$$

$$3:\quad dim\left(\ydiagram{1,1,1}\right) = \frac{2\cdot4\cdot5\cdot6\cdot3\cdot4\cdot5\cdot2\cdot3\cdot4}{3\,!\,4\,!\,5\,!} = 20$$

2-2 - From oscillatory modes to counting of states

The form of the mass square operator – as a long distance approximation – not specified iwith respect to any basis , in particular for the oscillatory zero modes , as well as other constant contributions in the configuration space distances at large .

The next step can be inferred from the way the universal inverse Regge slope determines the mass-square operator

(82) $$\mathcal{M}^{\,2} \;=\; (\,2\,\Lambda\,)\,\sum_{\nu=1,\;k=1}^{\nu=2\;,k=3}\;\left[\,\overline{a}^{\;\dagger}_{\;\nu\;k}\,\overline{a}_{\;\nu\;k}\;+\;\tfrac{1}{2}\,\right]\quad;\quad 2\,\Lambda\;=\;1\,/\,\alpha^{\,\prime}$$

We introducuce the decomposition

(83) $$\begin{array}{l} \mathcal{M}^{\,2} \;=\; \Delta\,\mathcal{M}^{\,2}\;+\;\mathcal{M}^{\,2}_{\,(0)}\quad;\quad \mathcal{M}^{\,2}_{\,(0)}\;=\;(\,2\,\Lambda\,)\;C_{\,(0)} \\ \Delta\,\mathcal{M}^{\,2}\;=\;(\,2\,\Lambda\,)\;\sum_{\nu=1,\;k=1}^{\nu=2\;,k=3}\Big|_{\text{extended to spin and flavor}}\;\left[\,\overline{a}^{\;\dagger}_{\;\nu\;k}\,\overline{a}_{\;\nu\;k}\,\right] \end{array}$$

2-2-1 - Using the circular pair-mode basis for oscillators and wave functions

We first recall the structure of the oscillator operators in the circular pair-mode basis , repeating eq. 68 below

$\longrightarrow$

(84)
$$\zeta_{\mathbf{j}} = \frac{1}{\sqrt{2}} \left(\overline{\xi}_{2\mathbf{j}} + i \overline{\xi}_{\mathbf{1}\ \mathbf{j}} \right) \quad ; \quad x_{\mathbf{j}} = \overline{\xi}_{\mathbf{2}\ \mathbf{j}} , \ y_{\mathbf{j}} = \overline{\xi}_{\mathbf{1}\ \mathbf{j}}$$

$$a_{\mathbf{1}\ \mathbf{j}} = \frac{1}{\sqrt{2}} \left(\partial_{\zeta_{\mathbf{j}}} + \overline{\zeta}_{\mathbf{j}} \right) \quad ; \quad a_{\mathbf{2}\ \mathbf{j}} = \frac{1}{\sqrt{2}} \left(\partial_{\overline{\zeta}_{\mathbf{j}}} + \zeta_{\mathbf{j}} \right)$$

$$a^{\dagger}_{\mathbf{1}\ \mathbf{j}} = \frac{1}{\sqrt{2}} \left(- \partial_{\overline{\zeta}_{\mathbf{j}}} + \zeta_{\mathbf{j}} \right) \quad ; \quad a^{\dagger}_{\mathbf{2}\ \mathbf{j}} = \frac{1}{\sqrt{2}} \left(- \partial_{\zeta_{\mathbf{j}}} + \overline{\zeta}_{\mathbf{j}} \right)$$

$$\left[a_{\mathbf{1}\ \mathbf{j}} , a_{\mathbf{2}\ \mathbf{k}} \right] = \left[a_{\mathbf{1}\ \mathbf{j}} , a^{\dagger}_{\mathbf{2}\ \mathbf{k}} \right] = \left[a_{\mathbf{2}\ \mathbf{k}} , a^{\dagger}_{\mathbf{1}\ \mathbf{j}} \right] = 0 \ ; \ \mathbf{j} , \mathbf{k} = {}^{(X),(Y),(Z)}$$

Further we recall that the oscillator operators form – for 3 spatial dimensions – three-vectors as displayed in eq. 67 and in components in eqs. 68 $\longrightarrow$ 84

(85)
$$\vec{a}_{\mu} = \left(a_{\mu}^{(X)} , a_{\mu}^{(Y)} , a_{\mu}^{(Z)} \right) \ ; \ \mu = \mathbf{1,2}$$

$$\vec{a}^{\dagger}_{\mu} = \left(a_{\mu}^{\dagger\,(X)} , a_{\mu}^{\dagger\,(Y)} , a_{\mu}^{\dagger\,(Z)} \right) \ ; \ \mu = \mathbf{1,2}$$

Likewise the oscillator occupation numbers form 3-vectors in the association

(86)
$$\vec{n}_{\mu} = \left(\mathbf{n}_{\mu}^{(X)} , \mathbf{n}_{\mu}^{(Y)} , \mathbf{n}_{\mu}^{(Z)} \right)$$

$$\updownarrow$$

$$\vec{a}^{\dagger}_{\mu} = \left(a_{\mu}^{\dagger\,(X)} , a_{\mu}^{\dagger\,(Y)} , a_{\mu}^{\dagger\,(Z)} \right) \ ; \ \mu = \mathbf{1,2}$$

$\longrightarrow$

repeated below

$$\vec{n}_{\mu} = \left(\mathbf{n}_{\mu}^{(X)} , \mathbf{n}_{\mu}^{(Y)} , \mathbf{n}_{\mu}^{(Z)} \right) \equiv \mathbf{n}_{\mu}$$

$$\updownarrow$$

(87)

$$\frac{\vec{a}_{\mu}^{\dagger} = \left(a_{\mu}^{\dagger\,(X)} , a_{\mu}^{\dagger\,(Y)} , a_{\mu}^{\dagger\,(Z)} \right) \;\; ; \;\; \mu = \mathbf{1,2}}{\mathbf{n}_{\mu}^{\mathbf{k}} = 0 , 1 , \cdots \;\; ; \;\; \mathbf{k} = (X) , (Y) , (Z)}$$

form two [$\mu = \mathbf{1,2}$] 3-vectors with nonnegative integer components . From these two 3-vectors – $\vec{n}_{\mu}$; $\mu = \mathbf{1,2}$ – we form two sums

$$N_{\mathbf{1}} = \sum_{\mathbf{k}} \mathbf{n}_{\mathbf{1}}^{\mathbf{k}} \;\; ; \;\; N_{\mathbf{2}} = \sum_{\mathbf{k}} \mathbf{n}_{\mathbf{2}}^{\mathbf{k}}$$

(88)

$$N_{\mathbf{1,2}} = 0 , 1 , \cdots$$

$$N_{\mathbf{1}} + N_{\mathbf{2}} = N \quad : \quad \text{main oscillatory quantum number}$$

The 3-vector nonnegative integer valued quantities $\vec{n}_{\mu} \equiv \mathbf{n}_{\mu}$ defined in eq. 87 fully determine the wave function in the circular 3-pair-mode oscillator basis in eq. 76 , repeated below $\longrightarrow$

$$\psi_{\mathbf{n}_1,\mathbf{n}_2}\left(\vec{\zeta},\vec{\vec{\zeta}}\right) =$$

(89)
$$= \prod_{\mathbf{k}} \left(\left(\frac{2^{\left(n_{\mathbf{1}}^{\mathbf{k}} + n_{\mathbf{2}}^{\mathbf{k}}\right)}}{\pi \left(n_{\mathbf{1}}^{\mathbf{k}}!\right)\left(n_{\mathbf{2}}^{\mathbf{k}}!\right)} \right)^{\frac{1}{2}} \exp\left(i\left(n_{\mathbf{2}}^{\mathbf{k}} - n_{\mathbf{1}}^{\mathbf{k}}\right)\varphi_{\mathbf{k}}\right) \times \right. \\ \left. \times \left[\varrho_{\mathbf{k}}^{\left(\mathbf{n}_{\mathbf{1}}^{\mathbf{k}} + \mathbf{n}_{\mathbf{2}}^{\mathbf{k}}\right)} \exp\left(-\varrho_{\mathbf{k}}^2\right) \right] \right)$$

$$\varrho_{\mathbf{j}} = \left|\zeta_{\mathbf{j}}\right| \;\; ; \;\; \varphi_{\mathbf{j}} = arg\left(\zeta_{\mathbf{j}}\right) \;\; ; \;\; \mathbf{j} = (X),(Y),(Z)$$

The wave function $\psi_{\mathbf{n}_1,\mathbf{n}_2}\left(\vec{\zeta},\vec{\vec{\zeta}}\right)$ displayed in eq. 89 [and eq. 76] is obtained applying the $\vec{n}_\mu \equiv \mathbf{n}_\mu -$ dependent product of creation operators to the ground state

(90)
$$\psi_{\mathbf{n}_1,\mathbf{n}_2}\left(\vec{\zeta},\vec{\vec{\zeta}}\right) = \Pi_{\mathbf{k}} \left(\frac{1}{\left(n_{\mathbf{1}}^{\mathbf{k}}!\right)\left(n_{\mathbf{2}}^{\mathbf{k}}!\right)} \right)^{\frac{1}{2}} \left(a^{\dagger}_{\mathbf{1}\;\mathbf{k}}\right)^{n_{\mathbf{1}}^{\mathbf{k}}} \left(a^{\dagger}_{\mathbf{2}\;\mathbf{k}}\right)^{n_{\mathbf{2}}^{\mathbf{k}}} \psi_{\mathbf{0},\mathbf{0}}$$

In eq. 90 $\psi_{\mathbf{0}\,,\,\mathbf{0}}$ denotes th ground state wave function

$$\psi_{\mathbf{0}\,,\,\mathbf{0}}\left(\vec{\zeta}\,,\,\vec{\vec{\zeta}}\right) = \prod_{\mathbf{k}} \left(\left(\frac{1}{\pi}\right)^{\frac{1}{2}} \left[\exp\left(-\varrho_{\mathbf{k}}^{2}\right)\right] \right) \tag{91}$$

$$\varrho_{\mathbf{j}} = \left|\zeta_{\mathbf{j}}\right| \;\; ; \;\; \mathbf{j} = (X), (Y), (Z)$$

→

3 - Harmony of numbers in QCD

The symmetric [56] and antisymmetric [20] representations of S_3 for $SU6 = SU\,(\,spin\;flavor\,)$, i.e. associated with the Young tableaux in figures Fig 1 and Fig 3 respectively , correspond to simple conditions on the wave functions formed from the basis in eq. 91

These conditions are twofold , expressed in the quantum numbers $\mathbf{n}_{\mathbf{1}}$; $\mathbf{n}_{\mathbf{2}}$ as defined in eq. 87 and 89 .

It is convenient to define the nonnegative integer quantities $N_{1,2}$ associated with $\mathbf{n}_{\mathbf{1,2}}$ respectively as well as their signed difference

(92)
$$N_1 = \sum_{\mathbf{k}} \mathbf{n}_{\mathbf{1}}^{\mathbf{k}} \;;\; N_2 = \sum_{\mathbf{k}} \mathbf{n}_{\mathbf{2}}^{\mathbf{k}} \;\; \text{with} \;\; N = N_1 + N_2$$
$$\Delta = N_1 - N_2$$

The two conditions for cases 1 and 3 in eq. 81 are

Condition 1

(93)
$$\Delta = 0 \pmod 3 \quad \text{for cases } 1 \text{ and } 3$$

Condition 2

$$\psi_{\mathbf{n}_1,\mathbf{n}_2}\left(\vec{\zeta},\vec{\bar{\zeta}}\right) \longrightarrow \begin{cases} \psi^{(+)}_{\mathbf{n}_1,\mathbf{n}_2} = \mathcal{N}^{+\frac{1}{2}}\left\{\psi_{\mathbf{n}_1,\mathbf{n}_2} + \psi_{\mathbf{n}_2,\mathbf{n}_1}\right\} & \text{for case } 1 \\ \psi^{(-)}_{\mathbf{n}_1,\mathbf{n}_2} = \mathcal{N}^{-\frac{1}{2}}\left\{\psi_{\mathbf{n}_1,\mathbf{n}_2} - \psi_{\mathbf{n}_2,\mathbf{n}_1}\right\} & \text{for case } 3 \end{cases}$$

(94)

⟶

In eq. 94 $\mathcal{N}^{\pm}$ denote normalization constants .

The two conditions defined in eqs. 93 and 94 determine the counting in accordance with overall Bose symmetry for the completeosonic wave function respecting $SU6 \; (\; fl \; \times \; spin \;)$ and oscillatory modes in 6 barycentric coordinates – for the two cases considered in this section .

3-1 – Associating the mixed 70-representation of $SU6 \; (\; fl \; \times \; spin \;)$ to oscillator mode wave functions in the circular pair-mode basis

It turns out that the association of the symmetric and antisymmetric representations of $SU6 \; (\; fl \; \times \; spin \;)$ show the way to determine the Bose statistics asociation of the mixed 70-representation of $SU6 \; (\; fl \; \times \; spin \;)$ to the oscillatory modes in the circular pair-mode basis , i.e. in the remaining case 2 in eq. 81 .

In fact there is only one condition $\Big($ modulo a restriction by a factor of $\frac{1}{2}$ $\Big)$ determining the sought association , complementing cases 1 and 3 in eqs. 93 and 94

Condition for case 2

(95) $$\Delta \; = \; 1 \, \& \, 2 \quad (\text{mod } 3) \quad \textbf{for case} \;\; \textbf{2}$$

For the counting the powers of the two sets

(96) $$\Delta \; = \; 1 \quad (\text{mod } 3) \;\; ; \;\; \Delta \; = \; 2 \quad (\text{mod } 3)$$

$\longrightarrow$

are the same , allowing the required (multiple) realization of the $\boxed{2}$ representation of S_3 on the wave functions in the circular pair-mode basis – intervening in case 2 . Given this realization the number of oscillator modes satisfying the condition in eq. 95 are then paired with the corresponding $\boxed{2}$ in the 70-plet of $SU6\ (\ fl\ \times\ spin\ )$ in eq. 81 as shown in the direct product decomposition in in ref. 1 , op.cit. , [eq. (192)] . Hence overall Bose symmetry restricts the number count of wave functions satisfying the condition in eq. 95 by a facor of $\frac{1}{2}$.

What remains to be done is the actual counting , according to the conditions formulated in eqs. 93 and 94 for cases 1 and 3 and eqs. 95 and 96 for case 2 . This is best done for a finite number of main oscillator quantum numbers N in a dedicated computer program , left to be worked out soon .

$\longrightarrow$

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