Extremal Black-Holes and Attractors

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On the occasion of P. Minkowski retirement.

In his long and lasting career Peter has often worked on application of group-theoretical techniques to the theory of particle interactions and in particular the role that "exceptional groups" may have in this context (for example G_2, E_6 ...).

Such groups, and in particular their non compact forms, remarkably appear in supergravity and superstring theory and they play an important role in some aspects of the physics of extremal black holes

SYMMETRIC SPACES BASED ON EXCEPTIONAL GROUPS (WHICH OCCUR IN SUPERGRAVITY)

d		$\mathcal{M}$ type	N. compact form	rank	H
14	g_2	Q	$g_{2,2}$	2	$su(2) \times su(2)$
52	f_4	Q	$f_{4,4}$	4	$usp(6) \times su(2)$
		$N=9, d=3$	$f_{4,-20}$	1	$so(9)$
78	e_6	$N=8, d=5$	$e_{6(6)}$	6	$usp(8)$
		$N=10, d=3$	$e_{6(-14)}$	2	$so(10) \times so(2)$
		Q	$e_{6(2)}$	4	$su(6) \times su(2)$
			$e_{6(-26)}$	2	f_4
133	e_7	$N=8, d=4$	$e_{7(7)}$	7	$su(8)$
		Q	$e_{7(-5)}$	4	$so(12) \times su(2)$
		SK	$e_{7(-25)}$	3	$e_6 \times so(2)$
248	e_8	$N=16, d=3$	$e_{8(8)}$	8	$so(16)$
		Q	$e_{8(-24)}$	4	$e_7 \times su(2)$

TABLE V

Irreducible Riemannian Globally Symmetric Spaces of Type I and Type III

	Noncompact	Compact	Rank	Dimension
A I	$SL(n, R)/SO(n)$	$SU(n)/SO(n)$	$n-1$	$\frac{1}{2}(n-1)(n+2)$
A II	$SU^*(2n)/Sp(n)$	$SU(2n)/Sp(n)$	$n-1$	$(n-1)(2n+1)$
A III	$SU(p, q)/S(U_p \times U_q)$	$SU(p+q)/S(U_p \times U_q)$	$\min(p, q)$	$2pq$
BD I	$SO_o(p, q)/SO(p) \times SO(q)$	$SO(p+q)/SO(p) \times SO(q)$	$\min(p, q)$	pq
D III	$SO^*(2n)/U(n)$	$SO(2n)/U(n)$	$[\frac{1}{2}n]$	$n(n-1)$
C I	$Sp(n, R)/U(n)$	$Sp(n)/U(n)$	n	$n(n+1)$
C II	$Sp(p, q)/Sp(p) \times Sp(q)$	$Sp(p+q)/Sp(p) + Sp(q)$	$\min(p, q)$	$4pq$
E I	$(\mathfrak{e}_{6(6)}, \mathfrak{sp}(4))$	$(\mathfrak{e}_{6(-78)}, \mathfrak{sp}(4))$	6	42
E II	$(\mathfrak{e}_{6(2)}, \mathfrak{su}(6) + \mathfrak{su}(2))$	$(\mathfrak{e}_{6(-78)}, \mathfrak{su}(6) + \mathfrak{su}(2))$	4	40
E III	$(\mathfrak{e}_{6(-14)}, \mathfrak{so}(10) + R)$	$(\mathfrak{e}_{6(-78)}, \mathfrak{so}(10) + R)$	2	32
E IV	$(\mathfrak{e}_{6(-26)}, \mathfrak{f}_4)$	$(\mathfrak{e}_{6(-78)}, \mathfrak{f}_4)$	2	26
E V	$(\mathfrak{e}_{7(7)}, \mathfrak{su}(8))$	$(\mathfrak{e}_{7(-133)}, \mathfrak{su}(8))$	7	70
E VI	$(\mathfrak{e}_{7(-5)}, \mathfrak{so}(12) + \mathfrak{su}(2))$	$(\mathfrak{e}_{7(-133)}, \mathfrak{so}(12) + \mathfrak{su}(2))$	4	64
E VII	$(\mathfrak{e}_{7(-25)}, \mathfrak{e}_6 + R)$	$(\mathfrak{e}_{7(-133)}, \mathfrak{e}_6 + R)$	3	54
E VIII	$(\mathfrak{e}_{8(8)}, \mathfrak{so}(16))$	$(\mathfrak{e}_{8(-248)}, \mathfrak{so}(16))$	8	128
E IX	$(\mathfrak{e}_{8(-24)}, \mathfrak{e}_7 + \mathfrak{su}(2))$	$(\mathfrak{e}_{8(-248)}, \mathfrak{e}_7 + \mathfrak{su}(2))$	4	112
F I	$(\mathfrak{f}_{4(4)}, \mathfrak{sp}(3) + \mathfrak{su}(2))$	$(\mathfrak{f}_{4(-52)}, \mathfrak{sp}(3) + \mathfrak{su}(2))$	4	28
F II	$(\mathfrak{f}_{4(-20)}, \mathfrak{so}(9))$	$(\mathfrak{f}_{4(-52)}, \mathfrak{so}(9))$	1	16
G	$(\mathfrak{g}_{2(2)}, \mathfrak{su}(2) + \mathfrak{su}(2))$	$(\mathfrak{g}_{2(-14)}, \mathfrak{su}(2) + \mathfrak{su}(2))$	2	8

Non-compact groups appear in supersymmetric theories of gravity not only as space-time symmetries (Poincaré, Conformal symmetries and their super extensions) but also as "isometries" of non-linear σ-models

$$g_{ij}(\phi)\,\partial_\mu \phi^i \partial^\mu \phi^j$$

where ϕ^i are coordinates of $\mathcal{M}$.

If $\mathcal{M} = G/H$ either G is compact or G is non-compact and H is its maximal compact subgroup.

The nature of the σ-model depends on N (number of supersymmetries) and D (space-time dimension)

For $N > 2$ ($D = 4$) the σ-model is locally G/H. Not true for $N = 1, 2$

Famous example is the "moduli space" of Calabi-Yau compactifications.

Black-Hole physics depend on particular "discrete" points of the space corresponding to "Attractor varieties" (G. Moore)

Attractor mechanism
R. Kallosh, A. Strominger, S.F.

Supersymmetry seems to be related to dynamical systems with fixed points describing the equilibrium and stability. The particular property of long-term behaviour of dynamical flows in dissipative systems is the following: in approaching the attractors the orbits lose practically all memory of then initial conditions, even though the dynamics is strictly deterministic.

A point when the phase velocity $\vec{v}(x_{fix}) = 0$ is called fixed (critical) point. If in the motion $x(t)$, for $t \to \infty$ $x(t) \to x_{fixed}$ then x_{fixed} is called an attractor point.

Electric Magnetic Duality

$$\frac{1}{g^2} F_{\mu\nu} F^{\mu\nu} + \theta F_{\mu\nu} \tilde{F}^{\mu\nu}$$

$$t = -\frac{i}{g^2} + \theta$$

The combined field eqs + Bianchi identities are invariant under

$$t \to \frac{\alpha t + \beta}{\gamma t + \delta} \qquad \alpha\delta - \beta\gamma = 1$$

$$SL(2,\mathbb{R}) \sim Sp(2,\mathbb{R})$$

Generalization to many fields

$$G_{\Lambda\Sigma} F^\Lambda F^\Sigma + \theta_{\Lambda\Sigma} F^\Lambda \tilde{F}^\Sigma$$

$$\mathcal{N}_{\Lambda\Sigma} = -i G_{\Lambda\Sigma} + \theta_{\Lambda\Sigma}$$

$$Sp(2n,\mathbb{R}) \to \quad \mathcal{N}' = (C + D\mathcal{N})(A + B\mathcal{N})^{-1}$$

$$A^T C,\ B^T D \text{ sym},\quad A^T D - C^T B = \mathbb{1}$$

When the non-linear σ-model is coupled to n (abelian) vector fields, consistency of the field equations (and Bianchi identities) with electric-magnetic duality requires that the field strengths and their dual

$$\left(F^\Lambda = dA^\Lambda , \quad G_\Lambda = \frac{\delta \mathcal{L}}{\delta F^\Lambda} \right) \quad \Lambda = 1 \dots n$$

transform in a linear (symplectic) representation of G of dimension $2n$: R $\quad (R \times R \supset 1_A)$

$$\int_{S_2} F^\Lambda = m^\Lambda \qquad \int_{S_2} G_\Lambda = e_\Lambda$$

$$(m^\Lambda, e_\Lambda) = Q \qquad \text{charge vector}$$

Extremal (BPS) black-holes

$$M = |Z| = \sqrt{e^2+m^2}$$

(in the case of the Einstein-Maxwell theory, this is the (dyonic) Reissner-Nordström black-hole)

- Bekenstein-Hawking entropy

Relation between Entropy - Horizon area - mass (charge)

$$S = \frac{A}{4} = \pi |Z|^2 = \pi (e^2+m^2)$$

$$S = S(e,m)$$

When scalars are coupled to vectors the above formula generalizes as

$$S = S(e_\Lambda, m^\Lambda)$$

through the "attractor mechanism"

$$\phi^i(\phi_\infty, e_\Lambda, m^\Lambda, r) \to \phi^i_\infty$$
$$r \to \infty$$

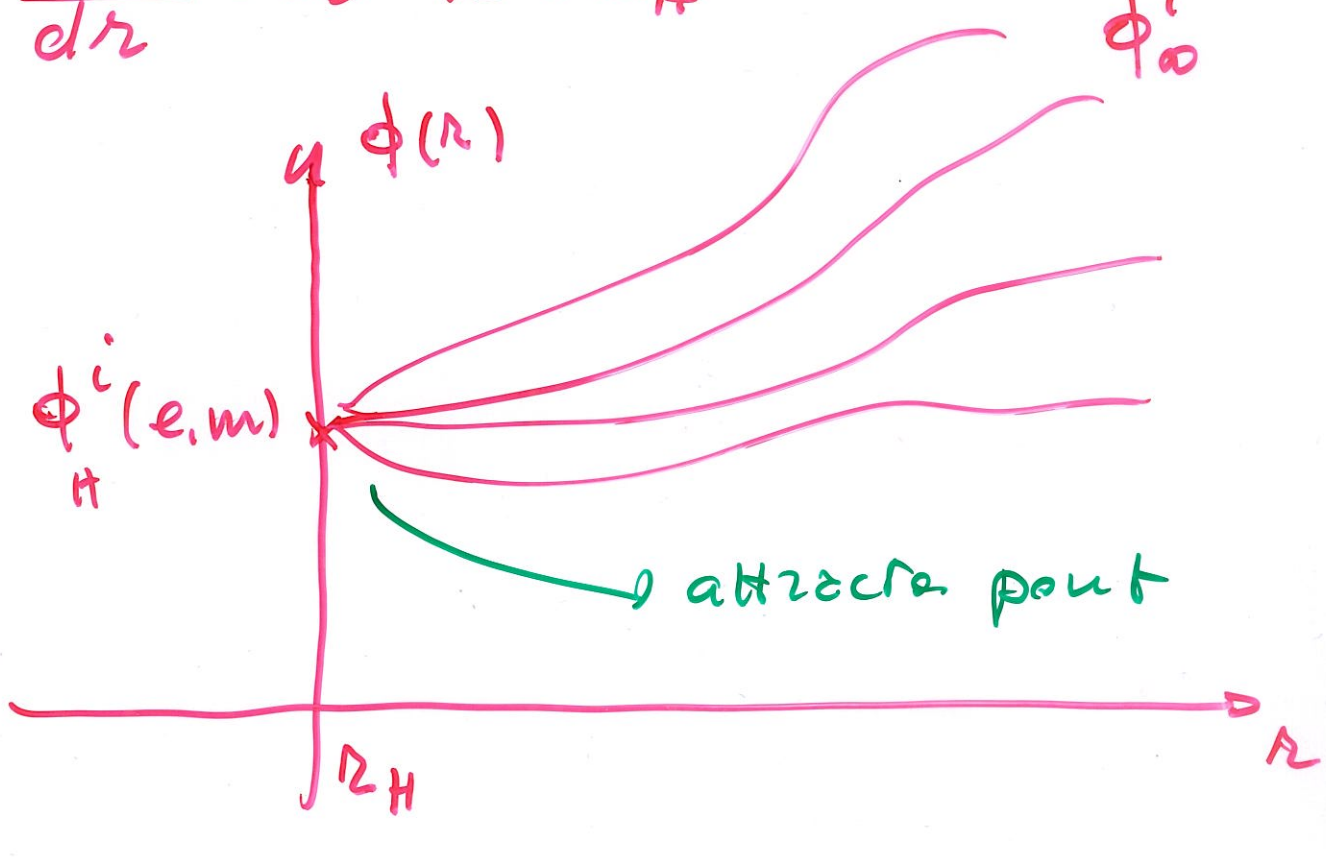
$$\phi^i(\phi_\infty, e_\Lambda, m^\Lambda, r) \to \phi^i_H(e_\Lambda, m^\Lambda)$$
$$r \to r_H$$

(Kallosh, Strominger, S.F.)

ϕ^i_H is an extremum of $V_{BH}(\phi, e, m)$

$\partial V_{BH} / \partial \phi = 0 \quad \phi = \phi^i_H$ so that

$$\frac{d\phi^i}{dr} \to 0 \quad r \to r_H$$



In the case of $\mathcal{M} = G/H$

$S(e,m)$ is a G invariant

$N=8$ supergravity at $D=4$

$(m^\Lambda, e_\Lambda) \in 56$ of $E_{7(7)}$

$$S(e,m) = \frac{A(e,m)}{4} = \pi \sqrt{I_4(e,m)}$$

In a $28+\overline{28}$ basis (of $SU(8)$) for the 56

$$I_4 = \mathrm{Tr}(Z\bar{Z})^2 - \frac{1}{4}(\mathrm{Tr}\, Z\bar{Z})^2 + 4(\mathrm{Pf}\, Z + \mathrm{Pf}\, \bar{Z})$$

$Z = Z^{AB} = -Z^{BA}$ 28 of $SU(8)$.

$$Z^{AB} = (\rho_i e^{i\phi/4}) \quad i = 1, 2, 3, 4$$

$N=8$ and $N=2$, $D=4$

$E_{7(7)}$, $E_{7(-25)}$

$56 \to (e_\Lambda, m^\Lambda)$

Black-Holes in $D=4$

$N=8 \longrightarrow \frac{1}{8}$ BPS $\quad |Z|^2 = \sqrt{I_4}$

$N=2 \longrightarrow \frac{1}{2}$ BPS $\quad |Z|^2 = \sqrt{I_4}$

$$I_4 = \left[(\rho_1+\rho_2)^2-(\rho_3+\rho_4)^2\right]\left[(\rho_1-\rho_2)^2-(\rho_3-\rho_4)^2\right]$$
$$+ 8\rho_1\rho_2\rho_3\rho_4(\cos\phi - 1)$$

$(\rho_1 \geq \rho_2 \geq \rho_3 \geq \rho_4)$

BPS $I_4 \geq 0$

Non BPS $I_4 < 0$

$A_4 \neq 0$ if $I_4 \neq 0$

$I_4 > 0$ $\quad E_{7(7)}/E_{6(2)}$ $(28^-, 27^+)$

$I_4 < 0$ $\quad E_{7(7)}/E_{6(6)}$ $(30^-, 25^+)$

(55 dimensional spaces)

$$E_{7(7)} \supset SO^*(12) \times SU(2)$$

$$E_{7(7)} \supset Sp(6,\mathbb{R}) \times G_{2(2)}$$

$$E_{7(7)} \supset SU(3,3) \times SO(2,1)$$

$$E_{7,7} \supset SU(1,1)^3 \times SO(4,4)$$