Light quark mass ratios $(m_u : m_d : m_s)$ from meson and baryon mass splittings

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Abstract

The basis of the material discussed is our work in collaboration with Arnulfo Zepeda from 1979 [1-1979] . The ingredients and consequenes of this work will be presented, and compared with results obtained from QCD sum rules and lattice simulations of QCD in accordance with chiral expansions . An up to date conclusion will not be possible in this presentation, but some comments towards such goal will be given in a concluding section.

Contribution to the

International Workshop on the Determination of the Fundamental Parameters of QCD ,

18.-21. March 2013 , Nanyang Executive Center , Nanyang Technical University , Singapore

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1 - Introduction : QCD embedding [a]

We face the theoretical abstraction of QCD with $N_{fl} = 6$, representing strong interactioans – adaptable to two or three light flavors (u , d , s) of quarks and antiquarks. $\leftrightarrow$

quarks : color is counted in $\pi^0 \rightarrow \gamma\gamma$ (assuming global color- and flavor-projections to commute) yet see ref. [2-2001]

spin and flavor are clearly seen in $q\overline{q}$ and $3q$, $3\overline{q}$ spectroscopy (a pre-condition to count color) .

$$\mathcal{L} = \left[\overline{q}^{\,\dot{c}'}_{\dot{\mathcal{S}}'\,\dot{f}} \left\{ \begin{array}{c} \frac{i}{2} \overleftrightarrow{\partial}_\mu \, \delta_{c'\dot{c}} \\ + W^{\,r}_\mu \left(\frac{1}{2} \lambda_r \right)_{c'\dot{c}} \end{array} \right\} \gamma^{\mu}_{\dot{\mathcal{S}}'\mathcal{S}} \, q^{\,c}_{\mathcal{S}\,f} \atop - m_f \, \overline{q}^{\,\dot{c}}_{\dot{\mathcal{S}}\,\dot{f}} \, q^{\,c}_{\mathcal{S}\,f} \right]$$

(1) $$- \frac{1}{4 g^2} B^{\mu\nu\,r} B^{\,r}_{\mu\nu} + \Delta\mathcal{L}$$

$W^{\,r}_\mu \equiv - v^{\,r}_\mu$: for identification of convention for potentials

quarks : c' , $c = 1, 2, 3$ color , $f = 1, \cdots, 6$ flavor

$\mathcal{S}', \mathcal{S} = 1, \cdots, 4$ spin , m_f mass

$\longrightarrow$

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[a] Abbreviated from ref. [3-2012] .

In eq. 1 , the $\mathcal{D}$ associated gauge connection fields – where $\mathcal{D} = \mathcal{D} (\mathcal{G})$ denotes a general , irreducible representation of the local gauge group $\mathcal{G} = SU3_c$ – appear in the form appropriate for quarks : $\mathcal{D} = \{ 3 \}$, and antiquarks : $\mathcal{D} = \{ \overline{3} \}$ respectively

(2)
$$\boxed{\begin{array}{l} (\mathcal{W}_\mu (\mathcal{D}))_{\alpha\beta} (x) = W_\mu^{\,r} (x) (d_r)_{\alpha\beta} \;\leftrightarrow\; \mathcal{W}_\mu (\mathcal{D}) = - \mathcal{W}_\mu (\mathcal{D})^\dagger \\ d_r = - d_r^\dagger = \frac{1}{i} J_r \in Lie (\mathcal{D}) \;\; ; \;\; [d_p , d_q] = f_{pqr} d_r \\ r , p , q = 1, \cdots, dim\, \mathcal{G} \;\; ; \;\; \alpha , \beta = 1, \cdots, dim\, \mathcal{D} \end{array}}$$

For $\mathcal{D} (SU3_c) = \left\{ 3 \, (\overline{3}) \right\}$ the representation matrices become (the Gell-Mann matrices [4-1964])

$$\left(d_r (3) = \frac{1}{i} \frac{1}{2} \lambda_r \right)_{\alpha\beta} \; ; \; r = 1, \cdots, 8 \; ; \; (\alpha , \beta) \leftrightarrow \left(c' , \dot{c} \right) = 1, \cdots, 3$$

(3)
$$d_r (\overline{3}) = \overline{d}_r (3)$$

with the ***conventional*** **normalization conditions :** $- \; tr \, d^{\,r} d^{\,s} = \frac{1}{2} \delta^{\,rs}$

The quantity proportional to the gauge potentials $W_\mu^{\,r}$ for the $\overline{q} \, q$ in eq. 1 is thus identified as

(4)
$$\left[W_\mu^{\,r} \left(\tfrac{1}{2} \lambda_r \right)_{c' \dot{c}} = i \left(\mathcal{W}_\mu (\mathcal{D} = \{ 3 \}) \right)_{c' \dot{c}} \right] (x)$$

Here we postpone the discussion of complete connections and extend the QCD Lagrangean density to include the term quadratic in the field strengths $B_{\mu\nu}^{\,r}$ and $\Delta \mathcal{L}$ in eq. 1, pertinent to Fermi gauges. $\rightarrow$

gauge bosons : $\mathcal{L}_B = -\frac{1}{4 g^2} B^{\mu\nu\, r} B^r_{\mu\nu}$

$$B^r_{\mu\nu} = \partial_\mu W^r_\nu - \partial_\nu W^r_\mu + f_{rst} W^s_\mu W^t_\nu \longleftarrow (W^r_\mu \equiv - v^r_\mu)$$

(5) $$r, s, t = 1, \cdots, dim (G = SU3_c) = 8$$

Lie algebra labels, $\left[\frac{1}{2} \lambda^r , \frac{1}{2} \lambda^s \right] = i f_{rst} \frac{1}{2} \lambda^t$

perturbative rescaling :

$$W^r_\mu = g W^r_{\mu\, pert} , B^r_{\mu\nu} = g B^r_{\mu\nu\, pert}$$

Degrees of freedom are seen in jets , in (e.g.) the energy momentum sum rule in deep inelastic scattering but not clearly identically in spectroscopy.

Completing $\Delta \mathcal{L}$ in Fermi gauges

$$\Delta \mathcal{L} = \left\{ \begin{array}{c} - \frac{1}{2 \eta g^2} (\partial_\mu W^{\mu\, r})^2 \\ + \partial^\mu \overline{c}^r (D_\mu c)^r \end{array} \right\} \; ; \; \eta : \text{gauge parameter}$$

(6)

ghost fermion fields : $c , \overline{c} \; ; \; (D_\mu c)^r = \partial_\mu c^r + f_{rst} W^s_\mu c^t$

gauge fixing constraint : $C^r = \partial_\mu W^{\mu\, r}$

$\longrightarrow$

perturbative renormalization rescaling

We begin with the renormalization of external operaters and gauge boson fields, denoting unrenormalized fields and operators by the suffix $^{(0)}$

(7)
$$\begin{aligned} J_{\alpha} &= (Z_J)^{-1} J_{\alpha}^{(0)} \quad , \quad \mathcal{O} = (Z_{\mathcal{O}})^{-1} \mathcal{O}^{(0)} \\ g &= (Z_3)^{3/2} (Z_1)^{-1} g^{(0)} \quad , \quad \eta = (Z_3)^{-1} \eta^{(0)} \\ W_{\mu\, pert}^{r} &= (Z_3)^{-\frac{1}{2}} \left(W_{\mu\, pert}^{r} \right)^{(0)} \end{aligned}$$

and continue with the ghost fields c^{r} , $\overline{c}^{s}$ as defined in $\Delta \mathcal{L}$ in eq. 6

(8)
$$\begin{aligned} c^{r} &= \left(\widetilde{Z}_2 \right)^{-\frac{1}{2}} (c^{r})^{(0)} \quad , \quad \overline{c}^{r} = \left(\widetilde{Z}_2 \right)^{-\frac{1}{2}} (\overline{c}^{r})^{(0)} \\ g &= (Z_3)^{1/2} \widetilde{Z}_2 \left(\widetilde{Z}_1 \right)^{-1} g^{(0)} \end{aligned}$$

The renormalization constant $Z_1 = Z_1 (W - 3)$ refers to the 3 gauge boson vertex of the derivative type $W\, W\, \partial\, W$ as shown in figure 1, while $\widetilde{Z}_1 = \widetilde{Z}_1 (\overline{c}\, c\, W)$ refers to the ghost-W vertex, shown in figure 2, while the 4 gauge boson vertex involves the analogous renormalization constant $Z_{1(4W)}$ and the $\overline{q}\, q\, W$ vertex $Z_{1(\overline{q}qW)}$, shown in figures 3 and 4 respectively below .

$\longrightarrow$

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4 irreducible vertex diagrams

W-3 : (i)

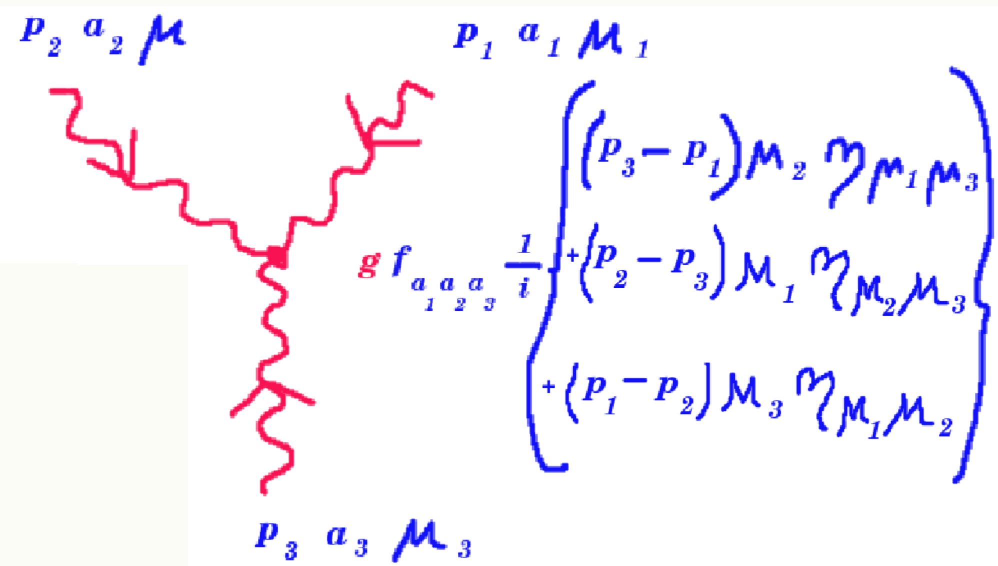

Fig. 1 : $Z_1 \leftrightarrow$ **W-3 vertex of the type** $W\,W\,\partial\,W$

$\longrightarrow$

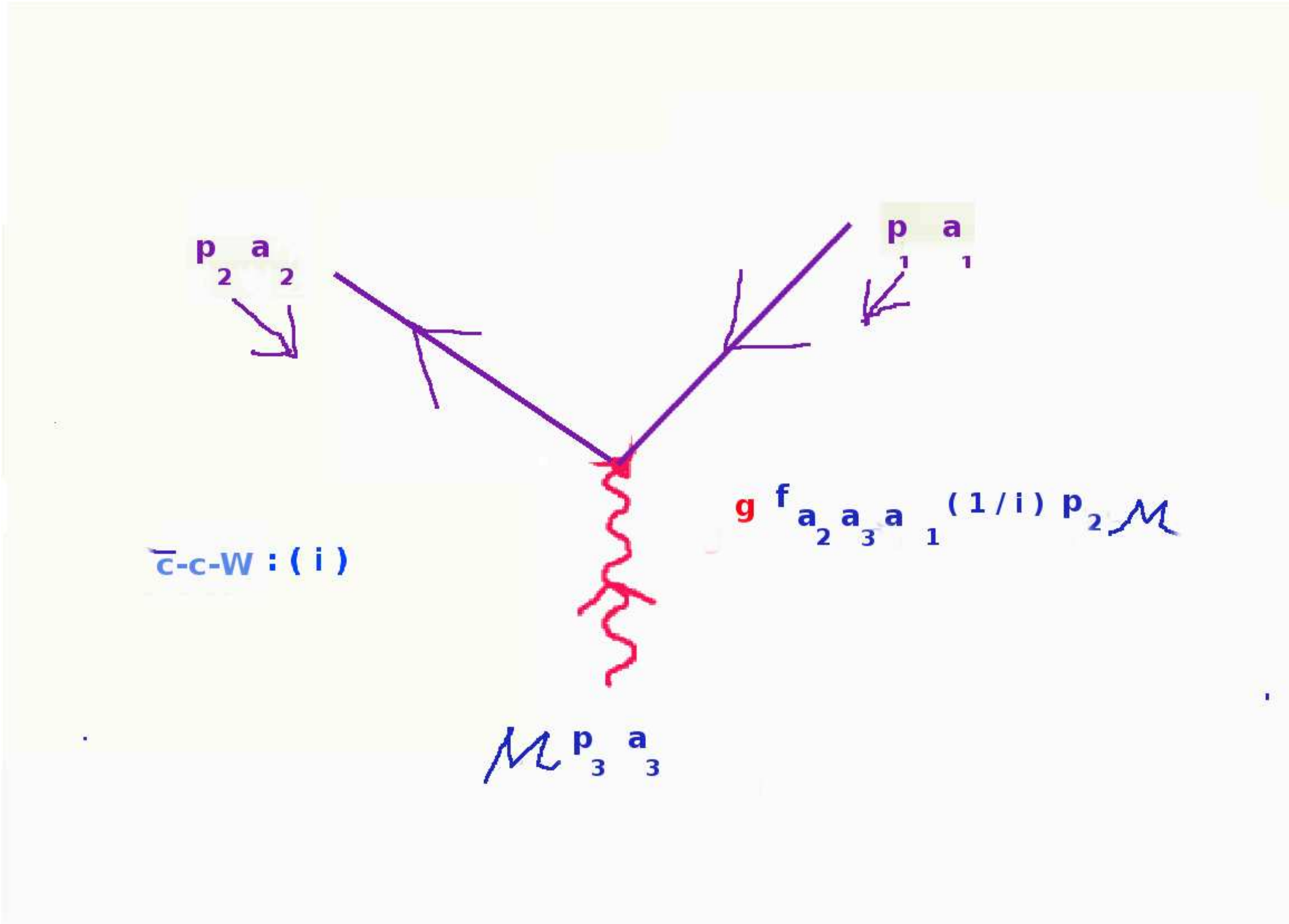

Fig. 2 : $\widetilde{Z}_1 \leftrightarrow$ cbar-c W-3 vertex of the type $\partial\,\overline{c}\,c\,W$

$\longrightarrow$

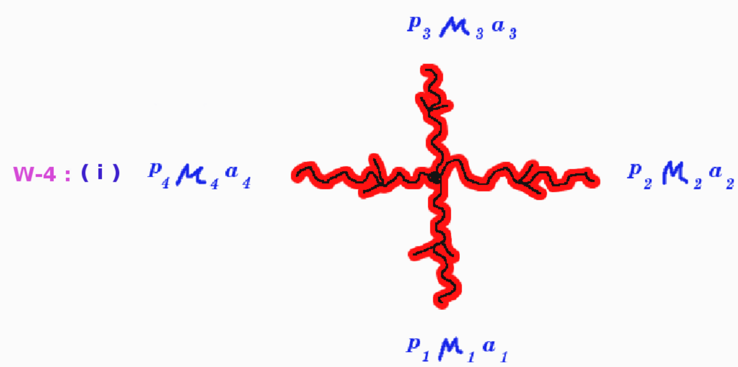

Fig. 3 : $Z_{1(4W)} \leftrightarrow$ **W-4 vertex for gauge fields**

$$
g^2 \left(\begin{array}{r} r_{[a_1 a_2][a_3 a_4]} \left[\begin{array}{r} \eta_{\mu_1 \mu_4} \eta_{\mu_2 \mu_3} \\ - \eta_{\mu_1 \mu_3} \eta_{\mu_2 \mu_4} \end{array} \right] \\ + r_{[a_1 a_3][a_2 a_4]} \left[\begin{array}{r} \eta_{\mu_1 \mu_4} \eta_{\mu_2 \mu_3} \\ - \eta_{\mu_1 \mu_2} \eta_{\mu_3 \mu_4} \end{array} \right] \\ + r_{[a_1 a_4][a_2 a_3]} \left[\begin{array}{r} \eta_{\mu_1 \mu_3} \eta_{\mu_2 \mu_4} \\ - \eta_{\mu_1 \mu_2} \eta_{\mu_3 \mu_4} \end{array} \right] \end{array} \right)^{\to r}
$$

$$
r_{[a_1 b_1][a_2 b_2]} = f_{d a_1 b_1} f_{d a_2 2_2}
$$

$\longrightarrow$

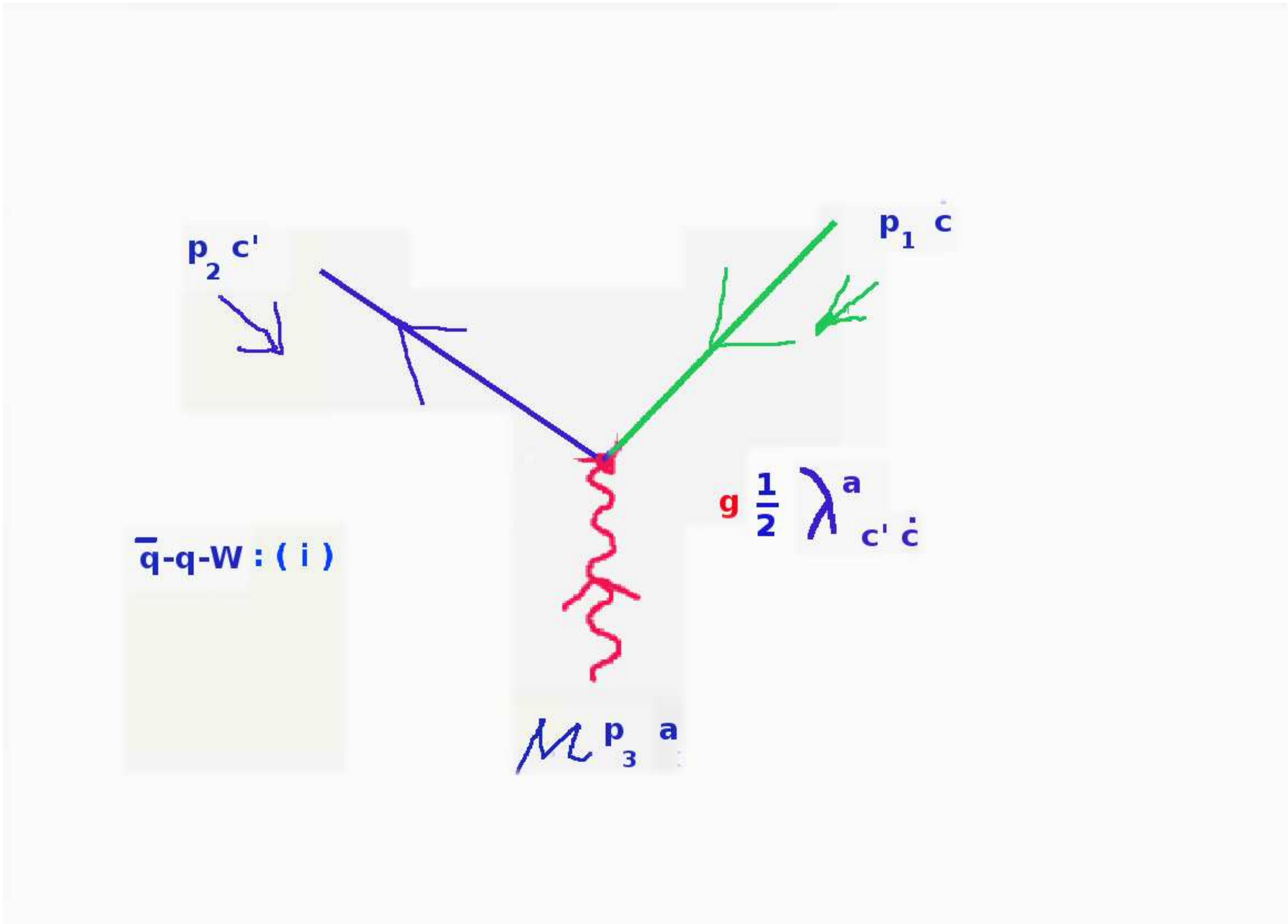

Fig. 4 : $Z_{1(\overline{q}qW)}$ $\leftrightarrow$ W-3 vertex for $\overline{q}\,q$ to gauge fields

→

the complete coupling constant renormalization equations

The minimal renormalization constants defined in eqs. 7 and 8, referring beyond the vertices shown in figures 1-4 to the two point functions for gauge connections , Z_3 , and ghost fields , $\widetilde{Z}_2$, thus need extension (also) to q , $\overline{q}$ fields and their two point function, Z_2 , as shown below

$$g = (Z_3)^{1/2} (Z_3 / Z_1) g^{(0)} \quad ; W^{r}_{\mu\, pert} = (Z_3)^{-\frac{1}{2}} \left(W^{r}_{\mu\, pert} \right)^{(0)}$$

$$g = (Z_3)^{1/2} \left(\widetilde{Z}_2 / \widetilde{Z}_1 \right) g^{(0)} \quad ; (c, \overline{c})^{r} = \left(\widetilde{Z}_2 \right)^{-\frac{1}{2}} ((c, \overline{c})^{r})^{(0)}$$

$$g^2 = Z_3 \left(Z_3 / Z_{1(4W)} \right) \left(g^{(0)} \right)^2$$

$$g = (Z_3)^{1/2} \left(Z_2 / Z_{1(\overline{q}qW)} \right) g^{(0)} \; ; \quad q^{c}_{\mathcal{S}\, f} = (Z_2)^{-\frac{1}{2}} \left(q^{c}_{\mathcal{S}\, f} \right)^{(0)}$$

(9) $$\overline{q}^{\dot{c}'}_{\dot{\mathcal{S}}'\, \dot{f}} = (Z_2)^{-\frac{1}{2}} \left(\overline{q}^{\dot{c}'}_{\dot{\mathcal{S}}'\, \dot{f}} \right)^{(0)}$$

Maintaining gauge invariance through renormalization thus implies the relations

(10) $$Z_3 / Z_1 = \widetilde{Z}_2 / \widetilde{Z}_1 = Z_2 / Z_{1(\overline{q}qW)} \; ; \; Z_3 / Z_{1(4W)} = (Z_3 / Z_1)^2$$

It remains to include quark mass renormalization and associated rescaling. →

renormalization equations for quark masses m_f

We perform quark mass renormalization 'at zero mass' [5-1973] , with the following relations , using the $q\,,\overline{q}$ fields and their field renormalization constant, Z_2 , as defined in eqs. 9 and 10, through the extension

(11) $$\left(\mathcal{L}_m\right)^{(0)} = -\sum_f Z_2 Z_m m_f \;\; ; \;\; m_f = \left(Z_m\right)^{-1} m_f^{(0)}$$

The coefficients of the quark mass rescaling functions are known to four loops as calculated by Chetyrkin and independently by others [6-1997] . They are given in eq. 93 in Appendix 1.

Here we subsum the complete set of renormalized rescaling functions, which follows from the renormalization constants defined in eqs. 8 - 10

$$\left(\begin{array}{l} \mu^2 \partial_{\mu^2} + \left(\beta(g)/g\right) g^2 \partial_{g^2} + \\ +\gamma_m m_f \partial_{m_f} - 2\gamma_3 \left(\eta \partial_\eta\right) - \gamma_{J\mathcal{O}} \end{array} \right) C_{J\mathcal{O}}^{T\,(\Pi)}\left(z\,;\mu\,,g\,,m_\beta\,,\eta\right) = 0$$

(12) $$\left\{ \begin{array}{l} \beta(g) = -g\,b\left(g^2\right) \\ \gamma_m\left(g^2\right) \equiv -\chi_m \\ \gamma_{J\mathcal{O}}\left(g^2\,,(\eta)\right) \\ \gamma_3\left(g^2\,,\eta\right) \end{array} \right\} = \mu^2 d/d\mu^2 \left\{ \begin{array}{c} 2\log\left(\left(Z_3\right)^{3/2}\left(Z_1\right)^{-1}\right) \\ \log\left(\left(Z_m\right)^{-1}\right) \\ \log\left(Z_{\mathcal{O}}/Z_J^2\right) \\ \log\left(\left(Z_3\right)^{1/2}\right) \end{array} \right\}$$

$\rightarrow$

We can always return to the standard form of the renormalization group equation

(13) $$\left(\begin{array}{l} \mu \, \partial_{\mu} + \beta(g)\, \partial_{g} + (2\gamma_{m})\, m_{f}\, \partial_{m_{f}} \\ -2\,(2\gamma_{3})\,(\eta\, \partial_{\eta}) - (2\gamma_{J\mathcal{O}}) \end{array} \right) C^{T\,(\Pi)}_{J\,\mathcal{O}}\,\left(z\,;\,\mu\,,\,g\,,\,m_{\beta}\,,\,\eta \right) = 0$$

without changing the definition of β, transforming the lower relation in eq. 12 , which amounts to double the coefficients of the quantities $\gamma_{m}\,,\,\gamma_{3}\,,\,\gamma_{J\mathcal{O}}$, as displayed in eq. 13

(14) $$\left\{ \begin{array}{l} \beta(g) = -\,g\,b(g^{2}) \\ 2\,\gamma_{m}(g^{2}) \equiv -\,2\,\chi_{m} \\ 2\gamma_{J\mathcal{O}}(g^{2},(\eta)) \\ 2\,\gamma_{3}(g^{2},\eta) \end{array} \right\} = \mu\, d/d\mu \left\{ \begin{array}{c} \log\left((Z_{3})^{3/2}\,(Z_{1})^{-1} \right) \\ \log\left((Z_{m})^{-1} \right) \\ \log\left(Z_{\mathcal{O}}/Z_{J}^{2} \right) \\ \log\left((Z_{3})^{1/2} \right) \end{array} \right\}$$

$\rightarrow$

rescaling equations in the $\overline{\text{MS}}$ scheme for coupling constant g and quark masses m_f
$\beta(g)$ and $\gamma_m(\kappa)$, $\kappa = g^2 / (16\pi^2)$

We turn to the scale (-square) evolution of the rationalized coupling constant and quark masses, introducing the variables

(15)
$$\tau = \log\left(\mu^2 / \mu_0^2\right) \quad ; \quad \kappa = g^2 / \left(16\pi^2\right) \quad \rightarrow$$
$$\left\{ \widehat{\beta} = (\beta / g)\,\kappa = \widehat{\beta}(\kappa)\,,\ \gamma_m(\kappa) \right\}$$

The renormalization group equation in the form given in eq. 12 then becomes

(16)
$$\left(\begin{array}{l} \partial_\tau + \widehat{\beta}\,\partial_\kappa + \gamma_m\, m_f\, \partial_{m_f} \\ -2\gamma_3\,(\eta\,\partial_\eta) - \gamma_{J\mathcal{O}} \end{array} \right) C_{J\mathcal{O}}^{T\,(\Pi)}\left(z\,;\mu\,,g\,,m_\beta\,,\eta\right) = 0$$

The partial derivative terms in brackets in eq. 16 determine the sliding scale equations with respect to the quantities $\tau\,,\overline{\kappa}\,,\overline{m}_f\,,\overline{\eta}$ in the sense of its initial value structure. We restrict the discussion to $\tau\,,\overline{\kappa}(\tau)\,,\overline{m}_f(\tau)$ here for simplicity .

(17)
$$\overset{\bullet}{\overline{\kappa}} = \widehat{\beta}(\overline{\kappa})\,,\ \overset{\bullet}{\overline{m}}_f = \gamma_m(\overline{\kappa})\,\overline{m}_f \quad ; \quad \bullet = d / d\tau$$
$$\overline{\kappa}(\tau = 0) = \kappa\,,\ \overline{m}_f(\tau = 0) = m_f$$

$\rightarrow$

sliding coupling constant : $\dot{\overline{\kappa}} = \widehat{\beta}\,(\,\overline{\kappa}\,)$

We associate integration variables , initial values and endpoint variables in the following way

(18) $$\theta \leftrightarrow [\,\tau\,,\,0\,]\ ,\ \lambda \leftrightarrow [\,\overline{\kappa}\,,\,\kappa\,]$$

The integration of the rescaling differential equation becomes

(19) $$\int_0^{\tau} d\,\theta = \tau = \int_{\kappa}^{\overline{\kappa}} d\,\lambda\,\left(\widehat{\beta}\,(\,\lambda\,)\right)^{-1} \longrightarrow \tau = \int_{\overline{\kappa}}^{\kappa} d\,\lambda\,\left(-\,\widehat{\beta}\,(\,\lambda\,)\right)^{-1}$$

The first two coefficients in the expansion of $\widehat{\beta}$ in powers of λ necessitate a twofold subtraction for $\lambda \downarrow 0$ for the integral on the right hand side of eq. 19 to converge at the lower limit of integration

(20) $$\widehat{\beta}\,(\,\lambda\,) = \sum_{n=0}^{\infty} \widehat{\beta}_{\,n}\,\lambda^{\,n+2}\ ;\ \widehat{\beta}_{\,n} \equiv -\,b_{\,n}$$

The first four coefficients $b_{\,n}\ ;\ n = 0, \cdots, 3$ are known , (ref. [7-1997]) and given in eq. 92 in Appendix 1 .

We perform this subtraction splitting $\widehat{\beta}$

(21) $$\widehat{\beta} = \widehat{\beta}_{\,(2)} + \Delta_{\,(2)}\,\widehat{\beta}\ ;\ \left[\begin{array}{l} \widehat{\beta}_{\,(2)} = -\,\left(b_{\,0}\lambda^{\,2} + b_{\,1}\,\lambda^{\,3}\right) \\ \Delta_{\,(2)}\,\widehat{\beta} = -\sum_{n=2}^{\infty} b_{\,n}\,\lambda^{\,n+2} \end{array}\right.$$

The subtraction, displayed in eq. 21 $\rightarrow$

gives rise to the substitutions

$$\zeta_{(2)} = \frac{\Delta_{(2)}\widehat{\beta}}{\widehat{\beta}_{(2)}} = b_2 b_0^{-1} \lambda^2 (1 + O(\lambda))$$

(22) $$\left(-\widehat{\beta}\right)^{-1} = \left(-\widehat{\beta}_{(2)}\right)^{-1} \Big/ (1 + \zeta_{(2)}) = \left(-\widehat{\beta}_{(2)}\right)^{-1} - \psi_{(2)}$$

$$\psi_{(2)} = \left(-\widehat{\beta}_{(2)}\right)^{-1} \frac{\zeta_{(2)}}{1 + \zeta_{(2)}} = b_2 (b_0)^{-2} (1 + O(\lambda))$$

ψ_2 as defined in eq. 22 has a regular power series expansion in λ, fully determined by $\widehat{\beta}$ and hence the sought subtractions involve only $\widehat{\beta}_{(2)}$, i.e. the beta function truncated to the first two terms.

We proceed in the reduction of $\left(-\widehat{\beta}\right)^{-1}$ using the substitutions in eq. 22

$$\left(-\widehat{\beta}\right)^{-1} = \left(-\widehat{\beta}_{(2)}\right)^{-1} - \psi_{(2)}$$

(23) $$\left(-\widehat{\beta}_{(2)}\right)^{-1} = b_0^{-1} \lambda^{-2} \left(1 + b_1^{(0)} \lambda\right)^{-1}; \quad b_1^{(0)} = b_1 / b_0$$

$$= b_0^{-1} \lambda^{-2} \left(1 - b_1^{(0)} \lambda\right) \left(1 - \left(b_1^{(0)} \lambda\right)^2\right)^{-1}$$

→

The last expression in eq. 23 decomposes into

(24)
$$\left(-\widehat{\beta}_{(2)}\right)^{-1} = b_0^{-1}\lambda^{-2} - b_0^{-2} b_1 \lambda^{-1} + \phi_{(2)}$$
$$\phi_{(2)} = b_0^{-1}\left(b_1^{(0)}\right)^2\left(1 - \left(b_1^{(0)}\lambda\right)^2\right)^{-1} \; ; \; b_1^{(0)} = b_1 / b_0$$

$\phi_{(2)}$ defined in eq. 24 , as ψ_2 defined in eq. 22 , has a regular power series expansion in λ .

Thus we rearrange the expressions in eq. 23

$$\left(-\widehat{\beta}\right)^{-1} = \left(-\widehat{\beta}_{sing.}\right)^{-1} + \left(-\widehat{\beta}_{reg.}\right)^{-1} \; ; \; \widehat{\beta} = \widehat{\beta}_{(2)} + \Delta_{(2)}\widehat{\beta}$$
$$\left(-\widehat{\beta}_{sing.}\right)^{-1} = b_0^{-1}\lambda^{-2} - b_0^{-2} b_1 \lambda^{-1} \; , \; \left(-\widehat{\beta}_{reg.}\right)^{-1} = \phi_{(2)} - \psi_{(2)}$$

---

(25)
$$\phi_{(2)} = b_0^{-1}\left(b_1^{(0)}\right)^2\left(1 - \left(b_1^{(0)}\lambda\right)^2\right)^{-1} \; ; \; b_1^{(0)} = b_1 / b_0$$
$$\psi_{(2)} = \left(-\widehat{\beta}_{(2)}\right)^{-1} \frac{\zeta_{(2)}}{1 + \zeta_{(2)}} = b_2 \left(b_0\right)^{-2} \left(1 + O(\lambda)\right)$$
$$\zeta_{(2)} = \frac{\Delta_{(2)}\widehat{\beta}}{\widehat{\beta}_{(2)}} = b_2 b_0^{-1} \lambda^2 \left(1 + O(\lambda)\right)$$

→

We recall that in the decomposition $\widehat{\beta} = \widehat{\beta}_{(2)} + \Delta_{(2)}\widehat{\beta}$, retained in eq. 25 , $\widehat{\beta}_{(2)}$ and $\Delta_{(2)}\widehat{\beta}$ denote the beta function truncated to the first wo terms and the remainder term respectively . It further follows from eq. 25

(26) $$\left(-\widehat{\beta}_{reg.}\right)^{-1} = b_0^{-1}\left((b_1/b_0)^2 - b_2/b_0\right)(1 + O(\lambda))$$

We return to eq. 19 and integrate the singular part $\left(-\widehat{\beta}_{sing.}\right)^{-1}$

$$\begin{aligned} \tau &= \int_{\overline{\kappa}}^{\kappa} d\lambda \left(-\widehat{\beta}(\lambda)\right)^{-1} \\ &= \int_{\overline{\kappa}}^{\kappa} d\lambda \left\{\left(-\widehat{\beta}_{reg.}(\lambda)\right)^{-1} + \left(-\widehat{\beta}_{sing.}(\lambda)\right)^{-1}\right\} \end{aligned}$$

$$\int_{\overline{\kappa}}^{\kappa} d\lambda \left(-\widehat{\beta}_{sing.}(\lambda)\right)^{-1} = b_0^{-1}\left[\begin{array}{c} \overline{\kappa}^{-1} - \kappa^{-1} \\ -b_0^{-1} b_1 \left(\log\left(\overline{\kappa}^{-1}\right) - \log\left(\kappa^{-1}\right)\right) \end{array}\right]$$

(27)

Eq. 19 thus allows the separation of variables $\overline{\kappa}$ and κ $\rightarrow$

(28)
$$\tau = F(\overline{\kappa}) - G_{reg.}(\overline{\kappa}) - (F(\kappa) - G_{reg.}(\kappa))$$
$$F(\overline{\kappa}) = b_0^{-1}\,\overline{\kappa}^{-1} - (b_1/b_0^2)\log(\overline{\kappa}^{-1})$$
$$G_{reg}(\overline{\kappa}) = \int_0^{\overline{\kappa}} d\lambda \left(-\widehat{\beta}_{reg.}(\lambda)\right)^{-1}$$

the substitution $\tau \to t = \tau - \log(\mu_0^2/\Lambda^2)$ and
inverting the functional relation $t = t(\overline{\kappa}) \longleftrightarrow \overline{\kappa} = \overline{\kappa}(t)$

We rewrite eq. 28 separating sliding scale parts and associated scale μ and initial value parts associated with scale μ_0, as indicated in eqs. 15 - 17

(29)
$$\left[\begin{array}{l} \tau + b_0^{-1}\kappa^{-1} - (b_1/b_0^2)\log\left(b_0^{-1}\kappa^{-1}\right) - G_{reg.}(\kappa) + \\ \qquad + (b_1/b_0^2)\log\left(b_0^{-1}\overline{\kappa}^{-1}\right) + G_{reg.}(\overline{\kappa}) \end{array}\right] = b_0^{-1}\,\overline{\kappa}^{-1}$$

The substitutions $\kappa \to b_0\,\kappa$ and $\overline{\kappa} \to b_0\,\overline{\kappa}$ in the inverse of the arguments of the logarithm terms in eq. 29 cancels out in the difference. Next we adjust the scale parameter $\mu_0 \to \Lambda$ $\rightarrow$

setting

$$- b_0^{-1} \kappa^{-1} + (b_1 / b_0^2) \log \left(b_0^{-1} \kappa^{-1} \right) + G_{reg.} (\kappa) = \log (\mu_0^2 / \Lambda^2)$$

(30) $$t = \tau - \log (\mu_0^2 / \Lambda^2) = \log (\Lambda^2 / \mu^2)$$

$$x = b_0 \overline{\kappa} \;\; ; \;\; G_{reg.}(x / b_0) = G (x) \;\; ; \;\; b_1 / b_0^2 = a$$

Further we substitute the sliding scale variable $\overline{\kappa} \longrightarrow b_0 \overline{\kappa} = x$. With these substitutions eq. 29 becomes

(31) $$t + a \log (x^{-1}) + G (x) = x^{-1}$$

$$G (x) = x \sum_{n=0}^{\infty} G_n x^n$$

In order to establish the asymptotic 'ultraviolet' expansion for $t \rightarrow \infty$ and $Y = x^{-1} \rightarrow \infty$ we introduce the notation , for clarity, including a change of variables $t = \exp (L) ; L = \log (t)$

(32) $$\log (t) = L \;\; ; \;\; x = x (L) \longrightarrow Y = Y (L) \equiv (x (L))^{-1}$$

$$\log (Y) = Z (L) \equiv \log \left((x (L))^{-1} \right)$$

$\rightarrow$

Eq. 31 takes the form

(33)
$$\frac{L + \log\left[\,1 + \dfrac{a\,Z(L) + G(x = Y^{-1})}{\exp(L)}\,\right] = Z(L)}{t = \exp(L)\;\;;\;\; Y = \exp(Z) \longleftarrow Z = Z(L)}$$

Eq. 33 can now be solved by iteration in the asymptotic limit $L \to \infty$

$$L = \log\left[\log\left(\Lambda^2/\mu^2\right)\right] \to \infty$$

$$Z(L) = lim_{n\to\infty} Z_n(L)$$

(34)
$$\frac{L + \log\left[\,1 + \dfrac{a\,Z_n(L) + G\left(x_n = Y_n^{-1}\right)}{\exp(L)}\,\right] = Z_{n+1}(L)}{\textbf{anchor :}\;\; Z_0(L) = L}$$

The first anchor takes care of the largest contribution to to Z , i.e. L . →

Thus we obtain $Z_1(L)$ **and** $Y_1 = \exp(Z_1)$ **from eq. 34**

(35)
$$Z_1(L) = L + \log\left[1 + a L \exp(-L) + \frac{G(\exp(-L))}{\exp(L)}\right]$$

$$Y_1(L) = e^L + a L + G(\exp(-L)) \quad ; \quad t = e^L$$

Comparing withe recursive equations (eq. 34 , repeated below

$$L = \log\left[\log\left(\mu^2/\Lambda^2\right)\right] \to \infty$$

$$Z(L) = lim_{n\to\infty} Z_n(L)$$

(36)
$$L + \log\left[1 + \frac{a Z_n(L) + G\left(x_n = Y_n^{-1}\right)}{\exp(L)}\right] = Z_{n+1}(L)$$

anchor : $Z_0(L) = L$

it follows that all contributions from the function G , i.e. resulting from the subleading terms proportinal in the beta function truncated by the first to terms $b_0 \kappa^2 + b_1 \kappa^3$ **give vanishing contributions to** $Y(L) \equiv (b_0 \overline{\kappa})^{-1} (L)$ **for** $L \to \infty$ **. This can be verified recursively.** →

Thus modulo terms vanishing for $L \to \infty$ we can solve the simpler functional equation, omitting G from eqs. 31, 35 and 36

(37) $$t + a \log\left(\widetilde{Y}\right) = \widetilde{Y} \;;\; \widetilde{Y} = x^{-1} \;;\; Y - \widetilde{Y} \to 0 \text{ for } t \to \infty$$

or equivalently

(38) $$L + \log\left(1 + a\,\widetilde{Z}\,e^{-L}\right) = \widetilde{Z} \;;\; \widetilde{Y} = e^{\widetilde{Z}} \;;\; t = e^{L}$$

The same argument which led to the elimination of the term proportional to G in eq. 36 as far as nonvanishing contributions for Y are concerned in the limit $L \to \infty$ imply that expanding the logarithm on the left hand side of eq. 38 only the first term need be retained

(39) $$\log\left(1 + a\,\widetilde{Z}\,e^{-L}\right) \sim a\,\widetilde{Z}\,e^{-L}$$

leading to the equivalent approximate functional equation simplifying eq. 38

(40) $$Z \sim \widetilde{Z} \sim L/\left(1 - a\,e^{-L}\right) \sim L + a\,e^{-L}\,L \;\longrightarrow$$
$$Y \sim \widetilde{Y} \sim e^{L}\left(1 + a\,e^{-L}\,L\right) = e^{L} + a\,L + [\,0\,] \text{ for } L \to \infty$$

In eq. 40 the $\sim$ symbol is meant to imply modulo additive terms to Y , vanishing for $L \to \infty$, proving the remarkable fact that the potential constant in the asymptotic expansion for Y vanishes . →

We conclude this subsection comparing side by side the functional dependence of $t \equiv \log\left(\mu^2/\Lambda^2\right) = t\left(\overline{\kappa}(\mu)\right)$ **and** $Y \equiv \left(b_0 \overline{\kappa}\right)^{-1} = Y(t)$**.**

The first relation is given in eq. 31

$$t = x^{-1} - a \log\left(x^{-1}\right) - G(x)$$

$$G(x) = x \sum_{n=0}^{\infty} G_n x^n \; ; \; x = b_0 \overline{\kappa}(\mu^2)$$

$$t = \log(\mu^2/\Lambda^2) \longrightarrow$$

---

(41)

$$\Lambda^2 = \mu^2 \left[\exp\left(-\frac{1}{b_0 \overline{\kappa}}\right)\right] \left[\frac{1}{b_0 \overline{\kappa}}\right]^a \exp\left[\, G(b_0 \overline{\kappa}) \,\right]$$

$$a = b_1 / b_0^2$$

The second relation is displayed in eq. 40 , as an asymptotic expansion for $t = e^L \rightarrow \infty$

---

(42)

$$Y = Y(L) \equiv \left(b_0 \overline{\kappa}\right)^{-1}(L) \text{ for } L = \log\left[\log\left(\mu^2/\Lambda^2\right)\right] \rightarrow \infty$$

$$\sim e^L + a L + [0]$$

$\rightarrow$

Eqs. 41 and 42 constitute the essence of this subsection, giving rise to some remarks :

1) The central scale Λ

with all subtle properties as defined, in the $\overline{\text{MS}}$ scheme, constitutes a renormalization group invariant finite mass scale, given a region of sliding scale coupling constant $\kappa = g^{2} / (16 \pi^{2})$ generically or $\overline{\kappa}$ in the perturbative region here, nonvanishing but appropriately small.

This scale does *not* depend on quark masses , only on the number of quark flavors , while derivations from measurable quantities (in the asymptiotic region) suffer from systematic errors , which do depend on quark massses .

The combined analysis of the sliding scale coupling constant , as of 2009 by Bethke , is shown in Fig. 5 below.

I quote here a discussion of Λ for $N_{fl} = 3$ by Bodenstein et al., ref. [8-2011] , from data in the mass scale region of the τ lepton

(43) $\Lambda_{N_{fl}=3} = 382 \pm 24$ MeV $\longleftrightarrow \alpha_{s} = 4\pi \kappa (\mu = m_{\tau}) = 0.344 \pm 0.014$

The reconstruction of Λ from the 'perturbatively accessible region' implies a hard breaking of scale invariance , and hence equivalently a nonvanishing trace of the co-renormalizable and renormalized energy momentum density operator. This shall be discussed in the next subsection.

$\rightarrow$

2) sliding scale quark masses

The rescaling equations (eqs. 15 - 17) extend to the quark mass parameters , again universally and quark mass independently rescaling m_f , keeping ratios invariant

(44) $$r_{f_1 f_2} = m_{f_1} / m_{f_2}$$

Here no detailed discussion of extracting sliding scale quark masses including heavy flavors c , b is given. For heavy flavor comparisons with initially the reaction $e^+ e^- \rightarrow f \overline{f} -$ flavored hadrons , the QCD sum rule approach is the main tool , as pioneered by Shifman, Vainshtein and Zakharov [9-1979] , together with lattice QCD .

The sliding quark masses as for 3 flavors are shown in figure 6 below , restricted to two loop approximation . References [8-2011] - [10-2011] are representative for present refinement to four loops in the rescaling functions and derived results , e.g.

(45) $$\overline{m}_b (\overline{m}_b) = \begin{cases} 4163 \pm 16 \text{ MeV} & \text{ref. [11-2010]} \\ 4171 \pm 7 \text{ MeV} & \text{ref. [12-2011]} \\ 4177 \pm 11 \text{ MeV} & \text{ref. [10-2011]} \end{cases}$$

$$\overline{m}_c (3 \text{ GeV}) = 986 \pm 13 \text{ MeV} , \text{ ref. [11-2010]}$$

$$\overline{m}_c (\overline{m}_c) = 1262 \pm 17 \text{ MeV} , \text{ ref. [10-2011]}$$

$\rightarrow$

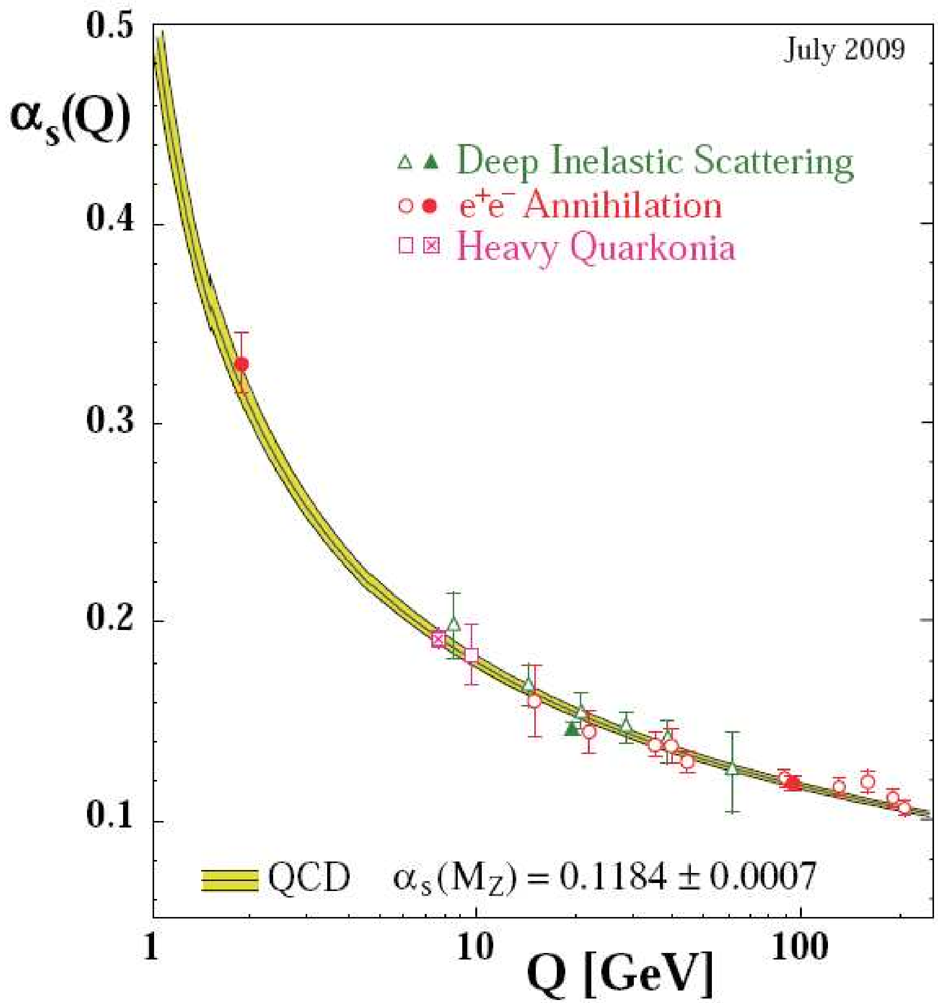

Fig. 5 : $\alpha_s(Q) = 4\pi\,\kappa(\mu = Q)$ **from ref. [13-2009] .**

1-1 – The two central anomalies alongside : scale- or trace- and U1-axial anomaly and the renormalization group invariant scale Λ

We display the sructure of the two 'central' anomalies in eq. 46 below

$$\left\{ \vartheta^{\mu}{}_{\mu} = \sum_f m_f \, S_{\dot{f}\, f} + \delta_0 \right\} (x)$$
$$\left\{ \partial^{\mu} \left(j^{5}_{\mu} \right)^{S} = 2 \, \langle m \rangle \, i \, P^{S} + \delta_5 \right\} (x)$$

(46)

$$\delta_0 = \left\{ - \left(-2 \beta(g) / g^3 \right) \left[\tfrac{1}{4} (:) \, B^{t}_{\mu \nu} \, B^{\mu \nu \, t} (:) \right] \right\}_{\rightarrow ren.gr.inv}$$
$$\delta_5 = \left\{ \left(2 \, N_{fl} \right) \tfrac{1}{8 \pi^2} \left[\tfrac{1}{4} (:) \, B^{t}_{\mu \nu} \, \widetilde{B}^{\mu \nu \, t} (:) \right] \right\}_{\rightarrow ren.gr.inv}$$

$$- \beta / g^3 = b_0 / (16 \pi^2) + O(\kappa) \; ; \; \kappa = g^2 / (16 \pi^2)$$

The predicate 'central' for the anomalies in eq. 46 stands for the property that in rendering the square coupling constant and the associated $\vartheta -$ parameter in the gauge boson *naively renormalized* Lagrangean density x dependent

(47)

$$\mathcal{L}_{g.b.} = - \tfrac{1}{g^2} \tfrac{1}{4} (:) \, B^{t}_{\mu \nu} \, B^{\mu \nu \, t} (:) + \vartheta \tfrac{1}{8 \pi^2} \tfrac{1}{4} (:) \, B^{t}_{\mu \nu} \, \widetilde{B}^{\mu \nu \, t} \longrightarrow$$
$$g^2 \rightarrow g^2 (x) \; ; \; \vartheta \rightarrow \vartheta (x)$$

maintains perturbative renormalizability and acts together with suitable boundary- – more generally – regularity conditions $\rightarrow$

as external sources for the scalar and pseudoscalar local field strength bilinears

(48) $$\frac{1}{4}\,(:)\,B^{\,t}_{\,\mu\,\nu}\,B^{\,\mu\,\nu\,t}\,(:)\quad,\quad\frac{1}{4}\,(:)\,B^{\,t}_{\,\mu\,\nu}\,\widetilde{B}^{\,\mu\,\nu\,t}\,(:)$$

We will use the following definitions relative to the rescaling function β

$$-\,\beta\,/\,g\;=\;\kappa\,B\,(\,\kappa\,)\quad;\quad B\,(\,\kappa\,)\;=\;b_{\,0}\,A\,(\kappa\,)$$

$$B\,(\,\kappa\,)\;\sim\;\sum_{\,n=0}^{\,\infty}\,b_{\,n}\,\kappa^{\,n}\quad,\quad A\,(\,\kappa\,)\;\sim\;\sum_{\,n=0}^{\,\infty}\,a_{\,n}\,\kappa^{\,n}$$

$$\kappa\;=\;g^{\,2}\,/\,(\,16\,\pi^{\,2}\,)\;\;\textbf{generic}\;\;\longrightarrow\;\;X\,,\,Y$$

(49) $$b_{\,0}\;=\;\frac{1}{3}\,\left(\,33\;-\;2\,N_{\,fl}\,\right)\quad,\quad a_{\,0}\;=\;1\quad,\quad a_{\,n}\;=\;b_{\,n}\,/\,b_{\,0}$$

$$b_{\,1}\;=\;\frac{2}{3}\,\left(\,153\;-\;19\,N_{\,fl}\,\right)$$

$$b_{\,2}\;=\;\frac{1}{54}\,\left(\,77139\;-\;15099\,N_{\,fl}\;+\;325\,N_{\,fl}^{\,2}\,\right)$$

$$b_{\,3}\;\sim\;29243\;-\;6946.3\,N_{\,fl}\;+\;405.089\,N_{\,fl}^{\,2}\;+\;1.49931\,N_{\,fl}^{\,3}$$

References in conjunction with this section (’1-1’) are presented in five (partial) collections :

1 : (R) directly related to the two central anomalies

2 : (rBsquare) establishing the one renormalization group invariant quantity of dimension $[\,M^{\,4}\,]$

3 : (r-sp-1) a recent paper by Guido Altarelli and references cited therein

4 : (r-A2x) a selection of papers and textbooks for the entire realm of QCD

5 : (r-condx) : Condensation phenomena and field theory realizations

$\rightarrow$

This concludes this introdctory layout of premises . The extended discssion of the sliding scale coupling constant and its relation to the renormalization group invariant scale Λ – as seen from the ultraviolet asymptotic region – is subsummed in eqs. 50 and 51 below

(50)
$$t = Z^{-1} - a \log \left(Z^{-1} \right) - G(Z) \; ; \; a = b_1 / b_0^2$$
$$G(Z) = Z \sum_{n=0}^{\infty} G_n Z^n \; ; \; Z = b_0 \overline{\kappa} (\mu^2)$$
$$t = \log (\mu^2 / \Lambda^2) \longrightarrow$$
$$\Lambda^2 = \mu^2 \left[\exp \left(- \frac{1}{b_0 \overline{\kappa}} \right) \right] \left[\frac{1}{b_0 \overline{\kappa}} \right]^a \exp \left[G (b_0 \overline{\kappa}) \right]$$

The inverted asymptotic expansion for $\zeta = \left(b_0 \overline{\kappa} (\mu^2) \right)^{-1}$ **for** $t = e^L \to \infty$ **is**

(51)
$$\zeta (L) \sim e^L + a L + [0] \text{ for } L = \log \left[\log \left(\mu^2 / \Lambda^2 \right) \right] \to \infty$$

→

2 – Direct consequences from the central anomalies

2-1 – Renormalizing composite local operators at sliding scale $\mu = \infty$

2-1-1 – rescaling equations in the $\overline{\text{MS}}$ scheme for coupling constant g and quark masses m_f

$\beta(g)$ and $\gamma_m(\kappa)$, $\kappa = g^2/\left(16\pi^2\right)$ *extended*

First we choose the (renormalization group invariant) scale Λ, which lies outside the perturbatively accessible region , and the sliding scale μ in the form shown in eq. 50 together with the rescaling functions β, γ_m governing the universal rescaling of coupling constant and quark masses $\overline{g}$, $\overline{m}_q$ respectively

(52)
$$t = \log\left(\mu^2/\Lambda^2\right) \; ; \; \overline{\kappa} = \overline{g}^2/\left(16\pi^2\right) \;\rightarrow$$
$$\left\{ \widehat{\beta} = (\beta/\overline{g})\,\overline{\kappa} = \widehat{\beta}(\overline{\kappa}) \,,\, \overline{m}_q \,,\, \gamma_m(\overline{\kappa}) \right\}$$

In eq. 52 the overlined quantities refer to the sliding scale μ, which shall be chosen $\mu \gg \Lambda$, such as to ensure perturbative accessibility

(53)
$$\overline{\kappa} = \overline{\kappa}(\mu) \; ; \; \overline{m}_q = \overline{m}_q(\mu) \text{ for quark flavor } q$$

with initial values for $\mu = \Lambda$:

$$\overline{\kappa}(\Lambda) = \kappa^* \,,\, \overline{m}_q(\Lambda) = m_q^*$$

$\rightarrow$

The rescaling equations can be cast into the form

$$\bullet = d/dt \;\; ; \;\; t = \log\left(\mu^2/\Lambda^2\right)$$

(54)

$$\frac{1}{\overline{\kappa}} \; \overset{\bullet}{\overline{\kappa}} = -\left[-\widehat{\beta}/\overline{\kappa}\right](\overline{\kappa}) \quad ; \quad \frac{1}{\overline{m}_q} \; \overset{\bullet}{\overline{m}}_q = -\left[-\gamma_m\right](\overline{\kappa})$$

Two remarks concerning the structure of the relations highlighted in eq. 54 are in order here :

1) the rescaling function γ_m

does not depend on the values of the quark masses, i.e. on flavor q , be these sliding scale masses $\overline{m}_q$ or renormalization group invariant ones , as e.g. $m_q^{N*} = \overline{m}_q(\mu = N\Lambda)$ for any fixed multiple $N\Lambda$. N can perfectly well be chosen large enough such that $N\Lambda$ lies within the perturbatively accessible region .

2) both sliding scale functions $\widehat{\beta}/\overline{\kappa}$ and γ_m

are negative at least for μ chosen large enough and dominated by the lowest term in the sliding scale expansion as made explicit below . The first four relevant powers in the expansion of the rescaling functions are given in Appendix 1 .

$\rightarrow$

The sliding scale quark mass, integrating eq. 54 within the limits $t_> \geq t' \geq t_<$ **, becomes**

$$\log\left(\frac{m_{>,q}}{\overline{m}_{<,q}}\right) = -\int_{t_<}^{t_>} d\,t'\,\gamma_m\left(\overline{\kappa}\left(t'\right)\right) \quad ; \quad d\,t' = -\left(\widehat{\beta}\left(\overline{\kappa}'\right)\right)^{-1} d\,\overline{\kappa}'$$

(55)

$$t_< \leftrightarrow m_{>,q} ; \overline{\kappa}_> \quad , \quad t_> \leftrightarrow \overline{m}_{<,q} ; \overline{\kappa}_<$$

Performing the substitution of integration variables $t' \to \overline{\kappa}'$ **it follows**

$$\log\left(\frac{m_{>,q}}{\overline{m}_{<,q}}\right) = \int_{\overline{\kappa}_<}^{\overline{\kappa}_>} d\,\overline{\kappa}'\,\frac{\gamma_m\left(\overline{\kappa}'\right)}{\widehat{\beta}\left(\overline{\kappa}'\right)} = \left(4 / b_0\right)\int_{\overline{\kappa}_<}^{\overline{\kappa}_>} d\,Y\,\frac{\Gamma(Y)}{Y\,A(Y)}$$

$$\gamma_m(Y) \simeq \gamma_0\;Y\;\Gamma(Y) \qquad \gamma_0 = -4$$

$$\widehat{\beta}(Y) \simeq \beta_0\,Y^2\,A(Y) \qquad \beta_0 = -b_0 = -\left(11 - \tfrac{2}{3}N_{fl}\right)$$

(56)

$$\gamma_m(Y) \simeq \sum_{n=0}^{\infty}\gamma_n\,Y^{n+1} = \gamma_0\,Y\,\Gamma(Y) \quad ; \quad \Gamma(Y) \simeq \sum_{n=0}^{\infty}\Gamma_n\,Y^n$$

$$\Gamma_n = \gamma_n / \gamma_0$$

$$\widehat{\beta}(Y) \simeq \sum_{n=0}^{\infty}\beta_n\,Y^{n+2} = -Y^2\,B(Y) \quad ; \quad B(Y) \simeq \sum_{n=0}^{\infty}b_n\,Y^n$$

$$B(Y) = b_0\,A(Y) \qquad b_n = -\beta_n$$

→

The integration region $\overline{\kappa}_{>} \geq \overline{\kappa}' \geq \overline{\kappa}_{<}$, specified in eqs. 55 and 56, is restricted to contain no nontrivial zero of $\widehat{\beta}(\overline{\kappa}')$, but can well extend to the perturbatively inaccessible region towards the upper value $\overline{\kappa}_{>}$.

In an abbreviated way performing the steps leading to the asymptotic relations for the sliding scale coupling constant for $\overline{\kappa}_{>} \rightarrow \infty$, given in eq. 50 it follows from eq. 56

$$\overline{m}_{<,q} \rightarrow 0 \ : \ \log\left(\frac{m_{>,q}}{\overline{m}_{<,q}}\right) \sim (4/b_0) \log\left(\frac{1}{b_0 \overline{\kappa}_{<}}\right) + \Phi(\overline{\kappa}_{>})$$

(57)

$$\frac{1}{b_0 \overline{\kappa}_{<}} \sim t_{>} = \log(\mu^2/\Lambda^2)$$

The key point in the asymptotic form of eq. 57 is that the function $\Phi(\overline{\kappa}_{>})$ does not depend on the quark flavor q, nor on the sliding scale $t_{>} \leftrightarrow \overline{\kappa}_{<}$, in particular in the limit $\mu \rightarrow \infty$, equivalent to $t_{>} \rightarrow \infty \leftrightarrow \overline{\kappa}_{<} \rightarrow 0$.

We choose a renormalization group invariant set of quark masses denoted m_q^{**} in the following, $\rightarrow$

such that eq. 57 asymptotically takes the form

(58)
$$\overline{m}_{<,q} \to 0 \quad : \quad \log\left(\frac{m_q^{**}}{\overline{m}_{<,q}}\right) \sim \left\{ \begin{array}{l} (4/b_0)\log\left(\frac{1}{b_0\,\overline{\kappa}_<}\right) + \\ + \left[\Phi(\overline{\kappa}_>) - \log\left(\frac{m_{>,q}}{m_q^{**}}\right)\right] \end{array} \right\}$$

$$\frac{1}{b_0\,\overline{\kappa}_<} \sim t_> = \log(\mu^2/\Lambda^2)$$

imposing the condition on the lower term in curly brackets in eq. 58 to vanish

(59)
$$\left[\Phi(\overline{\kappa}_>) - \log\left(\frac{m_{>,q}}{m_q^{**}}\right)\right] = 0$$

→

Then eq. 58 takes the form as shown in eq. 60 together with the structure of the renormalization group invariant scale Λ as derived from the asymptotic sliding coupling constant behaviour in eq. 50

$$\overline{m}_{<,q} \to 0 \;:\; \log \left(\frac{m_q^{**}}{\overline{m}_{<,q}} \right) \sim (4/b_0) \log \left(\frac{1}{b_0 \overline{\kappa}_<} \right)$$

$$m_q^{**} = \lim_{t \to \infty} \overline{m}_q(t) \left(\frac{1}{b_0 \overline{\kappa}(t)} \right)^{4/b_0} \;;\; \overline{\kappa}(t) \to \overline{\kappa} \searrow \tag{60}$$

$$\Lambda^2 = \mu^2 \left[\exp \left(- \frac{1}{b_0 \overline{\kappa}} \right) \right] \left[\frac{1}{b_0 \overline{\kappa}} \right]^a \exp \left[G(b_0 \overline{\kappa}) \right]$$

$$t = \log(\mu^2 / \Lambda^2) \sim Z^{-1} - a \log\left(Z^{-1}\right) \;;\; Z = b_0 \overline{\kappa}(t) \;;\; a = b_1 / b_0^2$$

$$b_0 = 11 - \tfrac{2}{3} N_{fl} \;;\; b_1 = \tfrac{2}{3} \left(153 - 19 N_{fl} \right)$$

This concludes main derivations in this subsection, subject to the following remarks $\rightarrow$

1) nonuniversality of the quantities Λ , m_q^{**}

whence extracted from the limiting values of $t \sim \infty$. The reason is illustrated in figure 6 below, which shows the flavor dependence of Λ as a function of N_{fl} contained in the coefiicients b_0 , b_1 determining this scale as defined through the above limit. Intuitively this can be traced back to the threshold behaviour of heavy quark (antiquark) flavors. The latter are represented also by the nonanomalous quark mass part of the QCD Lagrangean, and its renormalized counterpart in the trace of the energy momentum density tensor, dependent linearly on the quark mass parameters $\overline{m}_q^{**}$. The physical properties of these threshod effects however are by no means linear in these parameters.

2) The renormalization of composite operators at zero distance

is a useful tool especially for defining gauge field strength bilinears forming central anomalies. It however becomes mandatory to include all interactions up to infinite energies , not only unified gauge and associated interactions but ultimately also gravity. This is illustrated in Fig. 7 below including standard model interactions in the three sliding scale coupling constant rescaling equations to three loops by Mathias Steinhauser et al. in ref. [14-2012] .

3) Segregating the pure gauge degrees of freedom

in their combined infrared and ultraviolet role from quark flavors. This shall be the substrate of the next subsection. $\longrightarrow$

2-1-Fig. 6

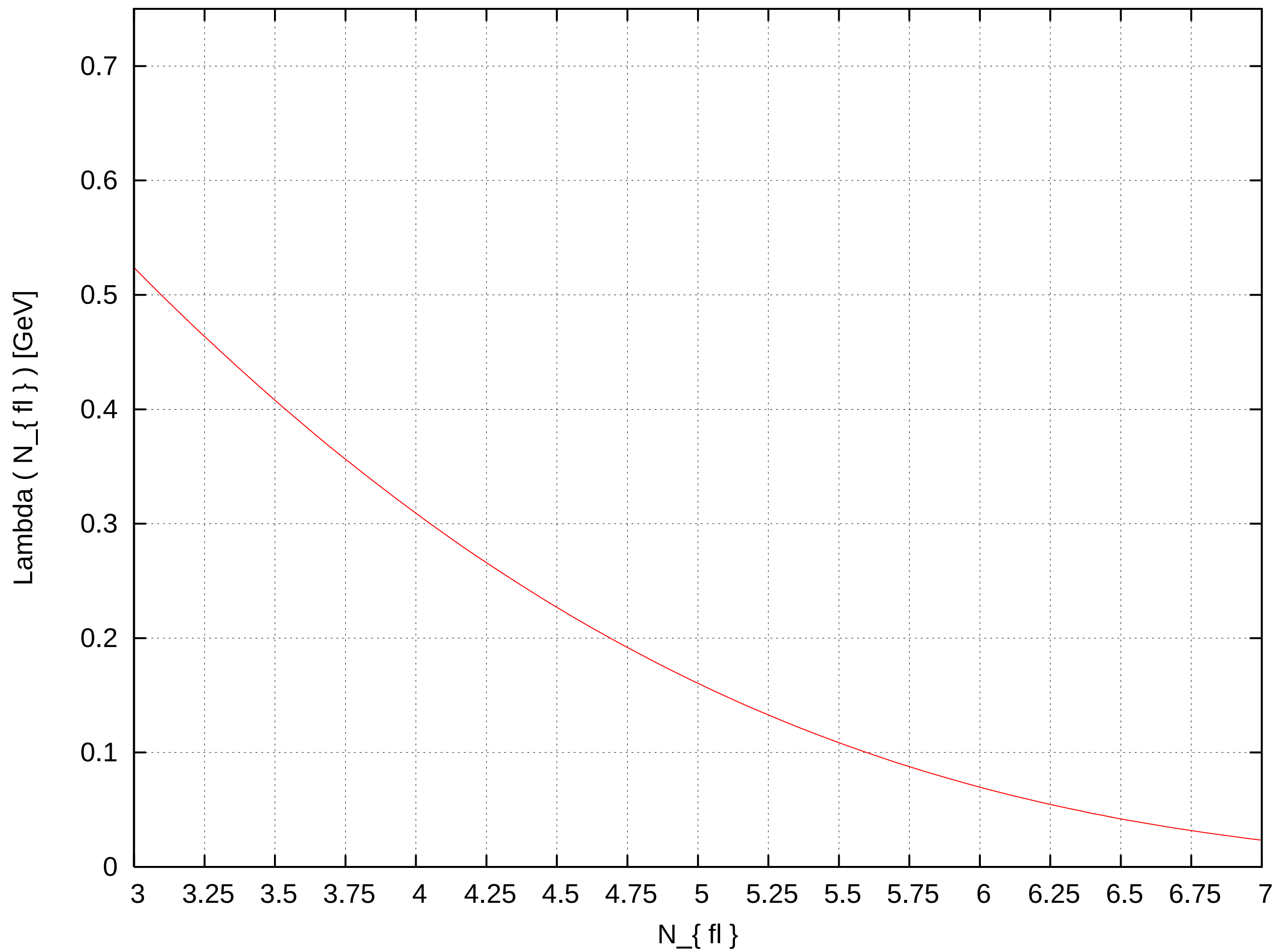

Fig. 6 : Scale Λ for fixed $\overline{\kappa}\,(\,m_{\,Z}\,)$ for continuous values of $N_{\,fl}$.
The input value $\alpha_{\,s}\,(\,m_{\,Z}\,)\;=\;0.1135\;(\,\pm\;\sim\;0.0009\,)$
is taken from André Hoang et al. , ref. [15-2010] .

$\rightarrow$

2-1-Fig. 7

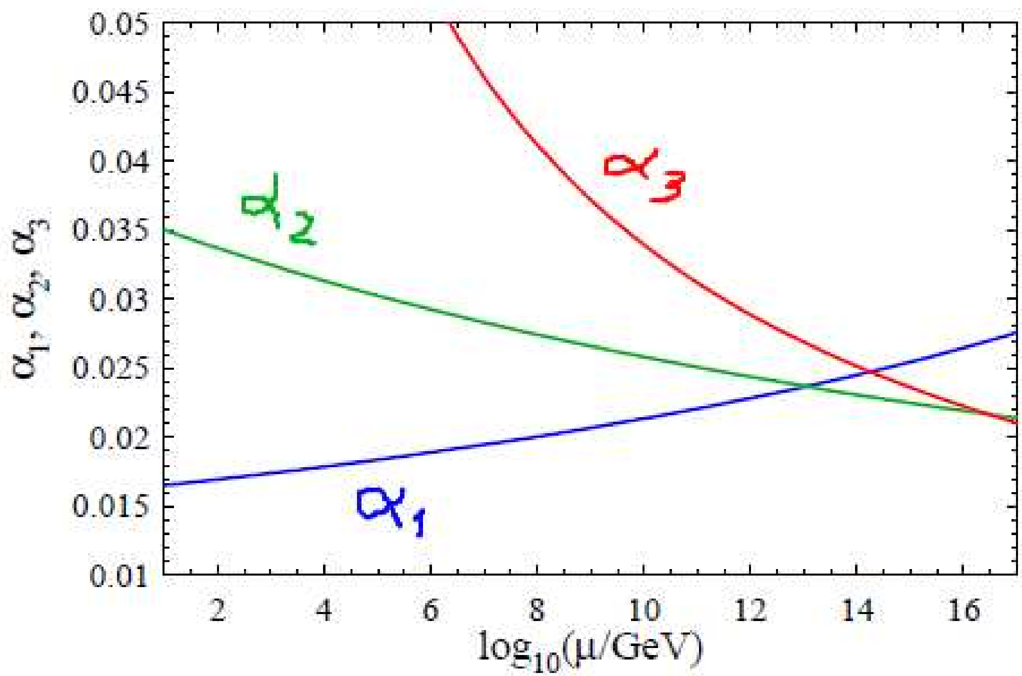

FIG. 3. The running of the gauge couplings at three loops. The curve with the smallest initial value corresponds to α_1, the middle curve to α_2, and the curve with the highest initial value to α_3.

to Fig. 7 : The figure is taken from Mathias Steinhauser et al. in ref. [14-2012] .

It is based as far as $\alpha_3 = \alpha_s = 4\pi\overline{\kappa}(m_Z)$ **is concerned**

on the value $\alpha_s(m_Z) = 0.1173\ (\pm \sim 0.00069)$

→

2-1-2 – reinterpreting the central anomalies

$$\tfrac{1}{4}\left[B^{\,r}_{\mu\nu}\,B^{\,\mu\nu\,r}\right]_{\infty} = \mathcal{B}^{\,+}_{\infty} \quad \text{\&} \quad \tfrac{1}{4}\left[B^{\,r}_{\mu\nu}\,\widetilde{B}^{\,\mu\nu\,r}\right]_{\infty} = \mathcal{B}^{\,-}_{\infty}$$

The central anomalies (eq. 46) become, using the field strength bilinears (re-)normalized at $\mu = \infty$, and separating gauge- and quark antiquatrk parts in the lowest order constant $b_0 = b_{0\,g.b.} - \frac{2}{3} N_{fl}$ of the coupling constant rescaling function $\beta(g)$

$$\left\{\begin{array}{ll} \vartheta^{\,\mu}{}_{\mu} & = \sum_q \left[m_q(\mu)\,\overline{q}\,q(\mu) + \frac{1}{12\pi^2}\,\mathcal{B}^{\,+}_{\infty}\right] - b_{0\,g.b.}\,\frac{1}{8\pi^2}\,\mathcal{B}^{\,+}_{\infty} \\ \partial^{\,\nu}\left(\overline{q}\,\gamma_\nu\,\gamma_5\,q\right) & = 2\,m_q(\mu)\,\overline{q}\,i\,\gamma_5\,q(\mu) \qquad\qquad + \quad 2\,\frac{1}{8\,\pi^2}\,\mathcal{B}^{\,-}_{\infty} \end{array}\right\}(x)$$

$$-\beta/g^3 = b_0/(16\,\pi^2) + O(\kappa) \;\; ; \;\; \kappa = g^2/(16\,\pi^2)$$

$$b_0 = b_{0\,g.b.} - \tfrac{2}{3} N_{fl} \;\; ; \;\; b_{0\,g.b.} = \tfrac{11}{3}\,C_2\left(adj\left(Lie - SU3_c\right)\right) = \tfrac{11}{3}\,3$$

(61)

In eq. 61 normal ordering symbols are omitted for brevity of notation. Furthermore the renormalization scale for the field strengths bilinears

$$\frac{1}{4}\left[B^{\,r}_{\mu\nu}\,B^{\,\mu\nu\,r}\right]_{\infty} = \mathcal{B}^{\,+}_{\infty} \;\; ; \;\; \frac{1}{4}\left[B^{\,r}_{\mu\nu}\,\widetilde{B}^{\,\mu\nu\,r}\right]_{\infty} = \mathcal{B}^{\,-}_{\infty}$$

is set to $\mu = \infty$ which does lead to a conceptual clarification of these quantities most importantly for a selfdual or anti-selfdual classical background field configuration. $\rightarrow$

The sliding scale quark masses $m_q(\mu)$ and contragredient $\overline{q}\,q$ scalar and pseudoscalar bilinears, for individual color triplet-antitriplet flavors q have been kept in eq. 61 , i.e. no attempt is made to fix μ in favor of a renormalization group invaraint scale.

2-1-2a – the catalytic effect of a quark triplet-antitriplet flavor with $\lim m_{q\,kat.}(\mu) = \infty \; ; \; \mu$ fixed

We enlarge the physical substrate of 6 quark-antiquark flavors u, d, s, c, b, t by an additional flavor $q\,catalytic$ which shall have an initially assigned finite but very heavy quark mass, subsequently increased to become infinite, as indicated in the title of this subsection. The anomaly relations in eq. 61 then take the form

$$\vartheta^{\mu}{}_{\mu} = \left[\begin{array}{l} \sum_{q=1}^{6} \left[m_q(\mu)\,\overline{q}\,q(\mu) + \frac{1}{12\pi^2}\mathcal{B}_{\infty}^{+} \right] - b_{0\,g.b.}\frac{1}{8\pi^2} \\ + \left[m_{q\,kat.}(\mu)\,\overline{q}_{kat.}\,q_{kat.}(\mu) + \frac{1}{12\pi^2}\mathcal{B}_{\infty}^{+} \right] \end{array} \right] \qquad (62)$$

$$\partial^{\nu}\left(\overline{q}_{kat}\,\gamma_{\nu}\,\gamma_5\,q_{kat.}\right) = 2\,m_{q\,kat.}(\mu)\,\overline{q}_{kat.}\,i\gamma_5\,q_{kat.}(\mu) + 2\,\frac{1}{8\,\pi^2}\mathcal{B}_{\infty}^{-}$$

$$\lim m_{q\,kat.}(\mu) \rightarrow \infty \quad ; \quad \mu \text{ fixed}$$

It follows that the catalytic flavors $q_{kat.}, \overline{q}_{kat.}$ decouple in the limit considered in a nontrivial way. $\rightarrow$

$$\lim_{m_{q\,kat.}(\mu)\to\infty} \; : \; \left[\begin{array}{c} m_{q\,kat.}(\mu)\,\overline{q}_{kat.}\,q_{kat.}(\mu) \\ 2\,m_{q\,kat.}(\mu)\,\overline{q}_{kat.}\,i\,\gamma_5\,q_{kat.}(\mu) \end{array} \right] = - \left[\begin{array}{c} \frac{1}{12\pi^2}\,\mathcal{B}^{+}_{\infty} \\ \frac{1}{4\,\pi^2}\,\mathcal{B}^{-}_{\infty} \end{array} \right]$$

(63) **μ fixed**

The space-time argument x ,of the composite operators in eq. 63 are suppressed for simplicity of notation.

As limiting operator identies the chiral properties of $\overline{q}_{kat.}\,\frac{1}{2}\,(\,¶\,\pm\,\gamma_5\,)\,)\,q_{kat.}$ become entangled with the positive and negative parity gauge boson field strength bilinears.

From the 'catalytic' limiting relations in eq. 63 taking vacuum expected values on both sides of it, choosing the positive parity relation we obatain

(64)
$$\lim_{m_{q\,kat.}(\mu)\to\infty} m_{q\,kat.}(\mu)\,\langle\,\Omega\,|\,\overline{q}_{kat.}\,q_{kat.}(\mu)\,|\,\Omega\,\rangle = -\,\frac{1}{12\pi^2}\,\langle\,\Omega\,|\,\mathcal{B}^{+}_{\infty}\,|\,\Omega\,\rangle$$

could *perfectly well* **vanish. This is a dynamical question, brought forth by Shifman, Vainshtein and Zakharov in 1979 [9-1979] in conjunction with the charmonium spectroscopy . We will take up this discussion in the next subsection.** $\rightarrow$

2-1-Fig. 8

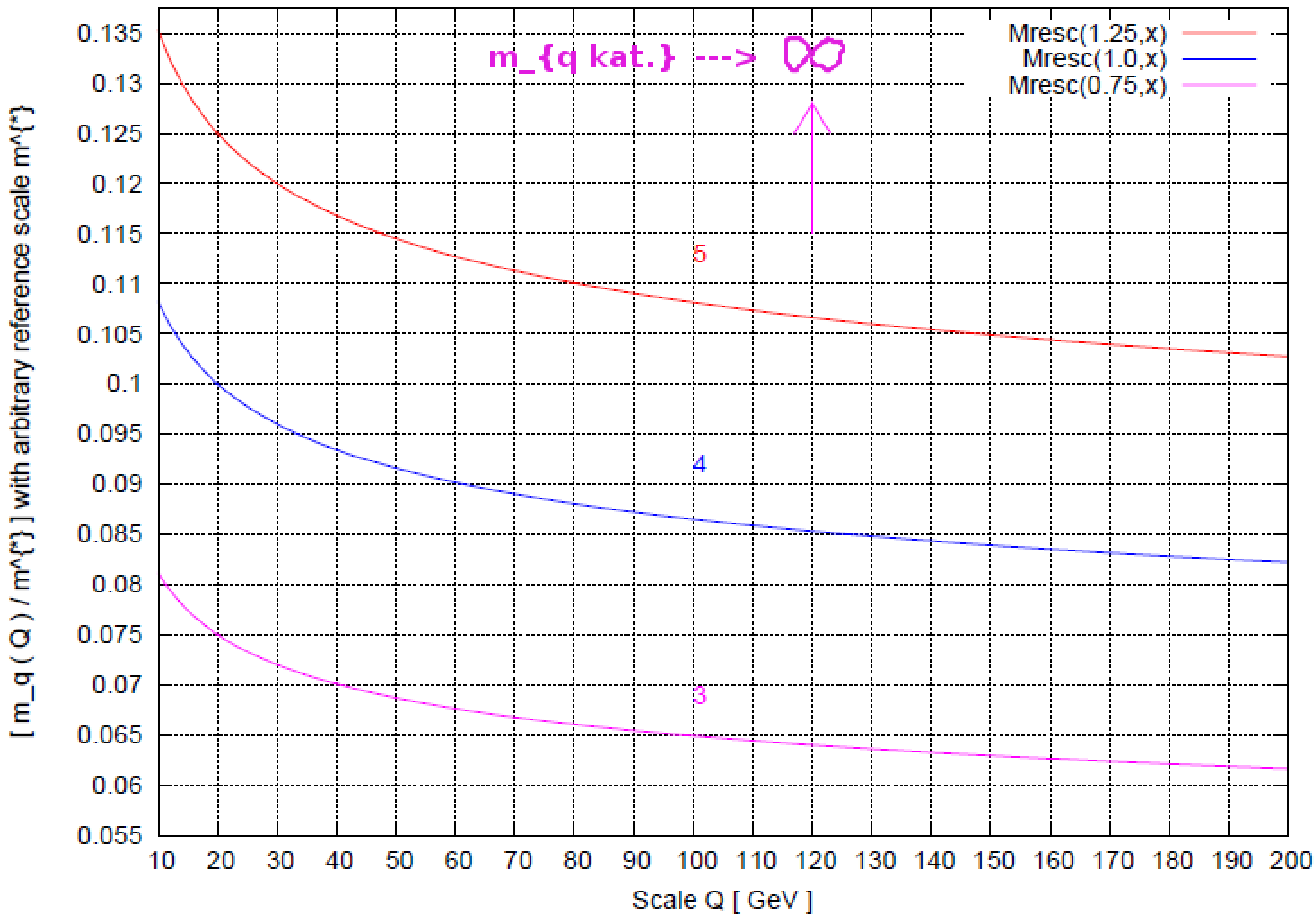

Fig. 8 : $m_q(Q)/m^*$ **with fixed ratio of rescaled quark masses**

$$m_u : \frac{1}{2}(m_d + m_u) : m_d = 3 : 4 : 5$$

→

3 – The filigran fabrics of gauge boson field complexes

(the word complex is used here in the association with 'complex chemistry')

3-1 – Shifting focus to the edge between perturbative and nonperturbative regions

3-1-1 – the nature of condensates and attempts to determine $\langle \Omega \,|\, \frac{1}{4} \left[B^{\,r}_{\mu\nu} B^{\mu\nu\, r} \right]_{\infty} |\, \Omega \rangle$ from QCD sum rules [9-1979]

Lets use as entry point the derivation of the scalar $\overline{u}\, u + \overline{d}\, d$ condensate (or vev) by Gell-Mann, Oakes and Renner [16-1968] and the axial current matrix element

$$\langle \Omega \,|\, j^{\,\varrho}_{5} (x) \left| \pi^{0}, q \right) \rangle = f_{\pi}\, i\, q^{\,\varrho}\, e^{-\,i\,x\,q} \;\; ; \; f_{\pi} \left(m^{2}_{\pi^{0}} \right) \sim 92.4 \text{ MeV}$$

(65) $$j^{\,\varrho}_{5} = \frac{1}{2} \left(\overline{u}\, \gamma^{\,\varrho} \gamma_{5}\, u - \overline{d}\, \gamma^{\,\varrho} \gamma_{5}\, d \right)$$

$$D_{5} (x) = \partial_{\varrho}\, j^{\,\varrho}_{5} (x) \;\longrightarrow\; \langle \Omega \,|\, D_{5} (x) \left| \pi^{0}, q \right) \rangle = f_{\pi}\, m^{2}_{\pi^{0}}\, e^{-i\,x\,q}$$

Associating a hermitian interacting field $\varphi (y)$ with the state $\left| \pi^{0}, q \right\rangle$ with the normalization and phase conditions leads to a special choice for the interpolating field φ

(66) $$\langle \Omega \,|\, \varphi (x) \left| \pi^{0}, q \right\rangle = e^{-i\,x\,q} \;\longrightarrow\; \varphi (y) = \frac{1}{f_{\pi}\, m^{2}_{\pi^{0}}} D_{5} (y)$$

Freezing the first argument – x – of D_{5} to $x = 0$ and using the Lehmann-Symanzik-Zimmermann reduction for the on shell pion we obtain →

(67) $$f_\pi m^2_{\pi^0} = i \int d^4y\, e^{-iyq} \langle \Omega | \, T \, \{ D_5(0) D_5(y) \} \, | \Omega \} \frac{m^2_{\pi^0} - q^2}{f_\pi m^2_{\pi^0}}$$

T denotes time ordered product, symmetric for Bose fields. Results from the LSZ-reduction involving a neutral field φ refer to one half of the difference between an incoming state with momentum q , as in eq. 65 and the associated outgoing one

(68) $$\left\langle \pi^0 , p \right| j^\varrho_5(x) \left| \Omega \right\rangle = f_\pi \, i p^\varrho e^{+ixp} \quad ; \quad p = -q$$

Eq. 67 can be written

(69) $$\begin{aligned} f^2_\pi m^2_{\pi^0} &= i \int d^4y\, e^{-iyq} \langle\Omega| \, T \, \{ D_5(y) D_5(0) \} \, |\Omega\rangle \left(1 - q^2 / m^2_{\pi^0} \right) \\ D_5(y) &= \partial_\varrho j^\varrho_5(y) = i \Big(m_u \overline{u} \gamma_5 u - m_d \overline{d} \gamma_5 d \Big) (y) \end{aligned}$$

Upon partial integration we obtain

(70) $$f^2_\pi m^2_{\pi^0} = \Big(1 - q^2 / m^2_{\pi^0} \Big) \int d^4y\, e^{-iyq} \left[\begin{array}{l} - q_\varrho \langle\Omega| \, T \, \{ j^\varrho_5(y) D_5(0) \} \, |\Omega\rangle \\ -i \, \delta\left(y^0 \right) \langle\Omega| \, \left[j^0_5(y) , D_5(0) \right] |\Omega\rangle \end{array} \right]$$

Next we extrapolate the right hand side of eq. 70 off shell to the value $q = 0$, which induces relative errors $o \left(m^2_{\pi^0} / \widetilde{\Lambda}^2 \right)$, where $\widetilde{\Lambda}$ denotes a hadronic scale not vanishing in the chiral limit . →

Chiral perturbative expansions within effective theories is a dedicated discipline, pioneered by Steven Weinberg, Heinrich Leutwyler and Jürg Gasser, [17-1966, 18-1984, 19-1985] .

Continuing the derivation laid out here, eq. 70 reduces to the form

(71) $$f_{\pi}^{2} m_{\pi^0}^{2} \sim \int d^4 y\, \delta\left(y^0\right) \langle \Omega | \left[j_5^0(y), \frac{1}{i} D_5(0) \right] | \Omega \rangle$$

The equal time commutator on the right hand side of eq. 71 leads to the contact term involving scalar quark-antiquark bilinears

$$j_5^{\varrho}(y) = \tfrac{1}{2}\left(\overline{u}\gamma^{\varrho}\gamma_5 u - \overline{d}\gamma^{\varrho}\gamma_5 d\right)(y)$$

$$\delta\left(y^0\right)\left[j_5^0(y), \left(m_u \overline{u}\gamma_5 u - m_d \overline{d}\gamma_5 d\right)(0)\right] =$$

$$\delta\left(y^0\right)\tfrac{1}{2}\left[\left(u^{\dagger}\gamma_5 u - d^{\dagger}\gamma_5 d\right)(y), \left(m_u u^{\dagger}\gamma_0\gamma_5 u - m_d d^{\dagger}\gamma_0\gamma_5 d\right)(0)\right]$$

(72)

$$= \delta\left(y^0\right)\tfrac{1}{2}\sum_{q=u,d} m_q \left[q^{\dagger}\gamma_5 q(y), q^{\dagger}\gamma_0\gamma_5 q(0)\right]$$

Finally we use the ***canonical structure*** **pertaining to quark flavors yielding for the equal time commutators**

(73) $$\delta\left(y^0\right)\left[q^{\dagger}\gamma_5 q(y), q^{\dagger}\gamma_0\gamma_5 q(0)\right] = \delta^4(y)\, q^{\dagger}\left\{\begin{array}{r}\gamma_5\gamma_0\gamma_5 \\ -\gamma_0\gamma_5\gamma_5\end{array}\right\} q(0)$$

→

For clarity we rewrite eq. 73 in condensed form

(74) $$\delta\,(y^0)\,\left[q^\dagger \gamma_5\, q(y),\, q^\dagger \gamma_0 \gamma_5\, q(0)\right] = \underbrace{-}\, 2\,\delta^4(y)\,\{\overline{q}\, q(0)\}$$

Eqs. 73 and 74 are operator identities compatible precisely with the Ward identities of $\overline{q}\,\gamma^{\varrho}\,\gamma_5\, q$ nonanomalous combinations of axial vector local quark currents and associated pseudoscalar and scalar densities, including elimination of the single $q\,(\overline{q})$ wave function renormalization factors in QCD. The - sign is not a convention , as made explicit in the derivation of eqs. 65 - 72 .

Integrating out the local contact term eq. 71 becomes

(75) $$\begin{aligned} f_\pi^2\, m_{\pi^0}^2 &\sim -\langle\Omega|\left\{\begin{matrix} m_u(\mu)\,\overline{u}\,u(\mu) + \\ m_d(\mu)\,\overline{d}\,d(\mu) \end{matrix}\right\}|\Omega\rangle \\ &\sim -\langle\Omega|\left\{\begin{matrix} (m_u + m_d)\,(\mu)\,\frac{1}{2}\left(\overline{u}\,u + \overline{d}\,d\right)(\mu) + \\ (m_d - m_u)\,(\mu)\,\frac{1}{2}\left(\overline{d}\,d - \overline{u}\,u\right)(\mu) \end{matrix}\right\}|\Omega\rangle \end{aligned}$$

In eq. 75 we dropped the space time dependence of the local operators and reintroduced the explicit contragredient sliding scale dependence of u , d quark masses and scalar densities $\overline{u}\,u$, $\overline{d}\,d$ in the $\overline{\text{MS}}$ sliding scale renormalization scheme.

The last term on the right hand side of eq. 75 is of $o\,(m_d - m_u)$ (of second order in a power expansion in $m_d - m_u$) , i.e. of the same order as the approximation involving →

the off shell extrapolation, and thus can be omitted without affecting the $\sim$ approximate relation. Thus using as abbreviation $(\overline{q}\, q)_{u \leftrightarrow d} = \frac{1}{2}\left(\overline{u}\, u + \overline{d}\, d\right)$ eq. 75 becomes

$$(\overline{q}\, q)_{u \leftrightarrow d} = \frac{1}{2}\left(\overline{u}\, u + \overline{d}\, d\right)$$

(76) $$f_{\pi}^{2}\, m_{\pi^{0}}^{2} \sim -\,(m_{u} + m_{d})\,(\mu)\,\langle\, \Omega \,|\, (\overline{q}\, q)_{u \leftrightarrow d}\,(\mu)\,|\, \Omega\,\rangle$$

$$\langle\, \Omega \,|\, (\overline{q}\, q)_{u \leftrightarrow d}\,(\mu)\,|\, \Omega\,\rangle \sim \langle\, \Omega \,|\, (\overline{u}\, u)\,(\mu)\,|\, \Omega\,\rangle \sim \langle\, \Omega \,|\, (\overline{d}\, d)\,(\mu)\,|\, \Omega\,\rangle$$

We attempt to reach the chiral limit relative to quark flavors u and d $\lim\,(m_{u}, m_{d})$ (μ fixed) $\rightarrow 0$ in an analogous but opposite way to the infinit mass limit limit of a catalytic quark flavor , discussed in subsection 2-1-2a .

To this end let me use recent results approaching the above limit by lattice simulations extending to pion mass(es) below the physical one(s), along exact $SU2_{u,d}$ symmetry, by Stephan Dürr et al. [20-2012] .

We set following ref. [20-2012] but with liberal errors

(77) $$f\,(0) = f_{\pi}\,(m_{\pi}^{2} = 0) = 87.2\ \text{MeV}\,(1 \pm 0.01) \rightarrow$$

$$f^{2}\,(0) = 0.006724\,(1 \pm 0.02)\ \text{GeV}^{2}$$

in order to test the asymptotic ($m_{u,d} \rightarrow 0$)) relation $\rightarrow$

$$f^2(0)\left[\frac{\partial m_\pi^2}{\partial(m_u+m_d)(\mu\ \textbf{fixed})}\right]_{m_u=m_d=0} = -\langle\Omega|\,(\overline{q}\,q)_{u\leftrightarrow d}(\mu)\,|\Omega\rangle$$

(78)

We do this estimate applying a linear interpolation to determine the quantity in eq. 78

(79) $$\left[\frac{\partial m_\pi^2}{\partial(m_u+m_d)(\mu\ \textbf{fixed})}\right]_{m_u=m_d=0} \sim \frac{m_\pi^2}{(m_u+m_d)(\mu\ \textbf{fixed})}$$

with the following specifications . We choose a reference mean u,d quark mass of 6 ± 1 **MeV**

(80) $$\widehat{m}(\mu_6^*) = 6\left(1 \pm \tfrac{1}{6}\right)\ \textbf{MeV} \ \leftarrow\ \widehat{m} = \tfrac{1}{2}(m_u+m_d)(\mu)$$

This can be compared with the $\overline{\widehat{m}}$ **at the scale** $\mu = 2$ **GeV in ref. [21-2011]**

$($ **Budapest-Marseille-Wuppertal Collaboration** $)$

(81) $$\overline{\widehat{m}}(2\ \textbf{GeV}) = 3.503(48)(49)\ \textbf{MeV}$$

Substituting eqs. 80 and 77 in eq. 76 we obtain $\rightarrow$

$$f^{2}(0)\left[\frac{m^{2}_{\pi^{0}}}{2\,\widehat{\overline{m}}(\mu^{*}_{6})}\right] \sim -\langle\Omega|\,(\overline{q}\,q)_{u\leftrightarrow d}\,(\mu^{*}_{6})\,|\Omega\rangle\Big|_{chir.lim.} \longrightarrow$$

---

(82)
$$f^{2}(0)\left[\frac{m^{2}_{\pi^{0}}}{2\,\widehat{\overline{m}}(\mu^{*}_{6})}\right] = 1.154\,(1\pm 0.02)\left(1\pm\tfrac{1}{6}\right)10^{-2}\,\text{GeV}^{3}$$
$$= 1.154\,(1\pm 0.19)\,10^{-2}\,\text{GeV}^{3}$$

$$f^{2}(0)\,m^{2}_{\pi^{0}} = 1.385\,(1\pm 0.02)\,10^{-4}\,\text{GeV}^{4}$$

For the sake of comparison we can rescale the condensate as defined in eq. 82 from μ_6^* to $\mu = 2$ GeV by the ratio of mean up and down quark masses

(83)
$$\langle\Omega|\,(\overline{q}\,q)_{u\leftrightarrow d}\,(\mu=2\,\text{GeV})\,|\Omega\rangle\Big|_{chir.lim.} =$$
$$= \frac{\widehat{\overline{m}}(\mu^{*}_{6})}{\widehat{\overline{m}}(\mu=2\,\text{GeV})}\,\langle\Omega|\,(\overline{q}\,q)_{u\leftrightarrow d}\,(\mu^{*}_{6})\,|\Omega\rangle\Big|_{chir.lim.}$$

→

This yields

(84) $\langle \Omega | \, (\overline{q} \, q \,)_{u \leftrightarrow d} \, (\mu = 2 \, \textbf{GeV} \,) \, | \Omega \rangle \, |_{chir.lim.} \; = \; - \, 1.979 \; (\, 1 \, \pm \, 0.19 \,) \; 10^{-2} \; \textbf{GeV}^{3}$

The quantity of the $\overline{q} \, q \, ; \, SU2_{fl}$ **chiral limit with reversed sign , relative to eq. 84**

$$- \, \langle \Omega | \, (\overline{q} \, q \,)_{u \leftrightarrow d} \, (\mu = 2 \, \textbf{GeV} \,) \, | \Omega \rangle \, |_{chir.lim.} \; = \; \Sigma \, (\, 2 \, \textbf{GeV} \,)$$

is denoted $\Sigma \, (\, 2 \, \textbf{GeV} \,)$ **in Stephan Dürr et al., op.cit. [20-2012] . The value determined is compared with the result derived here in eq. 85 below**

$$\Sigma \, (\, 2 \, \textbf{GeV} \,) \quad = \quad 2.020 \quad \pm 0.027 \, \pm 0.031 \quad . \, 10^{-2} \; \textbf{GeV}^{3}$$

(85)

$$- \, \langle \Omega | \left\{ \begin{array}{c} (\overline{q} \, q \,)_{u \leftrightarrow d} \\ (\, \mu = 2 \, \textbf{GeV} \,) \\ \textbf{chir. lim.} \end{array} \right\} | \, \Omega \, \rangle \quad = \quad 1.979 \quad \times \, (\, 1 \, \pm \, 0.19 \,) \quad . \, 10^{-2} \; \textbf{GeV}^{3}$$

This concludes main derivations in this subsection, subject to the following remarks →

1) The numerical agreement of the best values

for the negative of the $\overline{q}\, q\,(\,\mu)$ condensate $(\,\equiv\;\Sigma\,(\,\mu\,)\,)$ as displayed in eq. 85 is not due to a hidden 'mimicry' . The errors added in quadrature for ref. [20-2012] are 2 % against almost 20 % for my deivation.

2) The aim of the latter

is (was) to give an indication that the $\overline{q}\, q$ vacuum expected value remains different from zero also in the chiral limit, whence it represents not only a spontaneous parameter, but an intrinsic spontaneous symmetry breaking.

3) In conjunction with the katalytic relation in sbsection 2-1-2a

(86)
$$\begin{aligned} &\lim_{\,m_{\,q\;kat.}(\mu)\to\infty}\, m_{\,q\;kat.}\,(\,\mu\,)\;\langle\,\Omega\,|\;\overline{q}_{\,kat.}\,q_{\,kat.}\,(\,\mu\,)\;|\,\Omega\,\rangle \\ &=\;-\,\frac{1}{12\pi^{\,2}}\;\langle\,\Omega\,|\;\mathcal{B}^{\,+}_{\,\infty}\;|\,\Omega\,\rangle \end{aligned}$$

the proof of a unique sign of the $\overline{q}\, q$ vacuum expected value for all quark mass values brings the gauge boson pair ccondensate into unique focus, yet far from proving its nonvanishing value.

$\rightarrow$

3-1-Fig. 9

Fig. 9 : Filigran (latin filum = thread , granum = grain) fabric of gauge field complexes here supporting the wings of a dragon-fly

→

3-1-2 – attempts to determine $\langle\,\Omega\,|\,\frac{1}{4}\,[\,B^{\,r}_{\mu\nu}\,B^{\,\mu\nu\,r}\,]_{\,\infty}\,|\,\Omega\,\rangle$ from QCD sum rules [9-1979]

Let me start citing here a recent paper and result(s) by Stephan Narison [10-2011, 22-2011] in partcular with respect to the renormalization group invariant setting – in principle – of composite local field normalization

$$\alpha_{\,s}\,G^{\,2}\;=\;\pi^{\,-1}\,\mathcal{X}\;\;;\;\;\mathcal{X}\;\equiv\;\frac{1}{4}\,[\,B^{\,r}_{\mu\nu}\,B^{\,\mu\nu\,r}\,]_{\,\infty}$$

$$\langle\,\Omega\,|\,\alpha_{\,s}\,G^{\,2}\,|\,\Omega\,\rangle\;=\;(\,7.0\;\pm\;1.3\,)\,10^{\,-2}\;\textbf{GeV}^{\,4}$$

(87) $$=\;\pi^{\,-1}\,(\,0.22\;\pm\;0.04\,)\;\textbf{GeV}^{\,4}\qquad[\textbf{10-2011},\;\textbf{22-2011}]$$

$$\langle\,\Omega\,|\,\frac{1}{4}\,[\,B^{\,r}_{\mu\nu}\,B^{\,\mu\nu\,r}\,]_{\,\infty}\,|\,\Omega\,\rangle\;=\;(\,0.22\;\pm\;0.04\,)\;\textbf{GeV}^{\,4}\;=\;\left(\left(0.685\,{}^{+\,0.029}_{-\,0.036}\right)\;\textbf{GeV}\right)^{\,4}$$

The errors in the numerical values reported in ref. [10-2011, 22-2011] do correspond to a central hadronic scale for the gauge boson vacuum expected value driving the central anomaly.

Let us compare the above determination with the recent analysis of Cesareo Dominguez et al. in ref. [8-2011]

(88) $$\langle\,\Omega\,|\,\frac{1}{4}\,[\,B^{\,r}_{\mu\nu}\,B^{\,\mu\nu\,r}\,]_{\,\infty}\,|\,\Omega\,\rangle\;=\;(\,0.48\;\pm\;1.14\,)\;\textbf{GeV}^{\,4}\qquad[\textbf{8-2011}]$$

$\rightarrow$

3-1-11

To clarify conventions I repeat eq. 9 from op.cit. [8-2011] in eq. 89 below

(89) $$\langle \, \Omega \, | \; \frac{1}{4} \; \left[\, B^{\;r}_{\;\mu\nu} \, B^{\,\mu\nu \; r} \, \right]_{\,\infty} \, | \, \Omega \, \rangle \; = \; 3 \, C_{\,4} \, (\, O_{\,4} \,) \; - \; \left(\, 12 \, \pi^{2} \, \right) \, \langle \, \Omega \, | \, \sum_{\,q = u , d , s} \, \overline{q} \, q \; | \, \Omega \, \rangle$$

In evaluating the numerical values given in eq. 88 given the errors I have neglected the contribution of the u,d,s $\overline{q} \, q$ contributions . $\rightarrow$

4 - Concluding remarks and outlook

1) Perturbative accessibility of renormalization in asymptotically free theories :

While the entire renormalization procedure thus comes within *perturbative accessibility* **– as explained in textbooks [23] , [24-1982] – the associated renormalization group equation serves to restore renormalization group invariant properties, in particular such definitions of operators , as well as subtleties of definition and** $\overline{MS}$ **sliding scale dependence of quark masses and** $\overline{q}\ q$ **flavor specific bilinears, including extrapolations to chiral limits.**

2) Infrared instability

is associated with all physical scales *not* **accessible to perturbative approximations .**
The embedding of chiral symmetry depends in a nontrivial way on the strength of the *gauge field strength pair-* **Bose condensate as does the excitation of binary and higher gauge boson compounds ('glueballs')** *and* **the phase structure of QCD . While the above topics have not been directly addressed here , the main objective continues to be to acces on a deeper level the consequences of the trace anomaly associated with the gauge boson field strength bilinear**

(90) $$\frac{1}{4}\ \left[\ B_{\ \mu\nu}^{\ r}\ B^{\ \mu\nu\ r}\ \right]_{\ \infty}\ \equiv\ \mathcal{B}_{\ \infty}^{\ +}$$

and its nonvanishing vacuum expected value.

→

3) Canonical structure and valid estimates for $\langle\, \Omega \,|\, \mathcal{B}^{\,+}_{\,\infty} \,|\, \Omega \,\rangle$

These topics as well as many others – and dicussed by others – have not yet received analytically and/or experimentally satisfactory answers . ”Es bleibt noch viel zu tun” and this is fine .

4) It seems preferable to me, to concentrate on the embedding of quark mass ratiois into QCD , with the two central anomalies and the infrared instability as mentioned in items 1 - 4 above , here , than a rederivation of the steps in our paper with Arnulfo Zepeda ref. [1-1979] .

The result was , including an *estimate* of the theoretical and systematic errors combined

(91) $$m_{\,u} \,:\, m_{\,d} \,:\, m_{\,s} \,=\, 3 \,:\, 5 \,:\, 100 \,\pm\, 15\,\%$$

5) Outlook

The completion of QCD as a gauge field theory in uncurved space time remains a far goal .

Along the way let us keep in mind that (for all we know) the physical reality transcends much further, in particular to curved space time of unknown dimensionality .

What I could do to achieve some of the goals set , I tried to show you here.

— Thank you —

Appendix 1 - expansion coefficients of the rescaling functions $\widehat{\beta}, \gamma$ to four loops

$$-\beta/g = X\,B(X) \;;\; B(X) = b_0\,A(X)$$

$$B(X) \sim \sum_{n=0}^{\infty} b_n X^n \;,\; A(X) \sim \sum_{n=0}^{\infty} a_n X^n$$

$\kappa = g^2/(16\pi^2)$ **generic** X

(92)

$$b_0 = \frac{1}{3}\left(33 - 2N_{fl}\right) \;,\; a_0 = 1 \;,\; a_n = b_n/b_0$$

$$b_1 = \frac{2}{3}\left(153 - 19N_{fl}\right)$$

$$b_2 = \frac{1}{54}\left(77139 - 15099\,N_{fl} + 325\,N_{fl}^2\right)$$

$$b_3 \sim 29243 - 6946.3\,N_{fl} + 405.089\,N_{fl}^2 + 1.49931\,N_{fl}^3$$

→

$$-\gamma_m^0 = 4\,,\ -\gamma_m^1 = \frac{202}{3} - n_{fl}\frac{20}{9} \qquad \Big| \; -\gamma_m^l \equiv \chi_m^l$$

$$-\gamma_m^2 = 1249 - \left[\frac{2216}{27} + \frac{160}{3}\zeta(3)\right] N_{fl} - \frac{140}{81} N_{fl}^2$$

$$-\gamma_m^3 = \left\{ \begin{array}{l} \frac{4603055}{162} + \frac{135680}{27}\zeta(3) - 8800\,\zeta(5) + \\ + \left[-\frac{91723}{27} - \frac{34192}{9}\zeta(3) + 880\,\zeta(4) + \frac{18400}{9}\zeta(5) \right] N_{fl} + \\ + \left[\frac{5242}{243} + \frac{800}{9}\zeta(3) - \frac{160}{3}\zeta(4) \right] N_{fl}^2 + \\ + \left[-\frac{332}{243} + \frac{64}{27}\zeta(3) \right] N_{fl}^3 \end{array} \right\} \tag{93}$$

→

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