Gravitational and chargelike gauges

Peter Minkowski

Albert Einstein Center for Fundamental Physics - ITP, University of Bern

Topics taken from a talk at the Conference on Cosmology and Elementary Particle Physics

15.-19. December 2004, Coral Gables and Miami, Florida, USA

Conference on Cosmology, Gravitational Waves and Particles

6.-10. February 2017

Nanyang Executive Center, Nanyang Technological University, Singapore

On the apparent likeness of local gauges and their underlying physics

'natures way ... our way'

Peter Minkowski

Institute for Theoretical Physics, University of Bern

adapted and extended from : Miami, December 2005

$\longrightarrow$ **Beijing , September 2007**

Topics :

1) gauging orientation on a differentiable manifold

2) gauging a Lie tranformation group on a distinct fibre

3) Questions, conclusions and outlook

Extensions $t \geq$ 22.12.2005 :

e1) vierbeins on B , vielbeins on E and spin connections

e2) structure relations between vier(l)bein and Christoffel connections

e3) bridging remarks – concluding naturesway2005 as 1. part → naturesway2006...

natures way ... our way

Appendix 1 : minimal metric connections

Appendix 2 : Weyl transformations , an example of nonminimal symmetric Christoffel connection

natures way ... our way

1) gauging orientation on a differentiable manifold

(classical configurations)

We consider parallel transport of a (contravariant) vector v^{ϱ}

$$\delta_{\parallel} v^{\varrho} = - dx^{\kappa} \left(\Gamma_{\kappa} \right)^{\varrho}_{\ \sigma} (x) \, v^{\sigma}$$
$$\left(\Gamma^{(1)} = dx^{\kappa} \, \Gamma_{\kappa} \right)^{\varrho}_{\ \sigma} \quad (1)$$

: **matrix valued 1-form**

and along a curve C from x to y , giving rise to the (curve associated) parallel transport matrix , denoted $T (y \overset{C}{\leftarrow} x)$ [a] $\rightarrow$

---

[a] **Some still original works go back to Wolfgang Pauli [1] and Élie Cartan [2] - [3] .**

natures way ... our way

$$\left\{ T\,(\,y \overset{C}{\leftarrow} x\,) = P\ \exp - \int_x^y \Gamma^{(1)} \right\}^{\varrho}_{\ \sigma}$$

$$v_{\parallel}\,(\,y \overset{C}{\leftarrow} x\,) = T\,(\,y \overset{C}{\leftarrow} x\,)\,v \tag{2}$$

matrix notation

In eq. (2) P denotes ordering from left (further along) to right along the path C .

Now we imagine the same parallel transport done using other local coordinates

$$x^{\prime\,\varrho} = x^{\prime\,\varrho}\,(\,x\,) \longrightarrow \left\{ M^{\varrho}_{\ \sigma} = \partial_{\sigma}\, x^{\prime\,\varrho} \right\}(\,x\,) \tag{3}$$

natures way ... our way

Eq. (2) takes the (trans-) form

$$\left\{ T'(y' \overset{C}{\leftarrow} x') = P \exp - \int_{x'}^{y'} \Gamma'^{(1)} \right\}^{\varrho}_{\sigma}$$

$$v'_{\parallel}(y' \overset{C}{\leftarrow} x') = T'(y' \overset{C}{\leftarrow} x')\, v'$$

$$v'_{\parallel} = M(y)\, v_{\parallel}\,, \quad v' = M(x)\, v \tag{4}$$

and substituting one system relative to the other

$$M(y)\, T(y \overset{C}{\leftarrow} x)\, v = T'(y' \overset{C}{\leftarrow} x')\, M(x)\, v \qquad \longrightarrow \tag{5}$$

natures way ... our way

$$T'(y' \stackrel{C}{\leftarrow} x') = M(y)\,T(y \stackrel{C}{\leftarrow} x)\,(M(x))^{-1} \tag{6}$$

In eqs. (2 - 6)

$$\{ M(z) \mid \forall z \} \tag{7}$$

forms the family of local transformations , gauging orientation [a] **.**

---

[a] **They form the group** $GL(d, R)$**, where d is the (real) dimension of the manifold.**

natures way ... our way

The role of the entire set of parallel transport matrices $T\,(\,y \overset{C}{\leftarrow} x\,)$ is clear and perfectly covariant, while the local connection $\Gamma^{\,(1)}$ transforms inhomogeneously.

The parallel transported vectors along the path C, using a path parameter s

$$C \;:\; \{\, z\,(\,s\,)\,|\,z\,(1)\,=\,y\,;\,z\,(0)\,=\,x\,\} \tag{8}$$

satisfy the differential equation ($\dot{}\,=\,d\,/\,d\,s$)

$$\begin{aligned} \dot{v}\,(\,s\,) &= -\,\dot{z}^{\,k}\,(\,s\,)\,\Gamma_{\,k}\,(\,s\,)\,v\,(\,s\,) \\ v\,(\,s\,) &= T\,(\,z\,(\,s\,) \overset{C}{\leftarrow} x\,)\,v \end{aligned} \tag{9}$$

natures way ... our way

Comparing with the coordinate transformed equation and using $M(s) = M[z(s)]$

$$\dot{v}(s) = -\dot{z}^{k}(s)\,\Gamma_{k}(s)\,v(s)$$
$$\dot{v}'(s) = -\dot{z}'^{k}(s)\,\Gamma'_{k}(s)\,v'(s) \tag{10}$$
$$\begin{pmatrix} v'(s) \\ \dot{z}'(s) \end{pmatrix} = M(s)\begin{pmatrix} v(s) \\ \dot{z}(s) \end{pmatrix}$$

The second relation in eq. (10) thus becomes

$$M\dot{v} + \dot{M}v = -\dot{z}^{k} M^{r}{}_{k}\,\Gamma'_{r} M v \tag{11}$$

and substituting the first on →

natures way ... our way

$$\dot{z}^{k}\,\Gamma_{k}\,v \;=\; \dot{z}^{k}\,M^{r}{}_{k}\,M^{-1}\,\Gamma'_{r}\,M\,v \;+\; M^{-1}\,\dot{M}\,v$$

$$\dot{M} \;=\; \dot{z}^{k}\,\partial_{z^{k}}\,M \qquad\qquad \longrightarrow$$

$$M^{r}{}_{k}\,\Gamma'_{r} \;=\; M\,\Gamma_{k}\,M^{-1} \;+\; M\,\partial_{z^{k}}\,M^{-1}$$

$$\Gamma'_{r} \;=\{\, M\,\Gamma_{k}\,M^{-1} \;+\; M\,\partial_{z^{k}}\,M^{-1}\,\}\;(M^{-1})^{k}{}_{r} \tag{12}$$

From eq. (12) the transformation

natures way ... our way

of the one-form $\Gamma^{(1)} = d\,x^{\kappa}\,\Gamma_{\kappa}$ **(eq. 1) follows**

$$\Gamma'^{(1)} = d\,x'^{\kappa}\,\Gamma'_{\kappa}$$

$$\Gamma'^{(1)} = M\,\Gamma^{(1)}\,M^{-1} + M\,d\,M^{-1} \tag{13}$$

$$d\,F = d\,x^{\kappa}\,\partial_{x^{\kappa}}\,F \; ; \quad F \; : \; \textbf{matrix valued}$$

Torsion ... and ... , is it relevant ?

It shall remain relevant, until proven otherwise.

We proceed noting the one special feature of the connection transformation (eq. 12) ,

natures way ... our way

written in full , upon using

$$M \, d \, M^{-1} = - (d \, M) \, M^{-1}$$

$$\Gamma^{\ \ \ u'}_{r',\ \ t'} =$$

$$= \left[\begin{array}{c} M^{u'}_{\ u} \, \Gamma^{\ u}_{r \ \ t} \, (M^{-1})^{r}_{\ r'} \, (M^{-1})^{t}_{\ t'} + \\ + I^{\ \ \ u'}_{r',\ \ t'} \end{array} \right]$$

$$I^{\ \ \ u'}_{r',\ \ t'} =$$

$$= - (\partial_{r} M^{u'}_{\ u}) (M^{-1})^{r}_{\ r'} (M^{-1})^{u}_{\ t'}$$

$$\partial_{r} M^{u'}_{\ u} = \partial_{r} \partial_{u} x'^{u'} (x) = \partial_{u} M^{u'}_{\ r} \tag{14}$$

$\longrightarrow$

natures way ... our way

It follows that the inhomogeneous orientation gauging part is symmetric

$$\Gamma^{\;\;\prime}_{r^\prime}{}^{u^\prime}{}_{t^\prime} =$$

$$= \left[M^{u^\prime}_{\;\;u} \Gamma_{r}{}^{u}{}_{t} \left(M^{-1} \right)^{r}{}_{r^\prime} \left(M^{-1} \right)^{t}{}_{t^\prime} + + I^{\;\;\prime}_{r^\prime}{}^{u^\prime}{}_{t^\prime} \right]$$

$$I^{\;\;\prime}_{r^\prime}{}^{u^\prime}{}_{t^\prime} = I^{\;\;\prime}_{t^\prime}{}^{u^\prime}{}_{r^\prime} \tag{15}$$

Three things emerge →

natures way ... our way

a) the antisymmetric part of the connection defines a 3-tensor $T_{[\,r}{}^{u}{}_{t\,]}$: torsion $\surd$

$$T_{[\,r}{}^{u}{}_{t\,]} = \frac{1}{2}\left(\Gamma_{r}{}^{u}{}_{t} - \Gamma_{t}{}^{u}{}_{r} \right) \qquad (16)$$

b) it does *not* follow, that the symmetric part derives from a metric. $\longrightarrow$

natures way ... our way

c) a symmetric metric yields a symmetric Riemannian (minimal) connection

$$\overset{o}{\Gamma}{}_{\{ r \; \; t \}}^{\; \; u} =$$

$$= \frac{1}{2} g^{\, u \, v} \left[\partial_{\, r} g_{\, v \, t} + \partial_{\, t} g_{\, v \, r} - \partial_{\, v} g_{\, r \, t} \right]$$

$$\gamma_{\{ r}{}^{u}{}_{t \}} = \frac{1}{2} \left(\Gamma_{r}{}^{u}{}_{t} + \Gamma_{t}{}^{u}{}_{r} \right) - \overset{o}{\Gamma}{}_{\{ r \; \; t \}}^{\; \; u} \tag{17}$$

$\gamma_{\{ r}{}^{u}{}_{t \}}$ **defined in eq. (17) – if not vanishing – defines a symmetric 3-tensor, in addition to torsion .**

$\longrightarrow$

natures way ... our way

Notwithstanding the eventual presence of 3-tensors $T_{[\,r}{}^{u}{}_{t\,]}$ *and* $\gamma_{\{\,r}{}^{u}{}_{t\,\}}$ **the general 1-form , defined in eq. (1) with transformation properties given in eq. (13) (repeated below for clarity)**

$$\begin{aligned} &\left(\Gamma^{(1)} = d\,x^{\kappa}\,\Gamma_{\kappa} \right)^{\varrho}{}_{\sigma} \quad : \textbf{matrix valued 1-form} \\ &\Gamma^{\prime\,(1)} = d\,x^{\prime\,\kappa}\,\Gamma^{\prime}_{\kappa} \\ &\Gamma^{\prime\,(1)} = M\,\Gamma^{(1)}\,M^{-1} + M\,d\,M^{-1} \end{aligned} \tag{18}$$

generate a matrix valued 1-form curvature 2-form , $\rightarrow$

natures way ... our way

a 4-tensor

$$R^{(2)} = d\,\Gamma^{(1)} + \left(\Gamma^{(1)}\right)^2 \rightarrow \frac{1}{2}\left(R_{[\sigma\ \tau]}\right)^u{}_v\, d\,x^\sigma \wedge d\,x^\tau \tag{19}$$

as follows from the transformation properties (eq. 18) [a]

$$R'^{(2)} = M\,R^{(2)}\,M^{-1} \tag{20}$$

---

[a] **... well known yet remarkable ...** ↔

natures way ... our way

$$R'^{(2)} = \left[\begin{array}{lr} M \left(d\,\Gamma^{(1)} \right) M^{-1} & 1 \\ + (d\,M)\, M^{-1} M\, \Gamma^{(1)} M^{-1} & 2 \\ - \left(M\, \Gamma^{(1)} M^{-1} \right) M\, d\, M^{-1} & 3 \\ + (d\,M)\, d\, M^{-1} & 4 \\ + \left(M\, \Gamma^{(1)} M^{-1} \right) M\, d\, M^{-1} & 5 \\ + M\, (d\,M^{-1})\; M\, \Gamma^{(1)} M^{-1} & 6 \\ + (M\, d\, M^{-1})\, (M\, d\, M^{-1}) & 7 \\ + M \left(\Gamma^{(1)} \right)^2 M^{-1} & 8 \end{array} \right] \tag{21}$$

[a]

←

[a] **The red rows cancel .**

natures way ... our way

2) gauging a Lie tranformation group on a distinct fibre

In section 1) we did not introduce a special name for the manifold considered. Meanwhile the notation of fibre bundles distinguishes a base manifold B and a fibre F , combining their direct product to an extended manifold E

$$\begin{array}{l} (\ B \ , \ dim \ d_{\ B} \ ; \ F \ , \ dim \ d_{\ F} \ ) \ \rightarrow \\ E \ (\ B \ ; \ F \ , \ dim \ d_{\ B} \ + \ d_{\ F} \ ) \ \sim \ B \ \times F \end{array} \tag{22}$$

The fibre F shall be a homogeneous space :

natures way ... our way

right coset $F = G/H$ of a Lie group G modulo a Lie subgroup H , [a]

---

[a] **In the talk I gave an analogy of the fibre space a clearer word for this 'space' emphasizing its unresolved spatial extension – at present – could be Fibre → Filiput with the powder method of Debye and Scherrer [4] for the study of crystalline structure. Therein the property of the basic powder crystals to be 'invisible' directly is the common ingredient.**
Upon reflection on the above point I may remark that Filiput-space may well have discrete properties, beyond the differentiable manifold structure, conventionally imposed by the strict mathematical definition of fibre-space .

with transformations $a \in G$

$$a : f \rightarrow a \,.\, f \; ; \; f = \Big(f^{\,k} \; ; \; k = 1 \cdots dim\ F \Big)$$

$$a = (a^{\,\varrho} \; ; \; \varrho = 1 \cdots dim\ G)$$

$$(a \,.\, f)^{\,k} = \Omega^{\,k} (a \; ; \; f) \tag{23}$$

The Killing fields correspond to the infinitesimal transformations

$$h^{\,k}{}_{\varrho} (f) = \partial_{\,b\,\varrho}\, \Omega^{\,k} (b \; ; \; f) \Big|_{\,b\,=\,0} \tag{24}$$

The transformation $a : f \rightarrow a \,.\, f$ **on F allows to associate** $a \rightarrow Ad (a)$ **,**

natures way ... our way

where $Ad(a)$ **denotes the adjoint (**$dim\ G \times dim\ G$**) representation of G, through the relation**

$$h^k{}_\varrho(a.f) =$$

$$= \psi^k{}_l(a;f)\, h^l{}_\sigma(f)\, (Ad(a^{-1}))^\sigma{}_\varrho \tag{25}$$

$$\psi^k{}_l(a;f) = \partial_{f^l}\, \Omega^k(a;f)$$

$\psi^k{}_l(a;f)$ **defined in eq. (25) is the Jacobian of the coordinate transformation in F**

$$a : f \rightarrow a.f \tag{26}$$

The group property follows from the matrix form of eq. (25) $\longrightarrow$

natures way ... our way

$$h(a.f) = \psi(a;f)\,h(f)\,Ad(a^{-1})$$

$$h(b.a.f) =$$

$$= \left[\begin{array}{c} \psi(b;a.f)\,\psi(a;f)\,h(f) \times \\ \times\, Ad(a^{-1})\,Ad(b^{-1}) \end{array} \right] \tag{27}$$

$$= \psi(b.a;f)\,h(f)\,Ad((b.a)^{-1})$$

$$Ad(a^{-1})\,Ad(b^{-1}) = Ad((b.a)^{-1})$$

$$\psi(b;a.f)\,\psi(a;f) = \psi(b.a;f)$$

natures way ... our way

projecting on the adjoint connection

The base space B shall be described by coordinates

$$B \ : \ (\ x^{\ \mu} \ ; \ \mu \ = \ 1 \ \cdots \ dim \ B \ ) \tag{28}$$

Now we consider x-dependent group transformations from G on $F \ = \ G \ / \ H$

$$a \ \longrightarrow \ a \ (\ x \ ) \ : \ f \ \rightarrow \ a \ (\ x \ ) \ . \ f \tag{29}$$

As a consequence of eqs. (25 - 27) we project on the (adjoint Lie algebra-) **matrix valued** **connection on the base space B** $\longrightarrow$

natures way ... our way

$$(\mathcal{W}_{\mu})^{\sigma}{}_{\varrho} = W_{\mu}^{\kappa} (ad_{\kappa})^{\sigma}{}_{\varrho}$$

$$(ad_{\kappa})^{\sigma}{}_{\varrho} = f_{\sigma \kappa \varrho} \longrightarrow$$

$$[ad_{\alpha} , ad_{\beta}] = f_{\alpha \beta \gamma} \, ad_{\gamma} \qquad (30)$$

$$\mathcal{W}_{\mu} = \mathcal{W}_{\mu} (x) \leftrightarrow \Gamma_{\kappa}$$

$$\mathcal{W}^{(1)} = d x^{\mu} \mathcal{W}_{\mu} \leftrightarrow \Gamma^{(1)} = d x^{\kappa} \Gamma_{\kappa}$$

in clear analogy or relation with eqs. (1 - 2) .

Thus we proceed to construct the parallel transports as in eq. (2) $\longrightarrow$

natures way ... our way

$$\left\{ U \left(y \overset{C}{\longleftarrow} x \right) = P \exp - \int_x^y \mathcal{W}^{(1)} \right\}_\sigma^\varrho$$

$$\left\{ T \left(y \overset{C}{\longleftarrow} x \right) = P \exp - \int_x^y \Gamma^{(1)} \right\}_\sigma^\varrho \tag{31}$$

The analog of the orientation gauge transformation in eq. (6) corresponds for U to the local coordinate transformation on the fibre F , beyond B $\longrightarrow$

natures way ... our way

$$f \rightarrow a(z).f \rightarrow$$

$$U'(y \overset{C}{\leftarrow} x) =$$

$$U(y)\,U'(y \overset{C}{\leftarrow} x)\,U^{-1}(x)$$

$$U(z) = Ad(a(z))$$

$$T'(y' \overset{C}{\leftarrow} x') =$$

$$M(y)\,T(y \overset{C}{\leftarrow} x)\,(M(x))^{-1}$$

$$M(z) = \partial_z z' \tag{32}$$

natures way ... our way

Finally we compare the gauge transformations on local 1-forms (eq. 13)

$$\begin{aligned}
\mathcal{W}^{\prime\,(1)} &= U\,\mathcal{W}^{(1)}\,U^{-1} + U\,d\,U^{-1} \\
U(z) &= Ad(a(z)) \\
\Gamma^{\prime\,(1)} &= M\,\Gamma^{(1)}\,M^{-1} + M\,d\,M^{-1} \\
M(z) &= \partial_z\,z^{\prime}
\end{aligned} \tag{33}$$

and the curvature 2-form (eqs. 18 - 19)

$$\begin{aligned}
\mathcal{F}^{(2)} &= d\,\mathcal{W}^{(1)} + \left(\mathcal{W}^{(1)}\right)^2 \\
R^{(2)} &= d\,\Gamma^{(1)} + \left(\Gamma^{(1)}\right)^2
\end{aligned} \tag{34}$$

natures way ... our way

as well as the covariant transformation rules (eq. 20)

$$\begin{aligned} \mathcal{F}^{\,\prime\,(2)} &= U\,\mathcal{F}^{(2)}\,U^{-1} \\ R^{\,\prime\,(2)} &= M\,R^{(2)}\,M^{-1} \end{aligned} \tag{35}$$

natures way ... our way

3) Questions, conclusions and outlook

1) **Can the charge like local gauge structure be obtained as a reduction of the global (E-) extended coordinate transformation gauging ? Indeed** [a]

2) **Is the extension of general orientation gauging at the origin of the apparent similarities, or are these fortuitous ? I think not, but ...**

$\longrightarrow$

[a] $F = S_1$; $G = U1$ **: Kaluza and Klein (1921 , 1926) ,** $F = S_2$; $G = SU2$ **: Pauli (1953) , P. M. general** $F = G / H$ **(1977) , ...,** $B = R^4$ **or general [5] .**

natures way ... our way

3) There are definite problems with curved extra dimension , as $F = G / H$ can hardly prevent (spin 1/2) fermions to inherit heavy masses .

4) I hope that some of the ideas presented here , will despite obviously serious string theory solutions proposed, (may) lead to a broader understanding of gravity in more than four dimensions,as well as shed light on intermediary extension of chargelike gauges like SO10 .

Thank you .

natures way ... our way

e1) vierbeins on B , vielbeins on E and spin connections (standard)

Lets – first – restrict considerations to the base space B.

To guarantee causal conditions compatible with local Lorentz invariance we consider an indefinite metric of the restricted form

$$g_{\mu\nu} = e^{a}{}_{\mu}\, \eta_{ab}\, e^{b}{}_{\nu} \ ; \ det\ e > 0$$

$$\eta_{ab} = diag\ (1 , -1 , \cdots , -1)$$

$$\mu , \nu , a , b = 0 , 1 , \cdots , dim\ B - 1 \ (= 3) \tag{36}$$

natures way ... our way

To the covariant vierbein $e^{a}{}_{\mu}$ defined in eq. (36) there shall exist an inverse, the contravariant one

$$e^{\nu}{}_{b} \equiv \zeta^{\nu}{}_{b} \; ; \; \left[\begin{array}{ccc} e^{a}{}_{\mu}\, e^{\mu}{}_{b} & = & \delta^{a}{}_{b} \\ e^{\nu}{}_{a}\, e^{a}{}_{\mu} & = & \delta^{\nu}{}_{\mu} \end{array} \right] \tag{37}$$

By means of $e^{a}{}_{\mu}$ we can systematically assign to a (contravariant) vector and all extended tensors a tangent space equivalent. Choosing a pair of vectors

$$\left(\begin{array}{c} v^{\mu} \\ w^{\mu} \end{array} \right) \rightarrow \left(\begin{array}{ccc} V^{a} & = & e^{a}{}_{\mu}\, v^{\mu} \\ W^{a} & = & e^{a}{}_{\mu}\, v^{\mu} \end{array} \right) \tag{38}$$

natures way ... our way

we can express the metric scalar product in both bases

$$v\ .\ w\ =\ v^{\mu}\, g_{\mu\nu}\, w^{\nu} \quad \leftrightarrow \quad V \odot W\ =\ V^{a}\, \eta_{ab}\, W^{b}$$
$$v\ .\ w\ =\ V \odot W \tag{39}$$

We go back to the connection defined in eq. (1) repeated below

natures way ... our way

and the associated covariant derivative extendable to all tensors with generally mixed contra- and co-variant indices

$$\delta_{\parallel} v^{\varrho} = - d x^{\kappa} (\Gamma_{\kappa})^{\varrho}{}_{\sigma} (x) v^{\sigma} \quad \rightarrow$$

$$D_{\kappa} v^{\varrho} = (D v)_{\kappa}{}^{\varrho} = \partial_{\kappa} v^{\varrho} + (\Gamma_{\kappa})^{\varrho}{}_{\sigma} v^{\sigma} \tag{40}$$

The quantity $(D v)_{\kappa}{}^{\varrho}$ defined in eq. (40) is endowed with one contravariant index $^{\varrho}$ and one covariant one $_{\kappa}$, obeying the transformation rules [under invertible *diffeomorphisms* **(eqs. 3 , 10 - 12)]** $\rightarrow$

natures way ... our way

$$(Dv)'_{\kappa'}{}^{\varrho'} = M^{\varrho'}{}_{\varrho} (Dv)_{\kappa}{}^{\varrho} (M^{-1})^{\kappa}{}_{\kappa'}$$

$$M^{\varrho'}{}_{\varrho} = \partial_{x^{\varrho}} x'^{\varrho'} , \quad (M^{-1})^{\kappa}{}_{\kappa'} = \partial_{x'^{\varrho'}} x^{\varrho} \tag{41}$$

The chain rule fixes the covariant derivative acting on a covariant vector field $u_{\sigma}(x)$

$$D_{\kappa} u_{\sigma} = (Dv)_{\kappa \sigma} = \partial_{\kappa} u_{\sigma} - u_{\tau} (\Gamma_{\kappa})^{\tau}{}_{\sigma}$$

$$D_{\kappa} (u_{\sigma} v^{\sigma}) = \partial_{\kappa} (u_{\sigma} v^{\sigma}) \tag{42}$$

natures way ... our way

the (bosonic) spin connection

Eqs. (37 - 42) determine uniquely the *bosonic* spin connection from the defining connection $(\Gamma_{\kappa})^{\tau}{}_{\sigma}$

$$D_{\kappa}^{(\omega)} V^{a} = \Big(D^{(\omega)} V \Big)_{\kappa}^{a} =$$

$$= \partial_{\kappa} V^{a} + (\omega_{\kappa})^{a}{}_{b} V^{b}$$

using partial matrix notation :

$$D_{\kappa}^{(\omega)} V = \partial_{\kappa} V + \omega_{\kappa} V \; ; \; V = e \, v \; , \; v = \zeta \, V$$

$$(e)^{a}{}_{\varrho} = e^{a}{}_{\varrho} \; , \; (\zeta)^{\sigma}{}_{b} = (e^{-1})^{\sigma}{}_{b} = e^{\sigma}{}_{b} \tag{43}$$

requiring $\longrightarrow$

natures way ... our way

compatibility of tangent-space and tensorial quantities in eqs. (41 and 43) .

The former (eq. 41) is written in matrix form to simplify identification

$$D_{\kappa}\, v \;=\; \partial_{\kappa}\, v \;+\; \Gamma_{\kappa}\, v \rightarrow = \;\zeta\; D_{\kappa}^{(\omega)}\; V \qquad (44)$$

which implies

$$\begin{aligned} e\; D_{\kappa}\, v \;&=\; e\,\partial_{\kappa}\;(\;\zeta\, V\;)\;+\;e\,\Gamma_{\kappa}\,\zeta\, V\;= \\ &=\;\partial_{\kappa}\, V\;+\;(\;e\,\partial_{\kappa}\,\zeta\;)\;V\;+\;e\,\Gamma_{\kappa}\,\zeta\, V\;= \\ &=\;\partial_{\kappa}\, V\;+\;\omega_{\kappa}\, V \qquad\qquad \rightarrow \end{aligned} \qquad (45)$$

natures way ... our way

The sought identification thus becomes [a]

$$\omega_{\kappa} = e\,\Gamma_{\kappa}\,e^{-1} + e\,\partial_{\kappa}\,e^{-1}$$

$$\omega_{\kappa} = \omega_{\kappa}(\Gamma_{\kappa};\,e)$$

with the associated 1-forms :

$$\omega^{(1)} = d\,x^{\kappa}\,\omega_{\kappa}\;,\;\Gamma^{(1)} = d\,x^{\kappa}\,\Gamma_{\kappa} \qquad \rightarrow$$

$$\omega^{(1)} = e\,\Gamma^{(1)}\,e^{-1} + e\,d\,e^{-1} \tag{46}$$

[a] **We note that despite the availability of a metric – which defines the vier(l)bein modulo a Lorentz transformation – by no means the associated connections** ω_{k} **and** Γ_{k} **are related to the metric connection** $\overset{o}{\Gamma}_{\kappa}$ **in eq. (17)** – *in general* **.**

natures way ... our way

After the tentative theme, discussed in ref. [5] , it appears to me at this point, that the full 'tensorial redundancy' of the connection $\Gamma\ {}_{r}{}^{u}\ {}_{t}$ displayed in eq. (17) is at the heart of the short distance asymptotic freedom of dimensionally extended gravity and gauging of extended orientation

– to be defined in a consistent way –

An interesting variation of these ideas has been presented by Lee Smolin in 1979 [6] .

For the time beeing we concentrate on the minimal (metric) forms of connections $\longrightarrow$

natures way ... our way

minimal metric connections – gauge fixing due to coordinate transformations

We go back to eqs. (3 , 13 , 14) , for the Christoffel connection $\Gamma \, {}_{r}{}^{u} \, {}_{t}$, reproduced below

$$\begin{aligned}
\Gamma^{\,\prime \, (1)} &= d \, x^{\,\prime \, \kappa} \, \Gamma^{\,\prime}_{\,\kappa} \\
\Gamma^{\,\prime \, (1)} &= M \, \Gamma^{\,(1)} \, M^{\,-1} + M \, d \, M^{\,-1} \\
x^{\,\prime \, \varrho} &= x^{\,\prime \, \varrho} \, (\, x \,) \, \rightarrow \, \left\{ \, M^{\,\varrho}_{\,\sigma} = \partial_{\,\sigma} \, x^{\,\prime \, \varrho} \, \right\} \, (\, x \,)
\end{aligned} \tag{47}$$

The structure of the local transformation associated with the coordinate transformation

$x^{\,\prime} = x^{\,\prime} \, (\, x \,)$ $\longrightarrow$

natures way ... our way

seems **to identify the connection one-form** $\Gamma^{(1)}$ **in eqs. (13 , 47) as structure 1-form relative to the local transformation group**

$$\{ \ \ell \ \} \ = \ GL \ (\ dim \ B \ ) \ ; \ det \ \ell \ \neq \ 0 \tag{48}$$

Yet the constraint inherited from (differentiable) coordinate transformations

$$x^{\prime \varrho} = x^{\prime \varrho} (x) \ \rightarrow \ \left\{ M^{\varrho}_{\ \sigma} = \partial_{\sigma} x^{\prime \varrho} \right\} (x)$$
$$\rightarrow \quad \partial_{\tau} M^{\varrho}_{\ \sigma} = \partial_{\sigma} M^{\varrho}_{\ \tau} \tag{49}$$

does *not* **define a local subgroup of** $\{ \ \ell \ \}$ **(eq. 48) .**

natures way ... our way

This mismatch has to do with a distinction in coordinate tranformations between displacement and local reorientation , the latter keeping a given point – x say – invariant. [a]

The linear representation of the coordinate transformation group is a nonlocal one, as is shown by the last relation in eq. (27) , which we make explicit below, for clarity

$$T_1 \; : \; x' \leftarrow x \; ; \; T_2 \; : \; x'' \leftarrow x' \rightarrow \tag{50}$$

---

[a] **The above subtlety, which is obvious for the Euclidean motion group, in a given Euclidian space E^n or the Poincaré group in $\mathcal{M}^{4 \, (\rightarrow \, n)}$, appears hidden in various parts (tensor- and connection parts) here .**

natures way ... our way

$$T_{2\,1} = T_2 \,.\, T_1 \; : \; x'' \leftarrow x$$

$$\left[\, M_1 (x' , x) \,\right]^{\varrho_1}_{\;\sigma_1} = \partial_{x^{\sigma_1}} x'^{\varrho_1}$$

$$\left[\, M_2 (x'' , x') \,\right]^{\varrho_2}_{\;\sigma_2} = \partial_{x'^{\sigma_2}} x''^{\varrho_2}$$

$$\left[\, M_{2\,1} (x'' , x) \,\right]^{\varrho_3}_{\;\sigma_3} = \partial_{x^{\sigma_3}} x''^{\varrho_3} \qquad \longrightarrow$$

$$M_{21} (x'' , x) = M_2 (x'' , x') \,.\, M_1 (x' , x) \qquad (51)$$

in matrix notation

natures way ... our way

We report the first result on the spin connection (eqs. 83 and 81 from appendix 1) below

$$\overset{o}{\omega}_{\kappa}{}^{a\,b} = \eta^{bc}\,\overset{o}{\omega}_{\kappa}{}^{a}{}_{c} = -\,\overset{o}{\omega}_{\kappa}{}^{b\,a} =$$

$$= -\,\frac{1}{2}\,\Big(\,\tilde{\zeta}^{\varrho\,a}\,\partial_{\varrho}\,e^{b}{}_{\kappa} - \tilde{\zeta}^{\varrho\,b}\,\partial_{\varrho}\,e^{a}{}_{\kappa}\Big)$$

$$-\,\frac{1}{2}\,\tilde{\zeta}^{\varrho\,a}\,\tilde{e}_{d\,\kappa}\,[\,(\,\partial_{\varrho}\,e\,)\,e^{-1}\,]^{d\,b}$$

$$+\,\frac{1}{2}\,\tilde{\zeta}^{\varrho\,b}\,\tilde{e}_{d\,\kappa}\,[\,(\,\partial_{\varrho}\,e\,)\,e^{-1}\,]^{d\,a}$$

$$+\,\frac{1}{2}\,\Big(\,[\,(\,\partial_{\kappa}\,e\,)\,e^{-1}\,]^{b\,a} - [\,(\,\partial_{\kappa}\,e\,)\,e^{-1}\,]^{a\,b}\Big)$$

$$\tilde{\zeta}^{\varrho\,a} = \eta^{ab}\,\zeta^{\varrho}{}_{b}\;;\;\tilde{e}_{a\,\sigma} = \eta_{ab}\,e^{b}{}_{\kappa} \quad \longrightarrow \tag{52}$$

natures way ... our way

It follows that the spin connection matrices

$$\left(\overset{o}{\omega}_{\ \kappa} \right)^{a}_{\quad b} \tag{53}$$

have values in the Lie algebra of the Lorentz group in 4 (D) dimensions , i.e. satisfy the (matrix-) relation

$$\left(\overset{o}{\omega}_{\ \kappa} \right)^{T} \eta + \eta \left(\overset{o}{\omega}_{\ \kappa} \right) = 0 \ ; \ \forall \ \kappa \tag{54}$$

Thus the parallel transport pertaining to the vector connection in eqs. (2 , 6 and 31) extends to the spin connection in the following way $\longrightarrow$

natures way ... our way

$$\left(\overset{o}{\omega}{}^{(1)} \right)^{a}{}_{b} = dx^{\kappa} \left(\overset{o}{\omega}{}_{\kappa} \right)^{a}{}_{b}$$

$$\left\{ T^{(\omega)} (y \overset{C}{\leftarrow} x) = P \exp - \int_{x}^{y} \omega^{(1)} \right\}^{a}{}_{b}$$

$$T'^{(\omega')} (y' \overset{C}{\leftarrow} x') =$$

$$\Lambda (y) T^{(\omega)} (y \overset{C}{\leftarrow} x) (\Lambda (x))^{-1}$$

for $\omega' , \omega = \overset{o}{\omega}{}' , \overset{o}{\omega}$

[a] $\{ \Lambda (z) \}$: **local Lorentz structure group** (55)

[a] **While the structure group is Lorentzian for any spin connection satisfying eq. (54) and is 'standard' , in view of the preceding discussion this appears surprising.**

natures way ... our way

The structure of the minimal spin connection reduces its dependence on the first derivatives of the vier(l)bein to the field strength-like quantities, for which we introduce the notation (eqs. 93 - 95 in appendix 1)

$$e^{a}{}_{[\sigma\,\tau]} = \partial_{\tau} e^{a}{}_{\sigma} - \partial_{\sigma} e^{a}{}_{\tau} \tag{56}$$

The spin connection then takes the form

$$\overset{o}{\omega}{}_{\kappa}{}^{[a\,b]} = \\ = \frac{1}{2} \left[\begin{array}{c} \widetilde{\zeta}^{\,\varrho\,a} e^{b}{}_{[\varrho\,\kappa]} - \widetilde{\zeta}^{\,\varrho\,b} e^{a}{}_{[\varrho\,\kappa]} \\ + \ \widetilde{e}_{d\,\kappa} \widetilde{\zeta}^{\,\varrho\,a} \widetilde{\zeta}^{\,\sigma\,b} e^{d}{}_{[\varrho\,\sigma]} \end{array} \right] \tag{57}$$

natures way ... our way

We give the spin connections with same index types (eqs. 96 and 97 in appendix 1) .

$$\overset{o}{\omega}{}^{c\,[a\,b]} = \tilde{\zeta}^{\kappa\,c}\,\overset{o}{\omega}{}_{\kappa}{}^{[a\,b]} =$$
$$= \frac{1}{2}\left[\begin{array}{c} \tilde{\zeta}^{\varrho\,a}\,\tilde{\zeta}^{\sigma\,c}\,e^{b}{}_{[\varrho\,\sigma]} - \tilde{\zeta}^{\varrho\,b}\,\tilde{\zeta}^{\sigma\,c}\,e^{a}{}_{[\varrho\,\sigma]} \\ + \tilde{\zeta}^{\varrho\,a}\,\tilde{\zeta}^{\sigma\,b}\,e^{c}{}_{[\varrho\,\sigma]} \end{array}\right] \tag{58}$$

and

$$\overset{o}{\omega}{}_{\tau\,[\varrho\,\sigma]} = \tilde{e}_{a\,\varrho}\,\tilde{e}_{b\,\sigma}\,\overset{o}{\omega}{}_{\tau}{}^{[a\,b]} =$$
$$= \frac{1}{2}\left[\begin{array}{c} \tilde{e}_{a\,\varrho}\,e^{a}{}_{[\tau\,\sigma]} - \tilde{e}_{a\,\sigma}\,e^{a}{}_{[\tau\,\varrho]} \\ + \tilde{e}_{a\,\tau}\,e^{a}{}_{[\varrho\,\sigma]} \end{array}\right] \tag{59}$$

natures way ... our way

I report eq. (98) from appendix 1 as eq. (60)

$$
\begin{aligned}
A \;:&\qquad D_{\kappa}\, \eta_{\,a\,b} \;=\; 0 \\
B \;:&\qquad D_{\kappa}\, e^{\,a}{}_{\varrho} \;=\; 0
\end{aligned}
\tag{60}
$$

The two conditions (A and B) in eq. (60) , serve to resolove the puzzle in the footnote following eq. (55) . →

General spin connections fall into two classes , determined through condition A .

natures way ... our way

If A is satisfied we shall call the associated spin connections Lorentzian, containing the minimal metric one (class L) . The associated local structure group then is $SO(1, D-1)$ **(always restricting the discussion to the case of one time and D - 1 space dimensions) .**

The alternative (class G) corresponds to the larger local structure group $GL(D)$; $det > 0$ **.**

We proceed to consider condition B violated. $\longrightarrow$

natures way ... our way

Then the covariant derivative of the vier(l)bein defines a mixed tensor

$$
\begin{aligned}
& t_{\kappa}{}^{a}{}_{\varrho} \;=\; D_{\kappa}\, e^{a}{}_{\varrho} \;= \\
& =\; \partial_{\kappa}\, e^{a}{}_{\varrho} \;+\; \omega_{\kappa}{}^{a}{}_{c}\, e^{c}{}_{\varrho} \;-\; \Gamma_{\kappa}{}^{\tau}{}_{\varrho}\, e^{a}{}_{\tau} \;\;\Big|\; \widetilde{\zeta}^{\,\varrho\, b} \\
& t_{\kappa}{}^{a\, b} \;=\; \widetilde{\zeta}^{\,\varrho\, b}\, t_{\kappa}{}^{a}{}_{\varrho} \;= \\
& =\; \omega_{\kappa}{}^{[\, a\; b\,]} \;+\; \widetilde{\zeta}^{\,\varrho\, b}\, \partial_{\kappa}\, e^{a}{}_{\varrho} \;-\; \widetilde{\zeta}^{\,\sigma\, a}\, \widetilde{\zeta}^{\,\varrho\, b}\, \Gamma_{\kappa\, \sigma\, \varrho}
\end{aligned}
\tag{61}
$$

Eq. (46) of course holds in all generality, repeated below.

natures way ... our way

e2) structure relations between vier(l)bein and Christoffel connections

$$\omega_{\kappa} = e \, \Gamma_{\kappa} \, e^{-1} + e \, \partial_{\kappa} \, e^{-1}$$

$$\omega_{\kappa} = \omega_{\kappa} (\Gamma_{\kappa} ; e)$$

with the associated 1-forms :

$$\omega^{(1)} = d x^{\kappa} \omega_{\kappa} \; , \; \Gamma^{(1)} = d x^{\kappa} \Gamma_{\kappa} \qquad \rightarrow$$

$$\omega^{(1)} = e \, \Gamma^{(1)} \, e^{-1} + e \, d \, e^{-1} \tag{62}$$

It is maybe here, where reference to two mathematical textbooks is appropriate [7] , [8] .

natures way ... our way

Let me write eq. (62) in *components* **, transforming** $e \; d \; e^{-1} \; = \; - \; (\; d \; e \;) \; e^{-1}$

$$(\; \omega_{\kappa} \;)^{a}{}_{b} \; + \; (\; \partial_{\kappa} \; e^{a}{}_{\tau} \;) \; \zeta^{\tau}{}_{b} \; =$$

$$= \; e^{a}{}_{\sigma} \; (\; \Gamma_{\kappa} \;)^{\sigma}{}_{\tau} \; \zeta^{\tau}{}_{b} \quad | \; e^{b}{}_{\varrho} \; \rightarrow$$

$$\Gamma_{\kappa}{}^{a}{}_{\varrho} \; = \; e^{a}{}_{\tau} \; (\; \Gamma_{\kappa} \;)^{\tau}{}_{\varrho} \; , \; \omega_{\kappa}{}^{a}{}_{\varrho} \; = \; \omega_{\kappa}{}^{a}{}_{b} \; e^{b}{}_{\varrho}$$

$$\omega_{\kappa}{}^{a}{}_{\varrho} \; + \; \partial_{\kappa} \; e^{a}{}_{\varrho} \; = \; \Gamma_{\kappa}{}^{a}{}_{\varrho} \tag{63}$$

The last equation in eq. (63) contains the full set of structural relations between spin- and Christoffel- connections. $\rightarrow$

natures way ... our way

We separate symmetric and antisymmetric parts with respect to the indices $_{\kappa\ \varrho}$

$$\begin{aligned}
\omega_{\{\kappa}{}^{a}{}_{\varrho\}} &= \frac{1}{2}\left(\omega_{\kappa}{}^{a}{}_{\varrho} + \omega_{\varrho}{}^{a}{}_{\kappa}\right) \\
\Gamma_{\{\kappa}{}^{a}{}_{\varrho\}} &= \frac{1}{2}\left(\Gamma_{\kappa}{}^{a}{}_{\varrho} + \Gamma_{\varrho}{}^{a}{}_{\kappa}\right) \\
\partial_{\{\kappa}\, e^{a}{}_{\varrho\}} &= \frac{1}{2}\left(\partial_{\kappa}\, e^{a}{}_{\varrho} + \partial_{\varrho}\, e^{a}{}_{\kappa}\right) \\
\omega_{[\kappa}{}^{a}{}_{\varrho]} &= \frac{1}{2}\left(\omega_{\kappa}{}^{a}{}_{\varrho} - \omega_{\varrho}{}^{a}{}_{\kappa}\right) \qquad (64) \\
T_{[\kappa}{}^{a}{}_{\varrho]} &= \frac{1}{2}\left(\Gamma_{\kappa}{}^{a}{}_{\varrho} - \Gamma_{\varrho}{}^{a}{}_{\kappa}\right) \\
\partial_{[\kappa}\, e^{a}{}_{\varrho]} &= \frac{1}{2}\left(\partial_{\kappa}\, e^{a}{}_{\varrho} - \partial_{\varrho}\, e^{a}{}_{\kappa}\right) \\
&= \frac{1}{2}\, e^{a}{}_{[\varrho\,\kappa]}
\end{aligned}$$

natures way ... our way

The structural relation in eqs. (46 → 63) thus splits into antisymmetric and symmetric parts

$$\begin{array}{l} s \ : \ \omega_{\{ \kappa}{}^{a}{}_{\varrho \}} + \partial_{\{ \kappa} e^{a}{}_{\varrho \}} = \Gamma_{\{ \kappa}{}^{a}{}_{\varrho \}} \\ a \ : \ \omega_{[\kappa}{}^{a}{}_{\varrho]} + \partial_{[\kappa} e^{a}{}_{\varrho]} = T_{[\kappa}{}^{a}{}_{\varrho]} \end{array} \tag{65}$$

In eqs. (64 , 65) $T_{[\kappa}{}^{a}{}_{\varrho]}$ **denotes the mixed components of the torsion 3-tensor.**

The antisymmetric structural relation can be expressed in terms of 2-forms , and the vier(l)bein- and spin connection- 1-forms →

natures way ... our way

$$e^{(1)\ a} = d x^{\kappa} e^{a}{}_{\kappa} \ ; \ \omega^{(1)\ a}{}_{b} = d x^{\kappa} \omega_{\kappa}{}^{a}{}_{b}$$

$$d e^{(1)\ a} = \partial_{[\kappa} e^{a}{}_{\varrho]} \left(d x^{\kappa} \wedge d x^{\varrho} \right)$$

$$\omega^{(1)\ a}{}_{b} e^{(1)\ b} \equiv \omega^{(1)\ a}{}_{b} \wedge e^{(1)\ b} =$$

$$= \omega_{[\kappa}{}^{a}{}_{\varrho]} \left(d x^{\kappa} \wedge d x^{\varrho} \right)$$

$$T^{(2)\ a} \equiv \frac{1}{2} \vartheta^{(2)\ a} = \frac{1}{2} T_{[\kappa}{}^{a}{}_{\varrho]} \left(d x^{\kappa} \wedge d x^{\varrho} \right) \tag{66}$$

With the definitions and identifications given in eq. (66) the *antisymmetric* **structural relation – eq. (65 a) – becomes** →

natures way ... our way

$$a \;:\; d\, e^{(1)\;a} + \omega^{(1)\;a}{}_{b}\, e^{(1)\;b} = 2\, T^{(2)\;a} = \vartheta^{(2)\;a} \tag{67}$$

Yet the symmetrical structural relation (eq. 65 s) is just as important.

the two Bianchi identities (from case a)

It is from the antisymmetrical structural relation (eq. 67) that the two Bianchi identities derive. →

natures way ... our way

To simplify notation we suppress the tangent space indices in eq. (67)

$$a \;:\; d\, e^{\,(1)} + \omega^{\,(1)}\, e^{\,(1)} = \vartheta^{\,(2)} \qquad \longrightarrow$$

$$d \left(\omega^{\,(1)}\, e^{\,(1)} \right) + \omega^{\,(1)}\, d\, e^{\,(1)} + \left(\omega^{\,(1)} \right)^{2} e^{\,(1)} =$$

$$= d\, \vartheta^{\,(2)} + \omega^{\,(1)}\, \vartheta^{\,(2)} \equiv D^{\,(1)}\, \vartheta^{\,(2)}$$

$$D^{\,(1)} = d\, ¶ + \omega^{\,(1)} \equiv d + \omega^{\,(1)} \tag{68}$$

The – matrix valued – differential operator $D^{\,(1)}$ defined in eq. (68) $\longrightarrow$

natures way ... our way

together with the curvature 2-form

$$R^{(2)} = d\,\omega^{(1)} + \left(\omega^{(1)}\right)^2$$

$$\left(R^{(2)} = \tfrac{1}{2}\,(d x^{\sigma} \wedge d x^{\tau})\, R_{[\tau\sigma]} \right)^{a}_{\ \ b} \tag{69}$$

$$R_{[\tau\sigma]} = \partial_{\sigma}\,\omega_{\tau} - \partial_{\tau}\,\omega_{\sigma} + [\,\omega_{\sigma}\,,\,\omega_{\tau}\,]$$

$$[\,\omega_{\sigma}\,,\,\omega_{\tau}\,] = \omega_{\sigma}\,\omega_{\tau} - \omega_{\tau}\,\omega_{\sigma}$$

gives rise to the iterated operation , yielding on any (tangent space vector-) valued *tensorial* $(\nu) -$ **form**

$$\left(D^{(1)} \right)^2 X^{(\nu)} = R^{(2)}\, X^{(\nu)} \tag{70}$$

$\longrightarrow$

natures way ... our way

For $X^{(\nu)} = e^{(1)}$ we find the first Bianchi identity

$$
\begin{aligned}
B_I : R^{(2)} e^{(1)} &= D^{(1)} \vartheta^{(2)} \\
&= d\, \vartheta^{(2)} + \omega^{(1)} \vartheta^{(2)}
\end{aligned}
\tag{71}
$$

The second Bianchi identity says →

natures way ... our way

that the covariant derivative of the curvature 2-form vanishes

$$B_{II} \ :$$

$$\mathcal{D} \, R^{(2)} \equiv d \, R^{(2} + \omega^{(1)} \, R^{(2)} - R^{(2)} \, \omega^{(1)} = 0$$

$$\mathcal{D} \, R^{(2)} =$$

$$= \begin{bmatrix} d \left(\omega^{(1)} \right)^{2} \\ + \, \omega^{(1)} \, d \, \omega^{(1)} - \left(d \, \omega^{(1)} \right) \omega^{(1)} \\ + \left(\omega^{(1)} \right)^{3} - \left(\omega^{(1)} \right)^{3} \end{bmatrix} = 0 \tag{72}$$

natures way ... our way

e3) bridging remarks – concluding naturesway2005

as 1. part → naturesway2006...

Discussions , derivations and questions forming the *topics* of gauging orientation in D dimensions have taken their own special turn. Here is a good place to end the present survey forming the file – naturesway2005.pdf which however is building a bridge to the full *subject* to which these notes are devoted. →

I should like to thank all those finding a little time for pertinent discussions; in particular the organizers and participants of the conference in December 2005 in Miami, my colleagues and friends at Valencia , where this last part was written, at CERN and last but not least in Bern.

Valencia, 19. January 2006 **Peter Minkowski**

We repeat eq. (17) below, defining the metric (minimal) connection

$$\overset{o}{\Gamma}{}_{\{r}{}^{u}{}_{t\}} =$$

$$= \frac{1}{2} g^{uv} \left[\partial_r g_{vt} + \partial_t g_{vr} - \partial_v g_{rt} \right]$$

$$= \frac{1}{2} g^{uv} \begin{bmatrix} \partial_r (e^a{}_v \eta_{ab} e^b{}_t) + \\ + \partial_t (e^a{}_v \eta_{ab} e^b{}_r) - \\ - \partial_v (e^a{}_r \eta_{ab} e^b{}_t) \end{bmatrix} \tag{73}$$

natures way ... our way

I repeat the vier(l)bein and its inverse definition in eq. (43) – for definiteness of notation

$$
\begin{aligned}
& (\, e \,) \, ^{a} \, _{\varrho} \; = \; e \, ^{a} \, _{\varrho} \; , \; (\, \zeta \,) \, ^{\sigma} \, _{b} \; = \; (\, e \, ^{-1} \,) \, ^{\sigma} \, _{b} \; = \; e \, ^{\sigma} \, _{b} \\
& \rightarrow \quad e \, ^{\sigma} \, _{b} \; = \; \eta \, _{ba} \, g \, ^{\sigma \tau} \, e \, ^{a} \, _{\tau} \; = \; \zeta \, ^{\sigma} \, _{b} \\
& g \, ^{\varrho \sigma} \; = \; \zeta \, ^{\sigma} \, _{a} \, \eta \, ^{ab} \, \zeta \, ^{\tau} \, _{b} \; ; \; \eta \, ^{ab} \; = \; \eta \, _{ab}
\end{aligned}
\tag{74}
$$

Eq. (73) takes the form $\rightarrow$

natures way ... our way

$$\overset{o}{\Gamma}{}_{\{r}{}^{u}{}_{t\}} =$$

$$= \frac{1}{2}\,\zeta^{u}{}_{c}\,\eta^{cd}\,\zeta^{v}{}_{d}\left[\begin{array}{l} \partial_{r}\,(\,e^{a}{}_{v}\,\eta_{ab}\,e^{b}{}_{t}\,) + \\ +\,\partial_{t}\,(\,e^{a}{}_{v}\,\eta_{ab}\,e^{b}{}_{r}\,) - \\ -\,\partial_{v}\,(\,e^{a}{}_{r}\,\eta_{ab}\,e^{b}{}_{t}\,) \end{array}\right] \tag{75}$$

Next we partially transform the metric connection

natures way ... our way

to tangent space $^{u} \rightarrow {}^{a}$

$$\overset{o}{\Gamma}{}'^{\;\;a}_{\{r\;\;t\}} = e^{a}{}_{u}\;\overset{o}{\Gamma}{}^{\;\;u}_{\{r\;\;t\}} =$$

$$= \frac{1}{2}\,\eta^{ab}\,\zeta^{v}{}_{b}\left[\begin{array}{l} \partial_{r}\,(\,e^{c}{}_{v}\,\eta_{cd}\,e^{d}{}_{t}\,) + \\ + \partial_{t}\,(\,e^{c}{}_{v}\,\eta_{cd}\,e^{d}{}_{r}\,) - \\ - \partial_{v}\,(\,e^{c}{}_{r}\,\eta_{cd}\,e^{d}{}_{t}\,) \end{array}\right] \tag{76}$$

which yields differentiating the first and second row $\longrightarrow$

natures way ... our way

$$\overset{o}{\Gamma}{}'_{\{\, r \;\; {}^{a} \;\; t \,\}} =$$

$$= \; - \frac{1}{2} \eta^{\, ak} \zeta^{\, v}{}_{k} \, \partial_{\, v} \left(e^{\, c}{}_{r} \, \eta_{\, cd} \, e^{\, d}{}_{t} \right) +$$

$$+ \frac{1}{2} \eta^{\, ak} \left[\left(\partial_{\, r} \, e \right) e^{\, -1} \right]^{c}{}_{k} \, \eta_{\, cd} \, e^{\, d}{}_{t} + \frac{1}{2} \partial_{\, r} \, e^{\, a}{}_{t}$$

$$+ \frac{1}{2} \eta^{\, ak} \left[\left(\partial_{\, t} \, e \right) e^{\, -1} \right]^{c}{}_{k} \, \eta_{\, cd} \, e^{\, d}{}_{r} + \frac{1}{2} \partial_{\, t} \, e^{\, a}{}_{r} \tag{77}$$

Next we transform the index $_{t} \rightarrow \; _{b}$ $\longrightarrow$

natures way ... our way

$$\overset{o}{\omega}{}'{}_{r}{}^{a}{}_{b} = \overset{o}{\Gamma}{}'{}_{\{r}{}^{a}{}_{t\}} \, \zeta^{t}{}_{b} =$$

$$= -\frac{1}{2} \eta^{ak} \eta_{cb} \zeta^{v}{}_{k} \partial_{v} e^{c}{}_{r} + \frac{1}{2} \zeta^{v}{}_{b} \partial_{v} e^{a}{}_{r}$$

$$- \frac{1}{2} \eta^{ak} \zeta^{v}{}_{k} e^{c}{}_{r} \eta_{cd} \left[\left(\partial_{v} e \right) e^{-1} \right]^{d}{}_{b}$$

$$+ \frac{1}{2} \eta^{ak} \zeta^{v}{}_{b} e^{c}{}_{r} \eta_{cd} \left[\left(\partial_{v} e \right) e^{-1} \right]^{d}{}_{k}$$

$$+ \frac{1}{2} \eta^{ak} \eta_{cb} \left[\left(\partial_{r} e \right) e^{-1} \right]^{c}{}_{k}$$

$$+ \frac{1}{2} \left[\left(\partial_{r} e \right) e^{-1} \right]^{a}{}_{b} \tag{78}$$

Finally we repeat eq. (46) $\longrightarrow$

natures way ... our way

and obtain the metric spin connection ($_{r}$ $\longrightarrow$ $_{\kappa}$)

$$\omega_{\kappa} = e\,\Gamma_{\kappa}\,e^{-1} + e\,\partial_{\kappa}\,e^{-1}$$

$$\omega_{\kappa} = \omega_{\kappa}(\,\Gamma_{\kappa}\,;\,e\,)$$

with the associated 1-forms :

$$\omega^{(1)} = d\,x^{\kappa}\,\omega_{\kappa}\,,\ \Gamma^{(1)} = d\,x^{\kappa}\,\Gamma_{\kappa} \qquad \longrightarrow$$

$$\omega^{(1)} = e\,\Gamma^{(1)}\,e^{-1} + e\,d\,e^{-1} \tag{79}$$

natures way ... our way

$$\overset{o}{\omega}_{\kappa}{}^{a}{}_{b} = \overset{o}{\omega}{}'_{\kappa}{}^{a}{}_{b} - [\,(\partial_\kappa e)\, e^{-1}\,]^{a}{}_{b} =$$

$$= -\tfrac{1}{2}\,\eta^{aj}\,\eta_{cb}\,\zeta^{\varrho}{}_{j}\,\partial_{\varrho}\, e^{c}{}_{\kappa} + \tfrac{1}{2}\,\zeta^{\varrho}{}_{b}\,\partial_{\varrho}\, e^{a}{}_{\kappa}$$

$$- \tfrac{1}{2}\,\eta^{aj}\,\zeta^{\varrho}{}_{j}\, e^{c}{}_{\kappa}\,\eta_{cd}\,[\,(\partial_{\varrho} e)\, e^{-1}\,]^{d}{}_{b}$$

$$+ \tfrac{1}{2}\,\eta^{aj}\,\zeta^{\varrho}{}_{b}\, e^{c}{}_{\kappa}\,\eta_{cd}\,[\,(\partial_{\varrho} e)\, e^{-1}\,]^{d}{}_{j}$$

$$+ \tfrac{1}{2}\,\eta^{aj}\,\eta_{cb}\,[\,(\partial_{\kappa} e)\, e^{-1}\,]^{c}{}_{j}$$

$$- \tfrac{1}{2}\,[\,(\partial_{\kappa} e)\, e^{-1}\,]^{a}{}_{b} \tag{80}$$

In order to keep matrix notation untouched we define separate symbols,

natures way ... our way

to facilitate component expressions

$$\widetilde{\zeta}^{\,\varrho\, a} \;=\; \eta^{\,ab}\, \zeta^{\,\varrho}{}_{b} \;;\; \widetilde{e}_{\,a\,\sigma} \;=\; \eta_{\,ab}\, e^{\,b}{}_{\kappa} \qquad \longrightarrow \tag{81}$$

$$\begin{aligned} \overset{o}{\omega}{}_{\kappa}{}^{a}{}_{b} = & \\ = -\tfrac{1}{2}\, \widetilde{\zeta}^{\,\varrho\, a}\, \partial_{\varrho}\, \widetilde{e}_{\,b\,\kappa} \;+\; \tfrac{1}{2}\, \zeta^{\,\varrho}{}_{b}\, \partial_{\varrho}\, e^{\,a}{}_{\kappa} & \\ -\tfrac{1}{2}\, \widetilde{\zeta}^{\,\varrho\, a}\, \widetilde{e}_{\,d\,\kappa} \left[\left(\partial_{\varrho}\, e \right) e^{-1} \right]^{d}{}_{b} & \\ +\tfrac{1}{2}\, \eta^{\,aj}\, \zeta^{\,\varrho}{}_{b}\, \widetilde{e}_{\,d\,\kappa} \left[\left(\partial_{\varrho}\, e \right) e^{-1} \right]^{d}{}_{j} & \\ +\tfrac{1}{2}\, \eta^{\,aj}\, \eta_{\,cb} \left[\left(\partial_{\kappa}\, e \right) e^{-1} \right]^{c}{}_{j} & \\ -\tfrac{1}{2} \left[\left(\partial_{\kappa}\, e \right) e^{-1} \right]^{a}{}_{b} & \end{aligned} \tag{82}$$

natures way ... our way

For the remaining expressions we freely lower and raise tangent space indices

$$\overset{o}{\omega}_{\kappa}{}^{a\ b} = \eta^{bc}\ \overset{o}{\omega}_{\kappa}{}^{a}{}_{c} =$$

$$= -\tfrac{1}{2}\left(\widetilde{\zeta}^{\varrho\ a}\,\partial_{\varrho}\, e^{b}{}_{\kappa} - \widetilde{\zeta}^{\varrho\ b}\,\partial_{\varrho}\, e^{a}{}_{\kappa} \right)$$

$$- \tfrac{1}{2}\,\widetilde{\zeta}^{\varrho\ a}\,\widetilde{e}_{d\ \kappa}\left[\,(\,\partial_{\varrho}\, e\,)\, e^{-1}\,\right]^{d\ b}$$

$$+ \tfrac{1}{2}\,\widetilde{\zeta}^{\varrho\ b}\,\widetilde{e}_{d\ \kappa}\left[\,(\,\partial_{\varrho}\, e\,)\, e^{-1}\,\right]^{d\ a}$$

$$+ \tfrac{1}{2}\left(\left[\,(\,\partial_{\kappa}\, e\,)\, e^{-1}\,\right]^{b\ a} - \left[\,(\,\partial_{\kappa}\, e\,)\, e^{-1}\,\right]^{a\ b} \right) \tag{83}$$

As retained in the main text (eq. 52) it follows that

natures way ... our way

the local structure group of the minimal metric connection becomes the Lorentz group , in particular

$$\overset{o}{\omega}_{\kappa}{}^{a\, b} = \eta^{bc}\, \overset{o}{\omega}_{\kappa}{}^{a}{}_{c} = - \overset{o}{\omega}_{\kappa}{}^{b\, a} \tag{84}$$

on the Lorentz Lie algebra basis

Lets consider the basis set of $D\,(\,D - 1\,)\,/\,2$ – **distinct** $c < d$ – $(\,D \times D\,)$ **real matrices**

$$\left(\,\lambda_{[\,c\,d\,]}\,\right)^{a}{}_{b} = \delta_{d}{}^{a}\,\eta_{c\,b} - \delta_{c}{}^{a}\,\eta_{d\,b}$$

$$a\,,\,b\,,\,c\,,\,d = 0\,,\,\cdots\,,\,D - 1 \tag{85}$$

$$\lambda_{[\,c\,d\,]} = -\lambda_{[\,d\,c\,]}$$

natures way ... our way

$\lambda_{[cd]}$ **satisfy the commutation rules of the Lie algebra of** $SO(1, D-1)$

$$\left[\lambda_{[cd]}, \lambda_{[c'd']}\right] =$$

$$= \{\delta_d{}^a \eta_{cn} - (c \leftrightarrow d)\} \times$$

$$\times \left\{\delta_{d'}{}^n \eta_{c'b} - (c' \leftrightarrow d')\right\}$$

$$- [cd] \leftrightarrow \left[c' d'\right] \tag{86}$$

$$= \eta_{cd'} \delta_d{}^a \eta_{c'b} - \eta_{cc'} \delta_d{}^a \eta_{d'b}$$

$$- \eta_{dd'} \delta_c{}^a \eta_{c'b} + \eta_{dc'} \delta_c{}^a \eta_{d'b}$$

$$- [cd] \leftrightarrow \left[c' d'\right]$$

→

natures way ... our way

$$
\left[\lambda_{[cd]}, \lambda_{[c'd']} \right] =
$$

$$
\begin{aligned}
&= \eta_{cd'} \delta_d{}^a \eta_{c'b} - \eta_{dc'} \delta_{d'}{}^a \eta_{cb} \\
&\quad - \eta_{cc'} \delta_d{}^a \eta_{d'b} + \eta_{cc'} \delta_{d'}{}^a \eta_{db} \\
&\quad - \eta_{dd'} \delta_c{}^a \eta_{c'b} + \eta_{dd'} \delta_{c'}{}^a \eta_{cb} \\
&\quad + \eta_{dc'} \delta_c{}^a \eta_{d'b} - \eta_{cd'} \delta_{c'}{}^a \eta_{db}
\end{aligned}
\tag{87}
$$

and collecting terms →

natures way ... our way

$$\left[\lambda_{[cd]} , \lambda_{[c'd']} \right] =$$

$$= \left\{ \begin{array}{c} (-\eta_{cd'}) \lambda_{[dc']} \\ - (-\eta_{cc'}) \lambda_{[dd']} \\ - (-\eta_{dd'}) \lambda_{[cc']} \\ + (-\eta_{dc'}) \lambda_{[cd']} \end{array} \right\} \tag{88}$$

We note that the Lorentz Lie algebra is here constructed from the $4 \rightarrow D$ dimensional irreducible representation of $SO(1, D-1)$,

natures way ... our way

which is singled out in the bosonic restriction inherent to the metric and vier(l)bein concepts.

From eq. (86) we deduce the invariant traces

$$tr \; \lambda_{[\,c\,d\,]} \, \lambda_{[\,c'\,d'\,]} \;=\; -\,2\, G_{[\,c\,d\,]\,[\,c'\,d'\,]}$$

$$G_{[\,c\,d\,]\,[\,c'\,d'\,]} \;=\; (\,\eta_{c\,c'}\,\eta_{d\,d'} \;-\; \eta_{c\,d'}\,\eta_{d\,c'}\,) \tag{89}$$

We will come back to the Lorentz Lie algebra , but here shall use the base matrices $\lambda_{[\,c\,d\,]}$ for the projection →

natures way ... our way

$$
\begin{aligned}
&\frac{1}{2}\ \overset{o}{\omega}_{\kappa}{}^{[\,d\,c\,]}\left(\lambda_{[\,c\,d\,]}\right)^{a}{}_{b} = \\
&= \frac{1}{2}\ \overset{o}{\omega}_{\kappa}{}^{[\,d\,c\,]}\,(\,\delta_{d}{}^{a}\,\eta_{c\,b} - \delta_{c}{}^{a}\,\eta_{d\,b}\,) \qquad (90)\\
&= \overset{o}{\omega}_{\kappa}{}^{a}{}_{b}
\end{aligned}
$$

Note the opposite ordering of the indices $^{d\,c}$ **relative to** $^{c\,d}$ **in** $\overset{o}{\omega}_{\kappa}{}^{[\,d\,c\,]}$ **and** $\lambda_{[\,c\,d\,]}$ **in eq. (90) .**

natures way ... our way

reduction of the spin connection $\overset{o}{\omega}{}_{\kappa}{}^{[a\ b]}$ in eqs. (52 and 83)

We turn to eq. (83) reproduced below

$$\overset{o}{\omega}{}_{\kappa}{}^{[a\ b]} = \eta^{bc}\, \overset{o}{\omega}{}_{\kappa}{}^{a}{}_{c} =$$
$$= -\frac{1}{2}\left(\tilde{\zeta}^{\varrho a}\,\partial_{\varrho}\, e^{b}{}_{\kappa} - \tilde{\zeta}^{\varrho b}\,\partial_{\varrho}\, e^{a}{}_{\kappa}\right)$$
$$-\frac{1}{2}\,\tilde{\zeta}^{\varrho a}\,\tilde{e}_{d\kappa}\,[\,(\partial_{\varrho} e)\, e^{-1}\,]^{d\ b}$$
$$+\frac{1}{2}\,\tilde{\zeta}^{\varrho b}\,\tilde{e}_{d\kappa}\,[\,(\partial_{\varrho} e)\, e^{-1}\,]^{d\ a}$$
$$+\frac{1}{2}\left([\,(\partial_{\kappa} e)\, e^{-1}\,]^{b\ a} - [\,(\partial_{\kappa} e)\, e^{-1}\,]^{a\ b}\right) \tag{91}$$

→

natures way ... our way

and substitute the expressions

$$[\,(\partial_\kappa e)\, e^{-1}\,]^{b\,a} = \tilde{\zeta}^{\sigma\,a}\, \partial_\kappa e^{b}{}_{\sigma}$$
$$[\,(\partial_\varrho e)\, e^{-1}\,]^{d\,b} = \tilde{\zeta}^{\sigma\,b}\, \partial_\varrho e^{d}{}_{\sigma} \qquad (92)$$

The minimal spin connection takes the form

$$\overset{o}{\omega}_{\kappa}{}^{[a\,b]} =$$
$$= -\frac{1}{2}\left(\tilde{\zeta}^{\varrho\,a}\,\partial_\varrho e^{b}{}_{\kappa} - \tilde{\zeta}^{\varrho\,b}\,\partial_\varrho e^{a}{}_{\kappa}\right)$$
$$-\frac{1}{2}\,\tilde{e}_{d\,\kappa}\,\tilde{\zeta}^{\varrho\,a}\,\tilde{\zeta}^{\sigma\,b}\left(\partial_\varrho e^{d}{}_{\sigma} - \partial_\sigma e^{d}{}_{\varrho}\right)$$
$$+\frac{1}{2}\left(\tilde{\zeta}^{\varrho\,a}\,\partial_\kappa e^{b}{}_{\varrho} - \tilde{\zeta}^{\varrho\,b}\,\partial_\kappa e^{a}{}_{\varrho}\right) \qquad (93)$$

natures way ... our way

The structure of the minimal spin connection (eq. 93) reduces its dependence on the first derivatives of the vier(l)bein to the field strength-like quantities , for which we introduce the notation

$$e^{a}{}_{[\sigma\tau]} = \partial_{\tau} e^{a}{}_{\sigma} - \partial_{\sigma} e^{a}{}_{\tau} \tag{94}$$

The spin connection then takes the form

$$\overset{o}{\omega}{}_{\kappa}{}^{[ab]} = \\ = \frac{1}{2}\left[\begin{array}{c} \widetilde{\zeta}^{\varrho a} e^{b}{}_{[\varrho\kappa]} - \widetilde{\zeta}^{\varrho b} e^{a}{}_{[\varrho\kappa]} \\ + \widetilde{e}_{d\kappa} \widetilde{\zeta}^{\varrho a} \widetilde{\zeta}^{\sigma b} e^{d}{}_{[\varrho\sigma]} \end{array} \right] \tag{95}$$

natures way ... our way

Eqs. 94 and 95 are exported to the main text, as eqs. 56 and 57 .

It is instructive to give all three indices of the spin connection the same status. This can be done (at least) in two ways

$$\overset{o}{\omega}{}^{c\,[a\,b]} = \widetilde{\zeta}{}^{\kappa\, c}\, \overset{o}{\omega}{}_{\kappa}{}^{[a\,b]} =$$

$$= \frac{1}{2}\left[\widetilde{\zeta}{}^{\varrho\, a}\, \widetilde{\zeta}{}^{\sigma\, c}\, e^{b}{}_{[\varrho\,\sigma]} - \widetilde{\zeta}{}^{\varrho\, b}\, \widetilde{\zeta}{}^{\sigma\, c}\, e^{a}{}_{[\varrho\,\sigma]} + \widetilde{\zeta}{}^{\varrho\, a}\, \widetilde{\zeta}{}^{\sigma\, b}\, e^{c}{}_{[\varrho\,\sigma]} \right] \tag{96}$$

and $\longrightarrow$

natures way ... our way

$$\overset{o}{\omega}_{\tau\,[\varrho\,\sigma]} = \tilde{e}_{a\,\varrho}\,\tilde{e}_{b\,\sigma}\;\overset{o}{\omega}_{\tau}{}^{[a\,b]} =$$
$$= \frac{1}{2}\left[\begin{array}{c} \tilde{e}_{a\,\varrho}\,e^{a}{}_{[\tau\,\sigma]} - \tilde{e}_{a\,\sigma}\,e^{a}{}_{[\tau\,\varrho]} \\ + \tilde{e}_{a\,\tau}\,e^{a}{}_{[\varrho\,\sigma]} \end{array}\right] \tag{97}$$

Eqs. 96 and 97 are exported to the main text, as eqs. 58 and 59 .

a cross check of the structure of $\overset{o}{\omega}_{\kappa}{}^{[a\,b]}$ in eqs. (95 - 97)

I propose to check the structures derived before , using the requirement of covariant constancy

$$D_{\kappa}\,\eta_{a\,b} = 0 \;;\; D_{\kappa}\,e^{a}{}_{\varrho} = 0 \tag{98}$$

natures way ... our way

From the first condition in eq. (98) we deduce

$$- \overset{o}{\omega}_{\kappa}{}^{c}{}_{a}\, \eta_{c\,b} - \overset{o}{\omega}_{\kappa}{}^{c}{}_{b}\, \eta_{a\,c} = 0$$
$$\rightarrow \quad \overset{o}{\omega}_{\kappa}{}^{a\,b} = - \overset{o}{\omega}_{\kappa}{}^{b\,a} \quad \surd \tag{99}$$

The second one implies

$$\partial_{\kappa}\, e^{a}{}_{\varrho} + \overset{o}{\omega}_{\kappa}{}^{a}{}_{c}\, e^{c}{}_{\varrho} = (\Gamma_{\kappa})^{\tau}{}_{\varrho}\, e^{a}{}_{\tau} \;\; | \; \zeta^{\varrho}{}_{b}$$

$$\overset{o}{\omega}_{\kappa}{}^{a}{}_{b} =$$

$$= - \zeta^{\varrho}{}_{b}\, \partial_{\kappa}\, e^{a}{}_{\varrho} + \zeta^{\varrho}{}_{b}\, e^{a}{}_{\tau}\, g^{\tau\,\sigma}\, \Gamma_{\kappa\,\sigma\,\varrho}$$

$$= - \zeta^{\varrho}{}_{b}\, \partial_{\kappa}\, e^{a}{}_{\varrho} + \widetilde{\zeta}^{\sigma\,a}\, \zeta^{\varrho}{}_{b}\, \Gamma_{\kappa\,\sigma\,\varrho} \tag{100}$$

natures way ... our way

The last term on the right hand side of eq. (100) becomes

$$\tilde{\zeta}^{\sigma a}\,\zeta^{\varrho}{}_{b}\,\Gamma_{\kappa\sigma\varrho} =$$

$$= \frac{1}{2}\,\tilde{\zeta}^{\sigma a}\,\zeta^{\varrho}{}_{b}\left[\,\partial_{\kappa}\,g_{\sigma\varrho} + \partial_{\varrho}\,g_{\sigma\kappa} - \partial_{\sigma}\,g_{\kappa\varrho}\,\right]$$

$$= \frac{1}{2}\,\tilde{\zeta}^{\sigma a}\,\zeta^{\varrho}{}_{b}\,\eta_{cd}\left[\begin{array}{c} \partial_{\kappa}\left(e^{c}{}_{\sigma}\,e^{d}{}_{\varrho}\right) \\ +\,\partial_{\varrho}\left(e^{c}{}_{\sigma}\,e^{d}{}_{\kappa}\right) \\ -\,\partial_{\sigma}\left(e^{c}{}_{\kappa}\,e^{d}{}_{\varrho}\right) \end{array}\right] \tag{101}$$

going step by step →

natures way ... our way

$$\tilde{\zeta}^{\sigma a} \zeta^{\varrho}{}_{b} \Gamma_{\kappa \sigma \varrho} =$$

$$= \frac{1}{2} \left[\begin{array}{c} \tilde{\zeta}^{\sigma a} \partial_{\kappa} e_{b \sigma} + \zeta^{\varrho}{}_{b} \partial_{\kappa} e^{a}{}_{\varrho} \\ + \tilde{\zeta}^{\sigma a} \zeta^{\varrho}{}_{b} \tilde{e}_{d \kappa} \partial_{\varrho} e^{d}{}_{\sigma} \\ + \zeta^{\varrho}{}_{b} \partial_{\varrho} e^{a}{}_{\kappa} - \tilde{\zeta}^{\sigma a} \partial_{\sigma} e_{b \kappa} \\ - \tilde{\zeta}^{\sigma a} \zeta^{\varrho}{}_{b} \tilde{e}_{d \kappa} \partial_{\sigma} e^{d}{}_{\varrho} \end{array} \right] \quad (102)$$

and completing the spin connection (eq. 100) $\longrightarrow$

natures way ... our way

$$\overset{o}{\omega}{}_{\kappa}{}^{a}{}_{b} =$$

$$= \frac{1}{2} \left[\begin{array}{c} \tilde{\zeta}^{\sigma a} \, \partial_{\kappa} \, e_{b \sigma} - \zeta^{\varrho}{}_{b} \, \partial_{\kappa} \, e^{a}{}_{\varrho} \\ + \tilde{\zeta}^{\sigma a} \, \zeta^{\varrho}{}_{b} \, \tilde{e}_{d \kappa} \, \partial_{\varrho} \, e^{d}{}_{\sigma} \\ + \zeta^{\varrho}{}_{b} \, \partial_{\varrho} \, e^{a}{}_{\kappa} - \tilde{\zeta}^{\sigma a} \, \partial_{\sigma} \, e_{b \kappa} \\ - \tilde{\zeta}^{\sigma a} \, \zeta^{\varrho}{}_{b} \, \tilde{e}_{d \kappa} \, \partial_{\sigma} \, e^{d}{}_{\varrho} \end{array} \right] \tag{103}$$

Finally we exchange the summation indices

$\varrho \longleftrightarrow \sigma$

natures way ... our way

and raise the tangent space index $_b$

$$\overset{o}{\omega}_{\kappa}{}^{a\,b} =$$

$$= \frac{1}{2} \left[\begin{array}{c} \widetilde{\zeta}^{\varrho\,a}\,\partial_{\kappa}\,e^{b}{}_{\varrho} - \widetilde{\zeta}^{\varrho\,b}\,\partial_{\kappa}\,e^{a}{}_{\varrho} \\ + \widetilde{\zeta}^{\varrho\,a}\,\widetilde{\zeta}^{\sigma\,b}\,\widetilde{e}_{d\,\kappa}\,\partial_{\sigma}\,e^{d}{}_{\varrho} \\ + \widetilde{\zeta}^{\varrho\,b}\,\partial_{\varrho}\,e^{a}{}_{\kappa} - \widetilde{\zeta}^{\varrho\,a}\,\partial_{\varrho}\,e^{b}{}_{\kappa} \\ - \widetilde{\zeta}^{\varrho\,a}\,\widetilde{\zeta}^{\sigma\,b}\,\widetilde{e}_{d\,\kappa}\,\partial_{\varrho}\,e^{d}{}_{\sigma} \end{array} \right] \tag{104}$$

and compare with eq. (93) repeated below as eq. (105)

natures way ... our way

$$\overset{o}{\omega}{}_{\kappa}{}^{[\,a\,b\,]} =$$

$$= -\frac{1}{2}\left(\widetilde{\zeta}^{\varrho\, a}\,\partial_{\varrho}\, e^{b}{}_{\kappa} - \widetilde{\zeta}^{\varrho\, b}\,\partial_{\varrho}\, e^{a}{}_{\kappa}\right)$$

$$-\frac{1}{2}\,\widetilde{e}_{d\,\kappa}\,\widetilde{\zeta}^{\varrho\, a}\,\widetilde{\zeta}^{\sigma\, b}\left(\partial_{\varrho}\, e^{d}{}_{\sigma} - \partial_{\sigma}\, e^{d}{}_{\varrho}\right)$$

$$+\frac{1}{2}\left(\widetilde{\zeta}^{\varrho\, a}\,\partial_{\kappa}\, e^{b}{}_{\varrho} - \widetilde{\zeta}^{\varrho\, b}\,\partial_{\kappa}\, e^{a}{}_{\varrho}\right) \tag{105}$$

The six terms indeed match numbering from 1 - 6 comparing eqs. (104) $\leftrightarrow$ (105)

$$\begin{array}{l} 1 \leftrightarrow 5\,,\ 2 \leftrightarrow 6\,,\ 3 \leftrightarrow 4 \\ 4 \leftrightarrow 2\,,\ 5 \leftrightarrow 1\,,\ 6 \leftrightarrow 3 \end{array} \qquad \surd \tag{106}$$

$\longrightarrow$

natures way ... our way

We take the occasion to check also the field strength(- like) form of the spin connection from eq. (104) and compare with eqs. (94 - 95)

$$\overset{o}{\omega}_{\kappa}{}^{[\,a\,b\,]} = \\ = \frac{1}{2}\left[\begin{array}{c} \widetilde{\zeta}^{\,\varrho\, a}\, e^{\,b}{}_{[\,\varrho\,\kappa\,]} - \widetilde{\zeta}^{\,\varrho\, b}\, e^{\,a}{}_{[\,\varrho\,\kappa\,]} \\ + \widetilde{\zeta}^{\,\varrho\, a}\, \widetilde{\zeta}^{\,\sigma\, b}\, \widetilde{e}_{\,d\,\kappa}\, e^{\,d}{}_{[\,\varrho\,\sigma\,]} \end{array}\right] \tag{107}$$

We reproduce eq. (95) as eq. (108) below for comparison

natures way ... our way

$$\overset{o}{\omega}{}_{\kappa}{}^{[\,a\;b\,]} =$$

$$= \frac{1}{2}\left[\begin{array}{c} \widetilde{\zeta}^{\,\varrho\,a}\, e^{\,b}{}_{[\,\varrho\;\kappa\,]} - \widetilde{\zeta}^{\,\varrho\,b}\, e^{\,a}{}_{[\,\varrho\;\kappa\,]} \\ + \,\widetilde{e}_{\,d\;\kappa}\, \widetilde{\zeta}^{\,\varrho\,a}\, \widetilde{\zeta}^{\,\sigma\,b}\, e^{\,d}{}_{[\,\varrho\;\sigma\,]} \end{array}\right] \tag{108}$$

eq. (107) $\leftrightarrow$ eq. (95) = (108) $\surd$. [a]

---

[a] **"Il y a des circonstances dans lesquelles il ne faut pas faire des fautes ." (Bonatti, alpinist from Courmayeur, Italy) .**

natures way ... our way

Appendix 2 : Weyl transformations , an example of nonminimal symmetric Christoffel connection

We reproduce the structural equations between spin- and Christoffel connections (section e2 , eq. 65) below

$$
\begin{array}{ll}
s \; : \; \omega_{\{ \kappa}{}^{a}{}_{\varrho \}} + \partial_{\{ \kappa} e^{a}{}_{\varrho \}} = \Gamma_{\{ \kappa}{}^{a}{}_{\varrho \}} \\
a \; : \; \omega_{[\kappa}{}^{a}{}_{\varrho]} + \partial_{[\kappa} e^{a}{}_{\varrho]} = T_{[\kappa}{}^{a}{}_{\varrho]}
\end{array}
\tag{109}
$$

and consider in this appendix only the case of vanishing torsion , in the form of eq. (67) in section e2 (reproduced below) →

natures way ... our way

dropping the tangent space indices $^{a}\ ,\ {}^{a}{}_{b}$

$$a\ :\ d\,e^{(1)} + \omega^{(1)}\,e^{(1)} = \vartheta^{(2)} \rightarrow 0 \qquad (110)$$

Now we consider , following Hermann Weyl [9] , a (real) scalar field $\lambda(x)$ **, giving rise to a family of metrics , starting from a 'base' metric** $g_{\mu\nu}$ **, together with a family of (square-) distance differentials**

$$G_{\mu\nu}(\lambda) = \lambda^{2}\, g_{\mu\nu}$$

$$(d\,S_{\lambda})^{2} = \lambda^{2}\,(d\,s)^{2} \qquad (111)$$

$$(d\,s)^{2} = g_{\mu\nu}\, d\,x^{\mu}\, d\,x^{\nu}$$

natures way ... our way

Obviously geodesics pertaining to $G(\lambda)$ **vary with** λ **. Accordingly we consider the family of** *minimal metric* **Christoffel connections , relative to** $G(\lambda)$

$$\overset{\lambda}{\Gamma}{}_{\kappa}{}^{\varrho}{}_{\sigma} =$$

$$= \frac{1}{2} G^{\varrho \nu} \left(\partial_{\kappa} G_{\nu \sigma} + \partial_{\sigma} G_{\nu \kappa} - \partial_{\nu} G_{\kappa \sigma} \right)$$

$$\overset{\lambda}{\vartheta}{}_{[\kappa}{}^{\varrho}{}_{\sigma]} = \overset{\lambda}{\Gamma}{}_{\kappa}{}^{\varrho}{}_{\sigma} - \overset{\lambda}{\Gamma}{}_{\varrho}{}^{\kappa}{}_{\sigma} = 0 \tag{112}$$

The last relation in eq. (112) shows that torsion tensors vanish for all λ **.**

natures way ... our way

We remark that any one of the $\overset{\lambda}{\Gamma}$ connections is a valid , torsion free connection on the space with 'base' metric $g_{\mu\nu}$ and 'base' (minimal metric-) connection

$$\overset{0}{\Gamma}{}_{\kappa}{}^{\varrho}{}_{\sigma} \; (\leftrightarrow \; \lambda^{0} \equiv 1) =$$

$$= \frac{1}{2} g^{\varrho \nu} \left(\partial_{\kappa} g_{\nu \sigma} + \partial_{\sigma} g_{\nu \kappa} - \partial_{\nu} g_{\kappa \sigma} \right) \tag{113}$$

As a consequence we are here in case c (eq. 17) , where the difference →

natures way ... our way

of so constructed connections

$$\overset{\lambda}{\gamma}_{\{\kappa}{}^{\varrho}{}_{\sigma\}} = \overset{\lambda}{\Gamma}_{\kappa}{}^{\varrho}{}_{\sigma} - \overset{0}{\Gamma}_{\kappa}{}^{\varrho}{}_{\sigma} \tag{114}$$

is a 3-tensor symmetric with respect to the index pair $\kappa\ \varrho$. In the configuration space we are considering we chall choose $\lambda(x) > 0$, in order to guarantee a continuity of all $\overset{\lambda}{\gamma}$ tensors .

Thus , defining $L(x) = \log \lambda(x)$, we obtain

$$\overset{\lambda}{\gamma}_{\{\kappa}{}^{\varrho}{}_{\sigma\}} = \left[\begin{array}{c} \delta^{\varrho}{}_{\sigma}\, \partial_{\kappa} L + \delta_{\kappa}{}^{\varrho}\, \partial_{\sigma} L \\ - g_{\kappa\sigma}\, g^{\varrho\nu}\, \partial_{\nu} L \end{array} \right] \tag{115}$$

$\longrightarrow$

natures way ... our way

or transforming to three lower case indices

$$\overset{\lambda}{\gamma}_{\kappa\alpha\sigma} = g_{\alpha\varrho}\,\overset{\lambda}{\gamma}_{\{\kappa}{}^{\varrho}{}_{\sigma\}} =$$

$$= \left[\begin{array}{c} g_{\alpha\sigma}\,\partial_\kappa L + g_{\kappa\alpha}\,\partial_\sigma L \\ - g_{\kappa\sigma}\,\partial_\alpha L \end{array}\right] \tag{116}$$

$$\frac{1}{2}\left(\overset{\lambda}{\gamma}_{\kappa\alpha\sigma} + \overset{\lambda}{\gamma}_{\kappa\sigma\alpha}\right) = g_{\alpha\sigma}\,\partial_\kappa L$$

$$\rightarrow \quad \overset{\lambda}{\gamma}_{\{\kappa}{}^{\sigma}{}_{\sigma\}} = D\,\partial_\kappa L = \partial_\kappa \log \lambda^D$$

$\rightarrow$

natures way ... our way

The last relation was clearly realised by Hermann Weyl [9] to imply 'length gauging' ('Längen-Eichung' in german) but involving scalar fields , to be endowed with their own dynamics [a] .

---

[a] **I feel it is *instructive* and thus allow myself to comment on the historical 'comedy of errors' which contrasts with this clear logic. I do so quoting (translating) from op. cit. [9] , p. 300 ff. :**
"Besides the gravitational field – there exists in nature only the electromagnetic one. Since the general 'Infinitesimalgeometrie' gives us beyond the metric quadratic form the linear one ($\propto \; d\, x^{\kappa} \, \partial_{\kappa} \, L$ as in eq. 116 here) and the electromagnetic potentials define also a linear form, it is 'naheliegend' and tempting to – associate the two ..."
this is my free translation .

natures way ... our way

I think that the present discussion *illustrates* **the relevance of the 'symmetric case' (c and eq. 17)** [a]**.**

We shall now follow the associated family of vier(l)beins and their – torsion free – spin connections.

$$G_{\mu\nu}(\lambda) = \lambda^2 g_{\mu\nu} \leftrightarrow E^a{}_\mu(\lambda, \Lambda) = \lambda e^b{}_\mu \Lambda^a{}_b(x) \tag{117}$$

In eq. (117) $\Lambda(x)$ **denotes** →

[a] **I should like to thank Martin Schmid for his apparently unrelated presence and ensuing discussion to this particular issue.**

a local Lorentztransformation, guaranteeing the most general vier(l)bein compatible with the metric $G\ (\ \lambda\ )$ **.**

Next we construct the family of *minimal metric* **spin connections relative to** $G\ (\ \lambda\ )$
(eqs. 57 , 95 = 108 , 107) $\rightarrow$

natures way ... our way

$$\overset{\lambda}{\omega}{}_{\kappa}{}^{[ab]} =$$

$$= \frac{1}{2} \Big[\widetilde{Z}^{\varrho a} E^{b}{}_{[\varrho \kappa]} - \widetilde{Z}^{\varrho b} E^{a}{}_{[\varrho \kappa]} + \widetilde{E}_{d \kappa} \widetilde{Z}^{\varrho a} \widetilde{Z}^{\sigma b} E^{d}{}_{[\varrho \sigma]} \Big]$$

$$\widetilde{Z}^{\varrho a} = \eta^{a a'} (E^{-1})^{\varrho}{}_{a'}$$

$$\widetilde{E}_{d \kappa} = \eta_{d d'} E^{d'}{}_{\kappa} \tag{118}$$

We first identify $\widetilde{Z}$, $\widetilde{E}$ →

natures way ... our way

$$\begin{aligned}
\widetilde{Z}^{\,\varrho\, a} &= \lambda^{-1}\, \Lambda^{a}{}_{e}\, \widetilde{\zeta}^{\,\varrho\, e} \\
\widetilde{E}_{d\,\kappa} &= \lambda\, \eta_{d\,d'}\, \Lambda^{d'}{}_{c}\, \eta^{c\,c'}\, \widetilde{e}_{c'\,\kappa} \\
&= \lambda\, \left(\Lambda^{-1}\right)^{c}{}_{d}\, \widetilde{e}_{c\,\kappa}
\end{aligned} \tag{119}$$

Substituting the expressions in eq. (119) in eq. (118) $\longrightarrow$

natures way ... our way

and using matrix-vector notation we find

$$\overset{\lambda}{\omega}{}_{\kappa}{}^{[\,a\;b\,]} =$$

$$= \frac{1}{2\,\lambda} \left[\begin{array}{c} \left(\Lambda\, \widetilde{\zeta}^{\,\varrho} \right)^{a} E^{\,b}{}_{[\,\varrho\;\kappa\,]} \\ -\left(\Lambda\, \widetilde{\zeta}^{\,\varrho} \right)^{b} E^{\,a}{}_{[\,\varrho\;\kappa\,]} \\ +\left(\widetilde{e}\,{}^{T}_{\kappa}\, \Lambda^{-1} \right)_{d} \left(\Lambda\, \widetilde{\zeta}^{\,\varrho} \right)^{a} \times \\ \times \left(\Lambda\, \widetilde{\zeta}^{\,\sigma} \right)^{b} E^{\,d}{}_{[\,\varrho\;\sigma\,]} \end{array} \right] \tag{120}$$

Next we go back to the definition of the fieldstrength-like quantities (56) reproduced below

$\longrightarrow$

natures way ... our way

$$e^{a}{}_{[\sigma\tau]} = \partial_\tau e^{a}{}_{\sigma} - \partial_\sigma e^{a}{}_{\tau}$$

$$E^{a}{}_{[\sigma\tau]} =$$

$$= [\partial_\tau (\lambda \Lambda e_\sigma)^a - \partial_\sigma (\lambda e_\tau)^a] \quad \longrightarrow$$

$$= \lambda \left[\begin{array}{c} \left(\Lambda e_{[\sigma\tau]} \right)^a \\ + \left[(\partial_\tau \Lambda) e_\sigma - (\partial_\sigma \Lambda) e_\tau \right]^a \\ + (\Lambda e_\sigma)^a \partial_\tau L - (\Lambda e_\tau)^a \partial_\sigma L \end{array} \right]$$

$$L = \log \lambda \tag{121}$$

natures way ... our way

Here we introduce the $\Lambda-$ related term

$$\Omega_\tau \; : \; (\Omega_\tau)^a{}_b = (\Lambda^{-1})^a{}_c \, \partial_\tau \Lambda^c{}_b \qquad (122)$$

Eq. (121) then takes the form

$$E^a{}_{[\sigma\tau]} = \lambda \, \Lambda^a{}_c \, D^c{}_{[\sigma\tau]}$$

$$D^c{}_{[\sigma\tau]} =$$

$$= \left[\begin{array}{c} e_{[\sigma\tau]} \\ + \Omega_\tau e_\sigma - \Omega_\sigma e_\tau \\ + (\partial_\tau L) e_\sigma - (\partial_\sigma L) e_\tau \end{array} \right]^c \qquad (123)$$

$\longrightarrow$

natures way ... our way

Thus we can express $\overset{\lambda}{\omega}\ (\ \Lambda\ )$ in eq. (120)

$$\overset{\lambda}{\omega}{}_{\kappa}{}^{[\ a\ b\ ]} = \Lambda^{a}{}_{c}\,\Lambda^{b}{}_{d}\ \overset{\lambda}{\omega}{}'{}_{\kappa}{}^{[\ c\ d\ ]}$$

$$\overset{\lambda}{\omega}{}'{}_{\kappa}{}^{[\ c\ d\ ]} = \tag{124}$$

$$= \frac{1}{2}\left[\begin{array}{c} \widetilde{\zeta}^{\,\varrho\ c}\, D^{d}{}_{[\ \varrho\ \kappa\ ]} - \widetilde{\zeta}^{\,\varrho\ d}\, D^{c}{}_{[\ \varrho\ \kappa\ ]} \\ +\ \widetilde{e}_{\,m\ \kappa}\,\widetilde{\zeta}^{\,\varrho\ c}\,\widetilde{\zeta}^{\,\sigma\ d}\, D^{m}{}_{[\ \varrho\ \sigma\ ]} \end{array}\right]$$

Next we look at the difference →

natures way ... our way

$$\left(\Delta\, \omega \;=\; \overset{\lambda}{\omega}{}' \;-\; \overset{0}{\omega} \;=\; \Delta\, \omega_{\,\Omega} \;+\; \Delta\, \omega_{\,L} \right)_{\kappa}{}^{[\, c\;\; d \,]}$$

$$(\, \Delta\, \omega_{\,\Omega} \,)_{\kappa}{}^{[\, c\;\; d \,]} \;=$$

$$= \; \frac{1}{2} \left[\tilde{\zeta}^{\,\varrho\; c} \left[\begin{array}{c} (\, \Omega_{\,\kappa} \,)^{d}{}_{n} \; e^{\,n}{}_{\varrho} \\ - \; (\, \Omega_{\,\varrho} \,)^{d}{}_{n} \; e^{\,n}{}_{\kappa} \end{array} \right] - (\, c \;\leftrightarrow\; d \,) \atop + \; \tilde{e}_{\,m\;\kappa} \; \tilde{\zeta}^{\,\varrho\; c} \; \tilde{\zeta}^{\,\sigma\; d} \left[\begin{array}{c} (\, \Omega_{\,\sigma} \,)^{m}{}_{n} \; e^{\,n}{}_{\varrho} \\ - \; (\, \Omega_{\,\varrho} \,)^{m}{}_{n} \; e^{\,n}{}_{\sigma} \end{array} \right] \right]$$

$$\Delta\, \omega_{\,L} \qquad\qquad \longrightarrow \tag{125}$$

natures way ... our way

Next we use the relation

$$e^{\,n}{}_{\varrho}\,\widetilde{\zeta}^{\,\varrho\,c} \;=\; \eta^{\,n\,c} \tag{126}$$

and transform the expression for $\Delta\,\omega_{\,\Omega}$

$$(\,\Delta\,\omega_{\,\Omega}\,)_{\,\kappa}{}^{[\,c\,d\,]} \;=$$

$$= \;\frac{1}{2}\left[\begin{array}{c} 2\,\Omega_{\,\kappa}{}^{[\,d\,c\,]} \\ -\,\widetilde{e}_{\,m\,\kappa}\,\widetilde{\zeta}^{\,\varrho\,c}\,\Omega_{\,\varrho}{}^{[\,d\,m\,]} + (\,c \leftrightarrow d\,) \\ -\,\widetilde{e}_{\,m\,\kappa}\,\widetilde{\zeta}^{\,\varrho\,d}\,\Omega_{\,\varrho}{}^{[\,c\,m\,]} + (\,c \leftrightarrow d\,) \end{array}\right]$$

$$\Omega_{\,\varrho}{}^{[\,c\,m\,]} \;=\; (\,\Omega_{\,\varrho}\,)^{\,c}{}_{\,n}\,\eta^{\,m\,n} \tag{127}$$

$\longrightarrow$

natures way ... our way

As a consequence of the antisymmetric $c \leftrightarrow m$ structure of $\Omega_{\varrho}{}^{[cm]}$ defined in eq. (127) the last four terms in the bracketed expression for $\Delta\omega_{\Omega}$ in eq. (127) cancel. $\longrightarrow$

$$(\Delta\omega_{\Omega})_{\kappa}{}^{[cd]} = -\Omega_{\kappa}{}^{[cd]} \tag{128}$$

We go back to eq. (124) which becomes

$$\overset{\lambda}{\omega}_{\kappa}{}^{[ab]} = \Lambda^{a}{}_{c}\Lambda^{b}{}_{d}\begin{pmatrix} \overset{0}{\omega}_{\kappa}{}^{[cd]} - \Omega_{\kappa}{}^{[cd]} \\ + (\Delta\omega_{L})_{\kappa}{}^{[cd]} \end{pmatrix} \tag{129}$$

and convert it to matrix coefficients $\longrightarrow$

natures way ... our way

$$\overset{\lambda}{\omega}_{\kappa}{}^{a}{}_{b} = \Lambda^{a}{}_{c}\,\widetilde{\Lambda}^{d}{}_{b}\left(\begin{array}{c}\overset{0}{\omega}_{\kappa}{}^{c}{}_{d} - \Omega_{\kappa}{}^{c}{}_{d} \\ + (\Delta\,\omega_{L})_{\kappa}{}^{c}{}_{d}\end{array}\right) \tag{130}$$

$$\widetilde{\Lambda}^{d}{}_{b} = \eta^{d\,m}\,\Lambda^{n}{}_{m}\,\eta_{n\,b} = (\Lambda^{-1})^{d}{}_{b}$$

Eq. (130) abbreviates to matrix notation

$$\overset{\lambda}{\omega}_{\kappa} = \Lambda\left(\begin{array}{c}\overset{0}{\omega}_{\kappa} - \Omega_{\kappa} \\ + (\Delta\,\omega_{L})_{\kappa}\end{array}\right)\Lambda^{-1} \tag{131}$$

We recall the form of Ω_{κ} defined in eq. (122) →

natures way ... our way

$$\Omega_{\tau} \; : \; (\Omega_{\tau})^{a}{}_{b} = (\Lambda^{-1})^{a}{}_{c} \, \partial_{\tau} \Lambda^{c}{}_{b}$$
$$\Omega_{\kappa} = \Lambda^{-1} \partial_{\kappa} \Lambda \tag{132}$$

Substituting eq. (132) into eq. (131) we obtain

$$\overset{\lambda}{\omega}_{\kappa} = \Lambda \left(\begin{array}{c} \overset{0}{\omega}_{\kappa} + \partial_{\kappa} \\ + (\Delta \omega_{L})_{\kappa} \end{array} \right) \Lambda^{-1} \tag{133}$$

The first line in bracket in eq. (133) shows the local gauge transformation pertinent to the spin connection(s) with gauge group the Lorentz group $(\{ \Lambda (x) \})$.

natures way ... our way

Next we recall eqs. (124 and 123) , reproduced below, turning towards the quantity $(\, \Delta \, \omega \, _{L} \,) \, _{\kappa}$

$$\overset{\lambda}{\omega} \, _{\kappa} \, ^{[\, a \, b \,]} = \Lambda \, ^{a} \, _{c} \, \Lambda \, ^{b} \, _{d} \, \overset{\lambda}{\omega} \, _{\kappa} \, ^{\prime \; [\, c \, d \,]}$$

$$\overset{\lambda}{\omega} \, _{\kappa} \, ^{\prime \; [\, c \, d \,]} = \tag{134}$$

$$= \frac{1}{2} \left[\begin{array}{c} \widetilde{\zeta} \, ^{\varrho \, c} \, D \, ^{d} \, _{[\, \varrho \, \kappa \,]} - \widetilde{\zeta} \, ^{\varrho \, d} \, D \, ^{c} \, _{[\, \varrho \, \kappa \,]} \\ + \, \widetilde{e} \, _{m \, \kappa} \, \widetilde{\zeta} \, ^{\varrho \, c} \, \widetilde{\zeta} \, ^{\sigma \, d} \, D \, ^{m} \, _{[\, \varrho \, \sigma \,]} \end{array} \right]$$

$$D \, ^{c} \, _{[\, \sigma \, \tau \,]} \, (\, L \,) = \tag{135}$$

$$= [\, (\, \partial \, _{\tau} \, L \,) \, e \, ^{c} \, _{\sigma} - (\, \partial \, _{\sigma} \, L \,) \, e \, ^{c} \, _{\tau} \,]$$

natures way ... our way

Substituting eq. (135) in eq. (134) we obtain

$$
\begin{aligned}
&(\Delta \, \omega_{L})_{\kappa}{}^{[c d]} = \\
&= \frac{1}{2} \left[\begin{array}{c}
\widetilde{\zeta}^{\varrho c} \left(\begin{array}{c} e^{d}{}_{\varrho} \, \partial_{\kappa} L - \\ - e^{d}{}_{\kappa} \, \partial_{\varrho} L \end{array} \right) \\
- \widetilde{\zeta}^{\varrho d} \left(\begin{array}{c} e^{c}{}_{\varrho} \, \partial_{\kappa} L - \\ - e^{c}{}_{\kappa} \, \partial_{\varrho} L \end{array} \right) \\
+ \widetilde{e}_{m \kappa} \, \widetilde{\zeta}^{\varrho c} \, \widetilde{\zeta}^{\sigma d} \left(\begin{array}{c} e^{m}{}_{\varrho} \, \partial_{\sigma} L - \\ - e^{m}{}_{\sigma} \, \partial_{\varrho} L \end{array} \right)
\end{array} \right]
\end{aligned}
\tag{136}
$$

natures way ... our way

Using the tangent space components of the gradient of L

$$L^{c} = \tilde{\zeta}^{\varrho c} \partial_{\varrho} L \tag{137}$$

eq. (136) reduces to

$$(\Delta \omega_{L})_{\kappa}{}^{[cd]} = \left[e^{c}{}_{\kappa} L^{d} - e^{d}{}_{\kappa} L^{c} \right] \tag{138}$$

We compare with eq. (116) reproduced below

$$\frac{1}{2} \left(\overset{\lambda}{\gamma}_{\kappa \alpha \sigma} + \overset{\lambda}{\gamma}_{\kappa \sigma \alpha} \right) = g_{\alpha \sigma} \partial_{\kappa} L$$
$$\rightarrow \quad \overset{\lambda}{\gamma}_{\{\kappa}{}^{\sigma}{}_{\sigma\}} = D \, \partial_{\kappa} L = \partial_{\kappa} \log \lambda^{D} \tag{139}$$

natures way ... our way

To this end we transform $\Delta\,\omega_L$ to the same type of indices as $\overset{\lambda}{\gamma}{}_{\{\kappa}{}^{\sigma}{}_{\sigma\}}$ in eq. (116 = 139)

$$
\begin{aligned}
(\Delta\,\omega_L)_{\kappa}{}^{\varrho}{}_{\sigma} &= \zeta^{\varrho}{}_{c}\,\widetilde{e}_{d\,\sigma}\,(\Delta\,\omega_L)_{\kappa}{}^{[c\,d]} \\
&= \delta_{\kappa}{}^{\varrho}\,\partial_{\sigma}\,L - g_{\kappa\,\sigma}\,g^{\varrho\,\tau}\,\partial_{\tau}\,L \quad \rightarrow \qquad (140) \\
(\Delta\,\omega_L)_{\kappa}{}^{\kappa}{}_{\sigma} &= (D-1)\,\partial_{\sigma}\,L
\end{aligned}
$$

From eqs. (138 and 140) the tensorial nature of $\Delta\,\omega_L$ becomes manifest [a] .

[a] **Yet I would not have guessed the factor $D - 1$ in the last relation in eq. (140) ...**

natures way ... our way

References

[1] Wolfgang Pauli, 'Relativitätstheorie', new edition of the original edition, "Encyclopädie der Wissenschaften", Vol. V, Art. 19 , 1921, Paolo Boringhieri, ed. , 1953 .

[2] Élie Cartan, 'Sur les variétés à connexion projective', Bull. Soc. Math. de France, 52 2 (1923) 205 .

[3] Élie Cartan, 'Sur une classe remarquable d'espaces de Riemann', Bull. Soc. Math. de France, 54 (1926) 214 , and 55 (1927) 114 .

[4] P. Debye and P. Scherrer, 'Kristallpulver', Gött. Nachr. (1916) 1 , 'Flüssigkeiten', Gött. Nachr. (1916) 16 and Phys. Z. 17 (1916) 277 .

[5] P. Minkowski, 'On the hypothesis that curvature freezes a set of space-like variables beyond the observed four at energies much below the Planck mass', Bern University preprint 17/1977, October 1977, unpublished ; see URL – http://www.mink.itp.unibe.ch/ .

[6] L. Smolin, 'Towards a theory of space-time structure at very short distances', HUTP-79/A010, April 1979, Nucl.Phys.B160 (1979) 253 .

[7] S. Kobayashi and K. Nomizu, 'Foundations of differential geometry', Interscience tracts in pure and applied mathematics, No. 15, Vols. I, II, L. Bers, R. Courant and J.J. Stoker eds., John Wiley and Sons, New York 1963,1969 .

[8] Charles Ehresmann – oeuvres complètes et commentées, 'Topologie algébrique et géometrie différentielle', parties I-1 et I-2, suppléments No. 1 et 2 au Vol. XXIV (1983) des 'Cahiers de topologie et géometrie différentielle', A.C. Ehresmann ed., Imprimerie Evrard, Amiens 1984 .

[9] **H. Weyl, 'Raum Zeit Materie', Julius Springer Verlag, Berlin 1923 .**