IL NUOVO CIMENTO VOL. 1 A, N. 1 1 Gennaio 1971

Regge Parametrization for Low-Energy $\pi\mathcal{N}$ Scattering.

P. MINKOWSKI (*)
SIN - Zürich

F. HALZEN (**)
CERN - Geneva

(ricevuto il 1° Giugno 1970)

Summary. — It is shown that the complementary aspect of high-energy Regge behaviour and low-energy resonance saturation known as « duality » from finite-energy sum rules can be advanced in a unifying direction. The Regge parametrization of low-energy partial waves is deduced reversing the path leading to asymptotic Regge behaviour in the crossed channel by means of the Khuri-Jones representation. The P_{11} and S_{11} waves in $\pi\mathcal{N}$ scattering are analysed. They show evidence for a pattern of trajectories characteristic for a model of oscillatory excitations.

1. – Introduction.

In Regge-pole theory the oscillatory band of resonances described by a linear angular momentum (J) — (mass)^2(s) relation is extended to interpolating, unphysical values of the angular momentum.

$\mathcal{N}_{\alpha,\beta,\gamma,\delta}$ denote the four trajectories

$$\alpha:\quad J^P = \tfrac{1}{2}^+,\ \tfrac{5}{2}^+,\ \tfrac{9}{2}^+,\ \ldots,$$
$$\beta:\quad J^P = \tfrac{1}{2}^-,\ \tfrac{5}{2}^-,\ \tfrac{9}{2}^-,\ \ldots,$$
$$\gamma:\quad J^P = \tfrac{3}{2}^-,\ \tfrac{7}{2}^-,\ \tfrac{11}{2}^-,\ \ldots,$$
$$\delta:\quad J^P = \tfrac{3}{2}^+,\ \tfrac{7}{2}^+,\ \tfrac{11}{2}^+,\ \ldots.$$

(*) Schweizerisches Institut für Nuklearforschung, Hüttenstrasse 34, 8006 Zürich.
(**) Aspirant NFWO (Belgium), on leave of absence from the University of Leuven, Leuven.

Dispersion relations provide the necessary tool (1) to carry out the above continuation in the s-channel scattering region corresponding to Fig. 1 *a*).

We try to present evidence here that the path to the backward-scattering region in the u-channel (Fig. 1*b*): $u \to +\infty$, $s < (m - \mu)^2$, m, μ the nucleon, pion

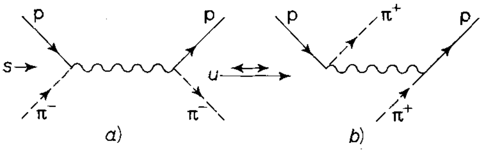


Fig. 1. – Baryon Regge exchange *a*) in the s-channel of π^-p scattering and *b*) in the u-channel of π^+p scattering.

mass respectively) is reversible. It relates leading as well as nonleading asymptotic Regge contributions to corresponding particle families and vice versa.

Thus we assume that background terms (cuts, strongly decreasing contributions to the amplitude) do not affect the physical parameters of resonances on a given trajectory.

However, a conventional Regge pole α does not provide a suitable resonance saturation model at low energies in the s-channel because of the following obvious technical difficulties:

parity doubling,

incorrect threshold behaviour,

incorrect cut structure.

The latter two difficulties are coupled and can be overcome in many different ways. The most appealing one is known as the Khuri-Jones representation (2). Our purpose is to study the first difficulty in this context and to show the power of the so obtained Regge amplitude describing in one low-energy phase shifts and high-energy scattering in the crossed channel.

This twofold aspect of Regge exchange enables one to study the low-lying trajectories in low-energy scattering in connection with the hypotheses of daughter trajectories, dual models, etc.

We want to emphasize in this paper that the low-energy aspect does emerge from $\pi\mathcal{N}$ phase shifts. This opens the interesting possibility to parametrize low-energy data by Regge parameters and *eo ipso* by resonance characteristics.

(1) M. FROISSART: *Nuovo Cimento*, **22**, 191 (1961); *Phys. Rev.*, **123**, 1053 (1961); V. N. GRIBOV: *Žurn. Èksp. Teor. Fiz.*, **41**, 677 (1961), English translation, *Sov. Phys. JETP*, **14**, 478 (1962).

(2) N. N. KHURI: *Phys. Rev.*, **130**, 429 (1963); E. JONES: Lawrence Radiation Laboratory Report UCRL-10700 (1963).

The experimental situation in $I = J = \frac{1}{2}$ ($J^P = \frac{1}{2}^{\pm}$) $\pi\mathcal{N}$ scattering provides a testing ground for the above project [3,4].

We take the following P_{11} and S_{11} resonances to mark the beginning of two corresponding series of parallel linear trajectories ($\mathcal{N}_\alpha$, $\mathcal{N}_\beta$) with separate and independent equal spacings Δ_P, Δ_S in the Chew-Frautschi plot [5].

The above analysis also extends to the recurrences of the $J = \frac{1}{2}$ resonances, *i.e.* to $\mathcal{N}^*_{\frac{5}{2}+}$ (1688) and to a hypothetical recurrence on the S_1 trajectory $\mathcal{N}^*_{\frac{5}{2}-}$ (2135). Further tests will thus be provided by an extension of $\pi\mathcal{N}$ phase-shift analysis above 2 GeV which on the other hand will require a reconsideration of resonance phenomena at these energies [*].

In Sect. **2** we construct a Khuri-Regge exchange amplitude consistent with general requirements and connect it to the low-energy P_{11}, S_{11} phases on the one hand, to π^+p high-energy backward scattering on the other hand.

In Sect. **3** the numerical analysis is discussed and the results are presented. The former can proceed along different lines:

a) After fitting $(\mathrm{d}\sigma/\mathrm{d}s)_{\pi^+p/s=0}$ at the available energies we have shown how to determine the parameters of the Roper resonance from low-energy P_{11} phase shifts [3].

b) In the simultaneous analysis of the MacDowell coupled P_{11}-S_{11} channels we fix the masses and widths of the resonances in Table I at their physical values and predict the phase shifts. A true prediction, however, would imply the knowledge of mass and width of the hypothetical $\mathcal{N}^*_{\frac{5}{2}-}$(2135) recurrence to determine an otherwise free parameter.

TABLE I. – *Patterns of $\mathcal{N}_\alpha$, $\mathcal{N}_\beta$ trajectories as observed in phase-shift analysis.* Ambiguities in the elastic widths arise from errors in both elasticity and total width [5].

Resonance	J^P	Elastic width (MeV)	Notation	Trajectory function
$\mathcal{N}$(938)	$\frac{1}{2}^+$	$[g^2/4\pi \simeq 14.6]$	P_1	$\alpha_{P_1} = \alpha^0_P + \alpha' s$
$\mathcal{N}^*$(1420)	$\frac{1}{2}^+$	~ 143	P_2	$\alpha_{P_2} = \alpha^0_P + \alpha' s - \Delta_P$
$\mathcal{N}^*$(1750)	$\frac{1}{2}^+$	~ 175	P_3	$\alpha_{P_3} = \alpha^0_P + \alpha' s - 2\Delta_P$
$\mathcal{N}^*$(1535)	$\frac{1}{2}^-$	~ 28	S_1	$\alpha_{S_1} = \alpha^0_S + \alpha' s$
$\mathcal{N}^*$(1700)	$\frac{1}{2}^-$	~ 182	S_2	$\alpha_{S_2} = \alpha^0_S + \alpha' s - \Delta_S$

(3) F. HALZEN and P. MINKOWSKI: *Lett. Nuovo Cimento*, **1**, 789 (1969).
(4) L. DURAND: Invited paper presented at the *Boulder Conference on High-Energy Physics, August 1969.*
(5) PARTICLE DATA GROUP: *Rev. Mod. Phys.*, **42**, 87 (1970).
(*) The newest results of the phase-shift analysis up to 2.8 GeV [6] seem to indicate the presence of further excitations in P_{11} and S_{11} as well as D_{15} ($\mathcal{N}^*_{\frac{5}{2}}$-(2135?)). However, a thorough analysis is needed to classify the loops into our pattern.
(6) R. AYED, P. BAREYRE and G. VILLET: *Phys. Lett.*, **31** B, 598 (1970).

c) Finally we vary the resonance parameters allowing for errors in their determination from phase-shift analysis within reasonable limits. This enables us to reproduce the $(0 \div 350)$ MeV solution of ROPER *et al.* (7) to a high accuracy (mean deviation $\sim 5\,\%$) in the region $(6 \div 240)$ MeV pion laboratory kinetic energy. We have chosen the phases of ROPER *et al.* (7) because this analysis aims at maximal likelihood in the region of interest to us.

In the last Section we give some thought to MacDowell symmetry, the oscillator models of Regge trajectories and the exponential behaviour of Regge residues.

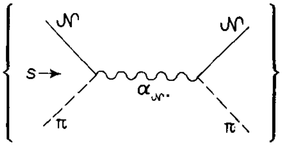
We hope to have demonstrated that it is worth-while to analyse low-energy data with Regge parameters even more so if inelastic effects could be included.

The extension to other spins and isospins is straightforward.

2. – MacDowell symmetric Khuri amplitude without parity doublets.

As pointed out in the Introduction we want to construct an amplitude for $\pi\mathcal{N}$ scattering corresponding to a summation of Regge exchange according to the trajectories given in Table I and Fig. 2:

(2.1) $$A = \sum_{\alpha_{\mathcal{N}^*}}$$



We restrict the discussion to the case of $I = J = \frac{1}{2}$ exchanges:

(2.2*a*) $$\alpha_{P_{i+1}} = \alpha_P^0 + \alpha' s - i\Delta_P\,, \qquad \alpha_{S_{i+1}} = \alpha_S^0 + \alpha' s - i\Delta_S\,, \qquad i = 0, 1, 2, \ldots\,.$$

For backward $\pi^+ p$ scattering $(u \to +\infty, s < (m - \mu)^2)$ we make the assumption that Δ exchange is suppressed. This is justified by isospin coefficients and by the fact that the $\pi^- p$ backward cross-section (pure Δ exchange) is experimentally an order of magnitude smaller than the $\pi^+ p$ backward cross-section.

From the Argand diagrams of P_{11}-S_{11} we conclude that

(2.2*b*) $$\Delta_P \simeq 2\Delta_S \simeq 1\,.$$

This regularity originally gave the impulse to our analysis and it will be given some thought in the last Section.

---

(7) L. D. ROPER, R. M. WRIGHT and B. T. FELD: *Phys. Rev.*, **138**, B 190 (1965).

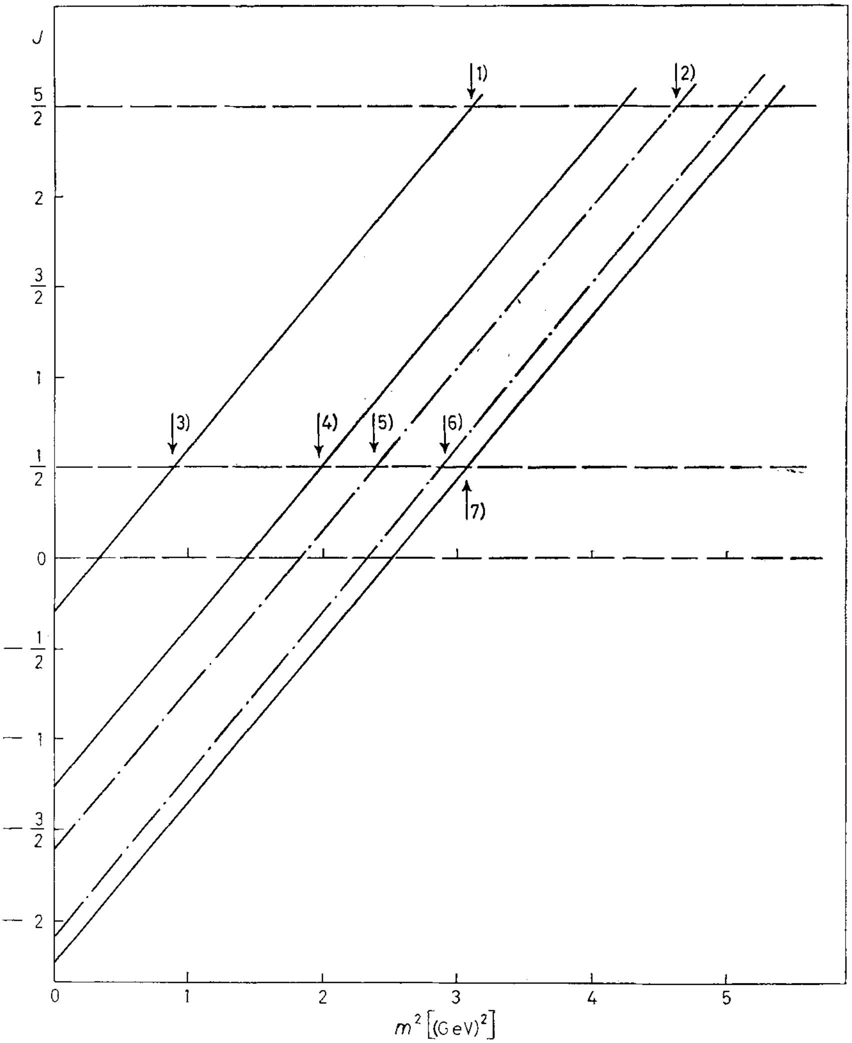


Fig. 2. – Chew-Frautschi plot emerging from $I = J = \frac{1}{2}$ resonances: —— P_{11} resonances; —·—·— S_{11} resonances. 1) $\mathcal{N}_{\frac{5}{2}^+}(1688)$, 2) $\mathcal{N}_{\frac{5}{2}^-}(2135)$, 3) $\mathcal{N}(938)$, 4) $\mathcal{N}(1400)$, 5) $\mathcal{N}(1535)$, 6) $\mathcal{N}(1700)$, 7) $\mathcal{N}(1705)$.

We require the following properties for the amplitude A:

i) explicit MacDowell symmetry and absence of parity doublets ([8]);

ii) correct cut structure in the s, t, u plane and correct threshold behaviour;

([8]) For a general discussion of this problem see *e.g.* H. Högaasen: Invited paper presented at the *Topical Conference on High-Energy Collisions of Hadrons, CERN* (Geneva, 1968).

iii) display of the trajectory pattern according to Table I and description of $I = J = \frac{1}{2}$ resonances in the zero-width approximation;

iv) correct description of high-energy backward π^+p scattering ($u \to \infty$, $s = 0$) by the leading nucleon trajectory (α_{P_1}).

The amplitude A contains information on the trajectory pattern in two ways:

a) by i)-iii) its properties are related to low-energy phase shifts (*) after partial-wave projection;

b) by iv) A is related to conventional analysis of asymptotic (high-energy) scattering.

Whereas Regge-pole parametrization is in general accepted in case *b*) we want to emphasize its importance for studying low-energy $\pi\mathcal{N}$ scattering. Equation (2.1) provides a means to parametrize low-energy phase shifts with physically relevant parameters.

The kinematics of $I_s = \frac{1}{2}$ $\pi\mathcal{N}$ scattering is described by

$$A = f_1 + (\hat{p}'\sigma)\cdot(\hat{p}\sigma) f_2 ,$$

where $\hat{p}$, $\hat{p}'$ represent unit vectors in the direction of centre-of-mass incoming and outgoing nucleon momenta. The partial-wave amplitudes are defined in the conventional way:

$$\left\{\begin{aligned} f_1(W, z) &= \sum_J \{t_j^+(W) P'_{J+\frac{1}{2}}(z) - t_j^-(W) P'_{J-\frac{1}{2}}(z)\} , \\ f_2(W, z) &= \sum_J \{t_j^-(W) P'_{J+\frac{1}{2}}(z) - t_j^+(W) P'_{J-\frac{1}{2}}(z)\} . \end{aligned}\right. \tag{2.3}$$

MacDowell symmetry reads

$$f_1(W, z) = -f_2(-W, z) , \qquad \text{or alternatively} \qquad t_j^+(-W) = -t_j^-(W) . \tag{2.4}$$

We use the conventional notations $z = \cos\theta s$, $W = \sqrt{s}$. $t_j^\pm$ are the partial-wave amplitudes distinguishing the possibilities $j = l \pm \frac{1}{2}$ due to nucleon spin. They are given by

$$\left\{\begin{aligned} t_j^+ &= \tfrac{1}{2}\int_{-1}^{+1} dz (f_1 \cdot P_{J-\frac{1}{2}} + f_2 \cdot P_{J+\frac{1}{2}}) , \\ t_j^- &= \tfrac{1}{2}\int_{-1}^{+1} dz (f_1 \cdot P_{J+\frac{1}{2}} + f_2 \cdot P_{J-\frac{1}{2}}) . \end{aligned}\right. \tag{2.5}$$

(*) Low-energy region means threshold region here where the phase shifts are small: $\delta_{I,J} \simeq \sin\delta_{I,J} \simeq \operatorname{tg}\delta_{I,J}$.

The P_i states correspond to $l = 1, 3, \ldots$ and are on positive-parity trajectories coupling t_j^-, the S_i states correspond to $l = 0, 2, \ldots$ or negative parity, they couple t_j^+. Both trajectories have positive signature. Performing a conventional Sommerfeld-Watson transformation on (2.3) we obtain neglecting the background integral

$$(2.6)\quad \begin{cases} f_1 = -\sum\limits_m \tau_{S_m}(\alpha_{S_m}, W)\cdot P'_{\alpha_{S_m}+\frac{1}{2}} \dfrac{\pi}{\sin(\pi[\alpha_{S_m}-\frac{1}{2}])} + \\ \qquad\qquad + \sum\limits_n \tau_{P_n}(\alpha_{P_n}, W)\cdot P'_{\alpha_{P_n}-\frac{1}{2}} \dfrac{\pi}{\sin(\pi[\alpha_{P_n}-\frac{1}{2}])}, \\ f_2 = -\sum\limits_n \tau_{P_n}(\alpha_{P_n}, W)\, P'_{\alpha_{P_n}+\frac{1}{2}} \dfrac{\pi}{\sin(\pi[\alpha_{P_n}-\frac{1}{2}])} + \\ \qquad\qquad + \sum\limits_m \tau_{S_m}(\alpha_{S_m}, W)\, P'_{\alpha_{S_m}-\frac{1}{2}} \dfrac{\pi}{\sin(\pi[\alpha_{S_m}-\frac{1}{2}])}, \end{cases}$$

with

$$\tau_{P_n}(\alpha_{P_n}[s], W) = \operatorname*{residue}_{J\to\alpha_{P_n}(s)} t_J^-(W)\,,$$

$$\tau_{S_m}(\alpha_{S_m}[s], W) = \operatorname*{residue}_{J\to\alpha_{S_m}(s)} t_J^+(W)\,.$$

Positive signature of P_i, S_i is incorporated by the replacements

$$P'_{\alpha+\frac{1}{2}}(z) \to \tfrac{1}{2}\,[P'_{\alpha+\frac{1}{2}}(z) + P'_{\alpha+\frac{1}{2}}(-z)]\,,$$

$$P'_{\alpha-\frac{1}{2}}(z) \to \tfrac{1}{2}\,[P'_{\alpha-\frac{1}{2}}(z) - P'_{\alpha-\frac{1}{2}}(-z)]\,.$$

Relations (2.4) induced by MacDowell symmetry would require

$$\alpha_{P_n}(s) = \alpha_{S_n}(s)$$

and

$$\tau_{P_n}(\alpha_{P_n}, -W) = -\tau_{S_n}(\alpha, W)\,.$$

As is manifest in Table I and Fig. 2, however, the physical P_i and S_i are far from being parity doubled. We enforce manifest MacDowell symmetry in two steps. We first introduce the MacDowell reflected parts of both P_i and S_i trajectories so that the symmetry of f_1, f_2 is explicit and then prevent the reflected trajectories from materializing.

Thus

$$(2.7)\quad f_2 = -\sum_n \left\{\tau_{P_n}(\alpha_{P_n}, W)\, P'_{\alpha_{P_n}+\frac{1}{2}} \frac{\pi}{\sin(\pi[\alpha_{P_n}-\frac{1}{2}])} + \right. \\ \left. + \tau_{P_n}(\alpha_{P_n}, -W)\, P'_{\alpha_{P_n}-\frac{1}{2}} \frac{\pi}{\sin(\pi[\alpha_{P_n}-\frac{1}{2}])}\right\} + \\ + \sum_m \left\{\tau_{S_m}(\alpha_{S_m}, W)\, P'_{\alpha_{S_m}-\frac{1}{2}} \frac{\pi}{\sin(\pi[\alpha_{S_m}-\frac{1}{2}])} + \tau_{S_m}(\alpha_{S_m}, -W)\, P'_{\alpha_{S_m}+\frac{1}{2}} \frac{\pi}{\sin(\pi[\alpha_{S_m}-\frac{1}{2}])}\right\}.$$

f_1 is obtained by the replacement

$$P_n \leftrightarrow S_n\,, \qquad \tau_{P_n} \leftrightarrow \tau_{S_n}\,.$$

In order to prevent the parity-doubled trajectories from materializing, the relevant residue functions must develop an organized set of zeros:

$$\left.\begin{aligned} \tau_{P_n}(\alpha_{P_n}, -W) &= 0 \qquad \text{for} \qquad \alpha_{P_n}(W^2) = \tfrac{1}{2} + 2i\,, \\ \tau_{S_m}(\alpha_{S_m}, -W) &= 0 \qquad \text{for} \qquad \alpha_{S_m}(W^2) = \tfrac{1}{2} + 2i\,, \end{aligned}\right\} \quad i = 0, 1, 2, \ldots\,.$$

In Regge fits it was customary to introduce the first few zeros of τ explicitly and to neglect all other conditions (9).

One of us discussed the class of functions consistent with general requirements achieving the above systematic cancellation (10).

Two possible forms for the functions for P_n, S_m are

$$(2.8) \qquad \tau(\alpha, W) = \begin{cases} \left[\prod_{J=0}^{\infty}\left(1 + \left(\dfrac{W}{W^J}\right)^3\right)\right]\cdot \varrho_1(W)\,, \\ \left[\prod_{J=0}^{\infty}\left(1 + \dfrac{W}{W^J}\right)\exp\left[\dfrac{1}{2}\left(\dfrac{W}{W^J}\right)^2 - \dfrac{W}{W^J}\right]\right]\cdot \varrho_2(W)\,, \end{cases} \qquad W^J = \left(\frac{2J + \frac{1}{2} - \alpha_0}{\alpha'}\right)^{\frac{1}{2}}.$$

An alternative mechanism to avoid parity doublets to appear as physical particles has recently been discussed by CARLITZ and KISLINGER (11).

We now work towards correct threshold behaviour and correct left- and right-hand cuts ($u \to$ l.h.c., $t \to$ r.h.c.).

First the residue functions are redefined:

$$(2.9) \qquad \begin{cases} \tau_{P_n} = \dfrac{E - m}{2W}\left(\dfrac{q^2}{q_0^2}\right)^{\alpha_{P_n} - \frac{1}{2}}\left[\prod_J \left(1 + \left(\dfrac{W}{W^J_{P_n}}\right)^3\right)\right]\varrho_{P_n}(W)\,, \\ \tau_{S_m} = \dfrac{E + m}{2W}\left(\dfrac{q^2}{q_0^2}\right)^{\alpha_{S_m} - \frac{1}{2}}\left[\prod_{J'} \left(1 + \left(\dfrac{W}{W^{J'}_{S_m}}\right)^3\right)\right]\varrho_{S_m}(W)\,. \end{cases}$$

Here q is the centre-of-mass momentum, q_0 is a unit momentum which will

(9) V. BARGER and C. CLINE: *Phys. Rev.*, **155**, 1792 (1967).
(10) P. MINKOWSKI: *Lett. Nuovo Cimento*, **3**, 503 (1970).
(11) R. CARLITZ and M. KISLINGER: *Phys. Rev. Lett.*, **24**, 186 (1970).

later be chosen 1 GeV/c. The particle masses are given by

$$W^J_{P_n} = \left(\frac{2J + \frac{1}{2} + (n-1)\Delta_P - \alpha^0_P}{\alpha'}\right)^{\frac{1}{2}},$$

$$W^{J'}_{S_m} = \left(\frac{2J' + \frac{1}{2} + (m-1)\Delta_S - \alpha^0_S}{\alpha'}\right)^{\frac{1}{2}},$$

$$n, m = 1, 2, \dots, \qquad J, J' = 0, 1, 2, \dots .$$

(n, m) label trajectories, (J, J') recurrences.

In (2.9) $\varrho(W)$ is the core of the residue which accompanies the asymptotic expansion of Legendre functions

$$\frac{\pi}{\sin(\pi(\alpha - \frac{1}{2}))} \frac{1}{2}\left[\left(\frac{u}{u_0}\right)^{\alpha-\frac{1}{2}} \pm \left(\frac{-u}{u_0}\right)^{\alpha-\frac{1}{2}}\right].$$

The amplitudes inherit the cuts from the Legendre functions beginning at $z = \pm 1$ and extending to $\pm\infty$. This is only asymptotically correct $(s \to \infty)$.

The lower ends should be shifted:

$$\text{r.h.c.:} \qquad z = 1 \to 1 + \frac{2\mu^2}{q^2} = \cosh \xi_1 ,$$

$$\text{l.h.c.:} \qquad -z = +1 \to -1 + \frac{s - m^2 - 2\mu^2}{2q^2} = \cosh \xi_2 .$$

A prescription to achieve this has been given by KHURI [2]:

$$P_\alpha(z) \to P^{\xi_1}_\alpha(z) ,$$

$$P_\alpha(-z) \to P^{\xi_2}_\alpha(-z) ,$$

$$P^{\xi_i}_\alpha(\pm z) = -\frac{\sin \pi\alpha}{\pi} \frac{1}{2\alpha + 1} \frac{1}{\sqrt{2}} \int\limits_{\xi_i}^{\infty} dx \frac{\exp[(\alpha + \frac{1}{2})x] \sinh x}{(\cosh x \pm z)^{\frac{3}{2}}} .$$

Upper and lower signs refer to $i = 1, 2$ respectively. For $u \to +\infty$, P_{ξ_i} behave as $P_\alpha(\pm z)$ respectively [12].

---

[12] For a short description of the Khuri-Jones representation see, *e.g.*, P. D. B. COLLINS and E. J. SQUIRES: *Regge Poles in Particle Physics*, *Springer Tracts in Modern Physics*, Vol. **45** (Heidelberg, New York, 1968).

Let us follow now the procedure for a typical term, say P_1, *i.e.* the nucleon,

$$f_1 = \begin{cases} -\tau(\alpha, -W)\dfrac{1}{2}[P'^{\xi_1}_{\alpha+\frac{1}{2}}(z) + P'_{\alpha-\frac{1}{2}}(-z)]\dfrac{\pi}{\sin(\pi[\alpha-\frac{1}{2}])}, \\ -\tau(\alpha, W)\dfrac{1}{2}[P'^{\xi_1}_{\alpha-\frac{1}{2}}(z) - P'^{\xi_2}_{\alpha-\frac{1}{2}}(-z)]\dfrac{\pi}{\sin(\pi[\alpha-\frac{1}{2}])}, \end{cases}$$

f_2 is given by MacDowell symmetry.

Using (2.5) the partial-wave projections take the simple form

$$t_J^- = \tau(\alpha, W)\frac{F^{\xi_1,\xi_2}_{J,\alpha}}{J-\alpha},$$

$$F^{\xi_1,\xi_2}_{J,\alpha} = \tfrac{1}{2}[\exp[(\alpha-J)\xi_1] + \exp[(\alpha-J)\xi_2]],$$

and the final partial-wave amplitudes read

$$(2.10)\qquad t_J^-(W) = \begin{cases} \sum\limits_n \tau_{P_n}(\alpha_{P_n}, W)\dfrac{F^{\xi_1,\xi_2}_{J,\alpha_{P_n}}}{J-\alpha_{P_n}}, \\ -\sum\limits_m \tau_{S_m}(\alpha_{S_m}, -W)\dfrac{F^{\xi_1,\xi_2}_{J,\alpha_{S_m}}}{J-\alpha_{S_m}}, \end{cases}$$

$$t_J^+(W) = -t_j^-(-W), \qquad J = \tfrac{1}{2}\ \tfrac{5}{2}\ \tfrac{9}{2}, \ldots,$$

with τ_{P_n}, τ_{S_m} given by (2.9).

3. – Interpretation of the Khuri amplitude and numerical evaluation.

We now want to emphasize the two aspects of the $\pi\mathcal{N}$ amplitude constructed in the preceding Section: its low-energy aspect or relation to the elastic couplings of the particles on the trajectories and its high-energy aspect or relation to π^+p backward scattering.

3·1. *Low energy.* – We believe the partial-wave projections (2.10) to be valid in the low-energy region. They contain the poles from particles on the trajectories in the narrow-width approximation in the spirit of the Khuri-Jones representation.

As we have no control over inelasticities $\eta_j^\pm$ we identify (2.10) with

$$(3.1)\qquad t_J^\pm = \frac{1}{2iq}(\eta_j^\pm \exp[2i\delta_j^\pm] - 1)$$

for $\delta_j^\pm \simeq \sin\delta_j^\pm \simeq \mathrm{tg}\,\delta_j^\pm$, which implies $\eta_j^\pm \simeq 1$.

For higher energies (2.10) is assumed to represent K-matrix elements

$$(3.2)\qquad t_j^\pm = \frac{1}{q}\mathrm{tg}\,\delta_j^\pm.$$

Thus

$$\text{(3.3)}\qquad \underset{W\to W^J_{P_n}}{\text{residue}}\, t^{\pm}_j = \frac{(\Gamma^J_{\text{el}})_{P_n}}{2q^J_{P_n}}\,, \qquad W^J_{P_n} = \left(\frac{J+(n-1)\Delta_P - \alpha^0_P}{\alpha'}\right)^{\frac{1}{2}}.$$

$q^J_{P_n}$ is the c.m. momentum corresponding to the c.m. energy $W^J_{P_n}$.

The above does not mean at all that inelastic effects are neglected. We describe the low-energy part of the scattering amplitude without assumption on extrapolations once we accept (2.10).

The following ansatz is made for the free Regge residues $\varrho(W)$ in (2.9):

$$\text{(3.4)}\qquad \varrho_{P_n}(W) = a_n\cdot\varrho_P(W)\,, \qquad \varrho_{S_m}(W) = b_m\cdot\varrho_S(W)\,,$$

with

$$\varrho_{P,S}(W) = \exp\left[\varphi_{P,S}(W)\right].$$

ϱ_P, ϱ_S are supposed to govern universally the contributions from all P_i, S_i. The relative coupling strengths are given by the proportionality constants a_n, b_m which are proportional to the elastic widths of the $J=\frac{1}{2}$ particles on P_i or S_i respectively.

The exponential behaviour of $\varrho_{P,S}(W)$ will be discussed in the next Section. It is a generally accepted procedure to parametrize the smooth core of the Regge residue by $\varrho_{P,S} = \exp\left[C_1 W^2 - C_2 W\right]$.

The a_n, b_m are readily connected to particle couplings to $\pi\mathcal{N}$ combining (2.10) and (3.13). To illustrate this we can write for instance the nucleon contribution to P_{11}

$$t^-_{\frac{1}{2}} = \frac{E-m}{2W}\,\varrho_P(W)\left\{\prod_J\left(1+\left(\frac{W}{W^J_{P_1}}\right)^3\right)\right\}\left(\frac{q^2}{q^2_0}\right)^{\alpha_{P_1}-\frac{1}{2}} F^{\xi_1,\xi_2}_{\frac{1}{2},\alpha_{P_1}}\,\frac{a_1}{J-\alpha_{P_1}} + \begin{pmatrix}\text{terms regular}\\ \text{at } W=m\end{pmatrix}.$$

3·2. *High energy.* – In the spirit of conventional Regge asymptotics (2.10) describes π^+p backward scattering for $s=0$, $u\to\infty$. We neglect the $I=\frac{3}{2}$ contribution. As we fit only in the strict $s=0$ direction cuts can safely be ignored.

The amplitudes f_1, f_2 given in (2.3) are Reggeized using the Mandelstam prescription ([13,14])

$$P_\alpha(z) \to -\frac{\operatorname{tg}\pi\alpha}{\pi}\,Q_{-\alpha-1}(z)\,.$$

([13]) S. Mandelstam: *Ann. of Phys.*, **19**, 254 (1962).

([14]) M. Gell-Mann, M. L. Goldberger, F. E. Low, E. Marx and F. Zachariasen: *Phys. Rev.*, **133**, B 145 (1964).

The calculation of the limit $s = 0$, $u \to \infty$ is straightforward and we obtain

$$(3.5)\quad \begin{cases} f_2 \underset{\substack{s=0\\ u\to\infty}}{\simeq} \tau_\alpha(W)\, c_\alpha \tfrac{1}{2}[(2z)^{\alpha-\frac{1}{2}} + (-2z)^{\alpha-\frac{1}{2}}] + \tau_\alpha(-W)\, c_{\alpha-1} \tfrac{1}{2}[(2z)^{\alpha-\frac{3}{2}} - (-2z)^{\alpha-\frac{3}{2}}]\,, \\ \alpha = \alpha_{P_1}(s) \text{ and } c_\alpha = 2\sqrt{\pi}\, \dfrac{\Gamma(-\alpha + \frac{1}{2})}{\Gamma(-\alpha)}\,, \end{cases}$$

$$(3.6)\qquad \tau_\alpha(W) \underset{W\to 0}{\simeq} \frac{E-m}{2W}\, \varrho_P(W=0)\, a_1 \left(\frac{q^2}{q_0^2}\right)^{\alpha-\frac{1}{2}}, \qquad f_1(W) = -f_2(-W)\,.$$

In their present form (3.5) and (3.6) contain kinematical singularities due to unequal-mass kinematics and the MacDowell symmetry term.

We suppose these to be cancelled by daughter trajectories that are kinematically induced [15] in a similar way as the MacDowell reflected trajectories. They thus do not materialize into particles.

3'3. *Results.* – For numerical calculation the parity doublet erasing functions can be approximated by their lowest dynamical zeros and we made the working hypothesis that $\varphi_{P,S}(W)$ are quadratic functions:

$$(3.7)\qquad \varrho_{P,S}(W) = \exp\,[C_1^{P,S}\cdot W^2 - C_2^{P,S}\cdot W]\,.$$

We started fitting $(\mathrm{d}\sigma/\mathrm{d}s)_P$ expressed by f_1, f_2 given in (3.5) to the backward $\pi^+ p$ measurements at 3.55, 5.9, 8, 9.9, 13.7, 17.1 GeV/c [16] and obtained a satisfactory fit for

$$(3.8)\qquad \varrho_P = \exp\,[-C_P(W-m)^2]\,.$$

The numerical values for a_1, C_P so obtained are consistent with the $\pi\mathcal{N}$ coupling constant (directly related to a_1) and the elastic width of the nucleon recurrence $\mathcal{N}^*_{15}(1688)$.

For $\varrho_S(W)$ we assume that the mean energy is equivalently given by the lowest resonance

$$(3.9)\qquad \varrho_S = \exp\,[-C_S(W-m_{S_1})^2]\,.$$

---

(15) D. Z. FREEDMAN and J. M. WANG: *Phys. Rev. Lett.*, **17**, 569 (1966).

(16) For a review of the backward $\pi\mathcal{N}$ data see, *e.g.*, M. DERRICK: *Backward peaks*, review talk presented at the *Topical Conference on High-Energy Collisions of Hadrons, CERN* (Geneva, 1968), and ref. (17).

(17) F. HALZEN, A. KUMAR, A. D. MARTIN and C. MICHAEL: to be published in *Phys. Lett.*, preprint TH-1155 CERN.

Feeding in Δ_P, Δ_S as obtained from the Chew-Frautschi plot and the different widths we predict P_{11}, S_{11} phase shifts in the low-energy region (pion laboratory kinetic energy range $(6\div240)$ MeV) as a function of one parameter C_S. This parameter is related to the width of the first recurrence on the S_1 trajectory $\mathcal{N}^*_{\frac{1}{2}}$-(2135).

In practice we varied the resonance parameters within narrow limits and determined C_S looking for a best approximation to the P_{11}-S_{11} phase shifts of ROPER *et al.* (7).

The existence of a sharp minimum in χ^2, all parameters lying inside their prescribed ranges, proves the significant dependence of phase shifts on our parametrization.

This is not obvious if we recall that the low-energy region prescribed by (3.1) is still a considerable distance away from the lowest-resonance position.

3·4. *Comments.* – One can turn this procedure around and extract resonance parameters from low-energy $\pi\mathcal{N}$ data using the above parametrization. This was successfully done for the Roper resonance in ref. (3) by fitting a simplified P_{11} amplitude to the phase shifts of Lovelace.

The scattering lengths $a_{P_{11}}$, $a_{S_{11}}$ are readily calculated. Since our solution coincides with the $(0\div350)$ MeV solution of ROPER *et al.* (7) (see Table II) our results correspond to their analysis:

$$a_{P_{11}} = -0.098\,, \qquad a_{S_{11}} = 0.180\,.$$

TABLE II. – *Results of the present analysis for the* P_{11}, S_{11} *waves, as compared with the* $(0\div350)$ MeV *solutions of* ROPER *et al.* (7). T_π: pion laboratory kinetic energy.

T_π	$\delta_{P_{11}}(0)$ ROPER (7)	$\delta_{P_{11}}(0)$ present analysis	$\delta_{S_{11}}(0)$ ROPER (7)	$\delta_{S_{11}}(0)$ present analysis
6	−0.082	−0.086	2.599	2.549
20	−0.429	−0.432	4.414	4.360
31	−0.755	−0.835	5.295	5.443
58	−1.549	−1.573	6.780	6.860
98	−2.249	−2.139	8.235	8.243
120	−2.189	−2.035	8.855	8.838
140	−1.762	−1.580	9.356	9.351
170	−0.318	−0.322	10.036	9.979
194	1.643	1.707	10.542	10.539
200	2.257	2.105	10.665	10.623
220	4.680	4.559	11.070	11.048
240	7.703	7.996	11.471	11.483

Looking at the numerical analysis for the detailed structure of (2.10) building up the P_{11} and S_{11} waves from additive contributions of P_1, P_2, P_3, S_1, S_2 (see Fig 3, 4) we conclude that the S_{11} resonances are not reflected to an appreciable amount in the P_{11} channel (Fig. 3), whereas both P_{11} and S_{11} particles combine in building up the S_{11} wave (Fig. 4).

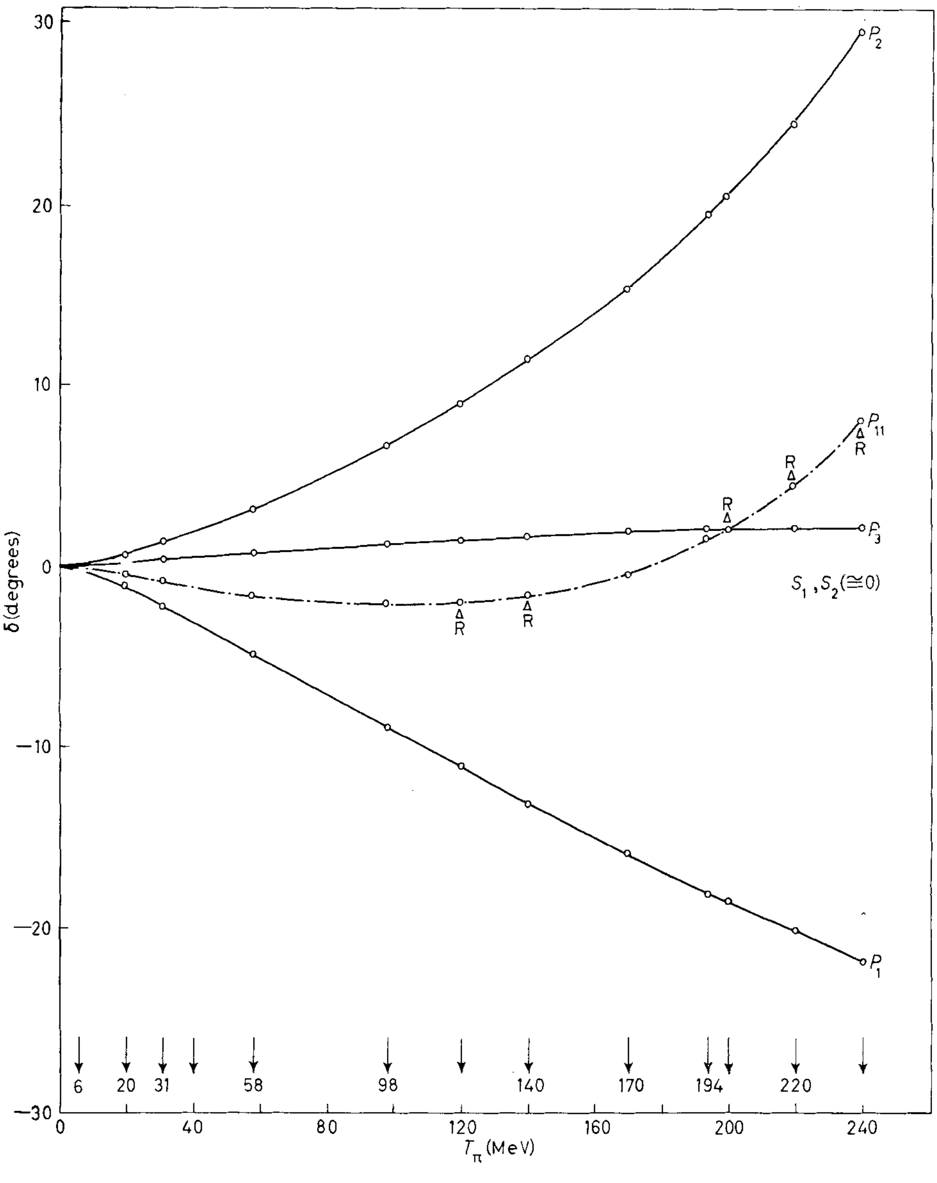


Fig. 3. – Contributions of P_1, P_2, P_3, S_1, S_2 to the P_{11} wave separately (full lines) and jointly (dot-dashed lines). The deviation from the results of ROPER *et al.* ([7]) are indicated by $\overset{\Delta}{\mathrm{R}}$ where this is possible on the given scale.

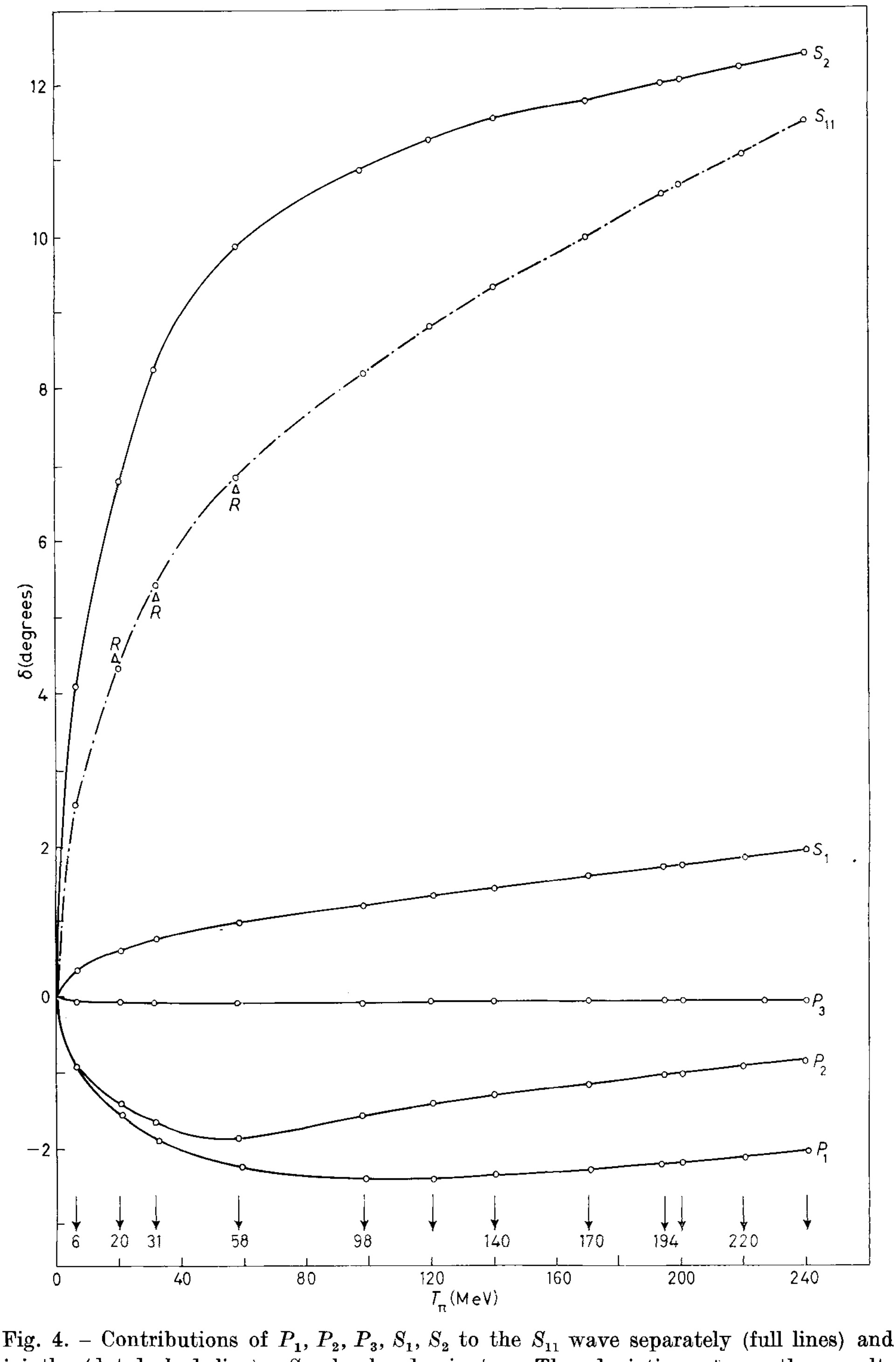


Fig. 4. – Contributions of P_1, P_2, P_3, S_1, S_2 to the S_{11} wave separately (full lines) and jointly (dot-dashed line). S_2 clearly dominates. The deviations from the results of ROPER *et al.* (7) are indicated as in Fig. 3.

If one does not accept the *ad hoc* parametrization (3.9) of ϱ_S, one can also vary m_{S_1} and relate its numerical value to widths of recurrences. The recurrence of S_1, $\mathcal{N}^*_{\frac{5}{2}}$-(2135), is predicted to couple weakly to $\pi\mathcal{N}$ from (3.9). Further information on phase shifts at higher energies will provide a tool to study the lower-lying trajectories in a more refined way.

TABLE III. – *Masses and elastic widths of the lowest resonances on the P_1, P_2, P_3, S_1, S_2 trajectories obtained from low-energy analysis compared to the results of phase-shift analysis and experiments.* The solution given in Table II corresponds to the following parameters for the relevant resonances.

Resonance	Mass (MeV)		Elastic width (MeV)	
	Result	Values from phase-shift analysis and experiments ([5])	Result	Values from phase-shift analysis and experiments ([5])
$\mathcal{N}$(938)	938 (input)	938	$g^2/4\pi = 14.6$ (input)	$g^2/4\pi = 14.6$
$\mathcal{N}$(1400) ([3,18])	1387	1435÷1505	171	120÷240
$\mathcal{N}$(1750)	1722	1750÷1860	3	90÷150
$\mathcal{N}$(1535)	1535 (input)	1500÷1600	20	16÷ 53
$\mathcal{N}$(1700)	1712	1605÷1765	100	70÷280

The displacements in angular momentum of the P_i, S_j trajectories are

$$\Delta_P = 0.94\,, \qquad \Delta_S = 0.48\,.$$

The two exponential cores of the residue functions ϱ_P, ϱ_S are given by

$$\varrho_P = \exp[-C_P(W-m)^2]\,, \qquad \varrho_S = \exp[-C_S(W-m_{1535})^2]\,,$$

$$C_P = 0.95\,, \qquad C_S = 6.00\,.$$

([18]) F. HALZEN and P. MINKOWSKI: *Nucl. Phys.*, **14** B, 522 (1969).

4. – General remarks.

In this Section we want to make a few final remarks connected with

a) parity doubling,

b) the phenomenologically observed pattern of Regge trajectories,

c) exponential behaviour of Regge residues.

a) The solution that CARLITZ and KISLINGER ([11]) propose to eliminate parity doublets moves the doubled pole to the unphysical sheet, introducing a square-root cut in the J-plane. A typical τ-function in this case becomes ((2.6), (2.7))

$$\tau_{\text{I}}(\alpha, W) = \underset{J\to\alpha}{\text{residue}} \frac{E-m}{8\pi W} (\alpha')^{\frac{1}{2}} \frac{W+\sqrt{J-\alpha_0}}{J-\alpha\sqrt{J-\alpha_0}} (q^2)^{J-\frac{1}{2}} g_J^2 \,, \tag{4.1}$$

$$\alpha = \alpha' s + \alpha_0 \,, \qquad \tau_{\text{I}}(\alpha, -W) = 0$$

and

$$\tau_{\text{I}}(\alpha, W) = -\tau_{\text{II}}(\alpha, W) \,, \qquad \text{Re } W > 0 \,.$$

I, II refer to the physical and unphysical J-sheets respectively. We prefer, however, to use the mechanism corresponding to (2.8) as (4.1) suffers (especially in the $I = \frac{1}{2}$ case) from not being unitary as shown in ref. ([17]). Especially extrapolation between scattering and particle region for the nucleon seems unreliable with (4.1), while our results here prove that it is qualitatively right using (2.8). Precisely this extrapolation is relevant to our calculations.

b) Both the Chew-Frautschi plot as well as our low-energy analysis show that isobaric Regge trajectories in the spirit of parallel linear trajectories follow a pattern

$$\alpha(s) \,, \qquad \alpha(s) - \Delta \,, \qquad \alpha(s) - 2\Delta \,, \quad \dots \,. \tag{4.2}$$

For the leading $\mathcal{N}_\alpha$ trajectory $\Delta \simeq 1$ for the $\mathcal{N}_\beta(1535)$ trajectory $\Delta \simeq \frac{1}{2}$. In the development of our analysis its motive has gradually changed to become mainly phenomenological search. However, for one of us the outset was to show that $\pi\mathcal{N}$ scattering provides evidence for an oscillatory excitation spectrum generating at the same time SU_3 multiplets and Regge trajectories in families of isobars ([19]) according to (4.2). Δ, the shift in angular momentum, is related to the two characteristic vibration frequencies ω_0, ω of timelike and

---

([19]) P. MINKOWSKI: University of Leuven preprint (unpublished).

spacelike modes respectively

$$\Delta = \frac{\omega_0}{\omega} \qquad \text{arbitrary}\,. \tag{4.3}$$

Since linear Regge trajectories unmistakably point towards the harmonic potential it is no surprise that the factorized N-point Veneziano amplitude ([20]) shows the excitation spectrum of oscillators (4.2) with the restriction $\Delta = 1$. The degeneracies of the two models do not coincide. For the dual amplitude it is in principle possible that a set of Ward identities for the vertex functions reduces the maximal degeneracy.

c) In the preceding Section the exponential form of the residue function is crucial. This is another independent evidence for exponential behaviour of the smooth core of Regge residues. Although this parametrization has been extensively used its crucial aspect in fitting asymptotic cross-section has been pointed out only in ref. ([17]). In the preceding Sections we tacitly assumed that u_0 is approximately 1 $(\text{GeV})^2$ like in the Veneziano approach.

A term $\exp[C\cdot S]$ in the residue is essentially a way to vary u_0 and in a Veneziano or 1 $(\text{GeV})^2$ normalization we expect $C\approx 0$. It has been shown that if an extrapolation between scattering and particle region makes sense this condition is not satisfied. From our low-energy point of view we would essentially sum Born terms in the amplitude (2.1) were it not for the structure introduced by Regge residues.

We tried instead to leave the residues constant in order to check the significance of our parameters to describe low-energy data. The phase shifts were not even qualitatively reproduced.

As a conclusion one might say that our calculation supports the hypothesis of oscillatory motion and independently suggests the interesting possibility to study or even fit low-energy scattering data with Regge parameters. To go beyond our simple calculation it is however crucial to incorporate inelastic channels. Besides this the above analysis provides a tool to study the properties of low-lying trajectories usually hidden from Regge asymptotics.

* * *

It is a pleasure to thank Dr. P. AUVIL for raising the pertinent question of whether the MacDowell coupled P_{11}-S_{11} channels can be treated on the same footing, and for a careful reading of the manuscript. We thank Dr. C. MICHAEL for constructive discussions and remarks.

([20]) S. FUBINI, D. GORDON and G. VENEZIANO: *Phys. Lett.*, **29** B, 679 (1969).

● RIASSUNTO (*)

Si dimostra che si può sviluppare in una direzione unificata l'aspetto di complementarità del comportamento di Regge alle alte energie e la saturazione delle risonanze a bassa energia nota come « dualità » dalle regole di somma di energia finita. Si deduce la parametrizzazione di Regge delle onde parziali alle basse energie invertendo la traiettoria che porta al comportamento di Regge asintotico nel canale incrociato tramite la rappresentazione di Khuri-Jones. Si analizzano le onde P_{11} e S_{11} nello scattering $\pi\mathcal{N}$. Si ottengono così indicazioni per un tipo di traiettorie caratteristiche per un modello di eccitazioni oscillatorie.

(*) *Traduzione a cura della Redazione.*

Параметризация Редже для $\pi\mathcal{N}$ рассеяния при низких энергиях.

Резюме (*). — Показывается, что дополнительный аспект поведения Редже при высоких энергиях и насыщение резонансов при низких энергиях, известное как « дуальность », могут быть развиты с единой точки зрения, исходя из правил сумм при конечной энергии. Выводится параметризация Редже для парциальных волн при низких энергиях, посредством обращения пути, приводящего к асимптотическому поведению Редже в перекрестном канале, с использованием представления Хури-Йонеса. Анализируются P_{11} и S_{11} волны в рассеянии. Они дают подтверждение для формы траектории, характерной для модели осцилляторных возбуждений.

(*) *Переведено редакцией.*