Ausgewählte Kapitel aus der Teilchenphysik

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1 The axial current anomaly in QCD+QED and ...

1a) $\eta'$, $\pi^0 \to 2 \gamma$ and related modes [1]

For definiteness we begin considering QCD with an arbitrary number of (anti-) quark flavors – $N_{fl} \leq 6$ – and arbitrary color neutral masses

$$m_f; f = 1, \ldots, N_{fl}$$

(1)

The original derivations of the anomalous divergence of the axial current are due to Stephen Adler [2] and independently John Bell and Roman Jackiw [3], see also [16].
We consider the matrix element of the VVA form, reducing two outgoing photons and an axial current according to the (Feynman-) rules or the Lehmann-Zimmermann-Symanzik LSZ reduction [5] in steps

\[
\langle \text{out} ; k_1 , k_2 \mid \text{in} \rangle = \\
\lim_{t_{ex} \to \pm \infty} \prod_{r'} ( w_{r'}^{* \text{out}} \cdot \Psi_{r'} ) ( t_{\text{out}} ) \\
\times \prod_{r} ( \Psi_{r}^{*} \cdot w_{r} ) ( t_{\text{in}} )
\]

\[ r' = (\gamma) , 1, 2 , \quad r = \pi_0 ( \text{or } \eta') \]

In eq. (2) w stands for an asymptotic
wave function, $\Psi$ for an 'interacting' field and $(\Psi^* \cdot w)(t)$ denotes a relativistically invariant scalar product, evaluated at a fixed time $t$

$$
(\Psi^* \cdot w)(t) = 
\int d^3x \, \Psi^*(t, \vec{x}) \, i \, \vec{\partial}_t \, w(p; t, \vec{x})
$$

(3)

Further, the argument $p$ of $w(p; t, \vec{x})$ denotes the asymptotic four momentum of the $\text{ex} = (\text{in}, \text{out})$ states respectively.

Strictly the form of scalar product in eq. (3) only applies for a scalar field $\Psi$,
where the symbol $\frac{\partial}{\partial t}$ denotes partial time derivative, acting with positive sign to the right and negative sign to the left.

generating asymptotic states ($t \to t_{ex} = \pm \infty$)

The in and out states are generated using the scalar products defined in eq. (3), simplest are just one particle states

\[
\begin{align*}
\lim_{t \to -\infty} (\Psi^* . w)(t) |\Omega\rangle &= |in; p, w\rangle \\
\lim_{t \to +\infty} (\Psi^* . w)(t) |\Omega\rangle &= |out; p, w\rangle
\end{align*}
\]

\[\text{(4)}\]
The next step is generalising the one particle ex states in eq. (4) to multiple particle ex states using the product(s) defined in eq. (2).

Also we specialise to the 'S matrix' situation, where the following matrix elements serve as definition, in accordance with eq. (2)

\[
\langle \text{out} ; \ p'_1 \ \cdots \ p'_m | \ \text{in} ; \ p_1 \ \cdots \ p_n \rangle = \langle \text{out} ; \ p'_1 \ \cdots \ p'_m | \ \hat{S} | \text{out} ; \ p_1 \ \cdots \ p_n \rangle
\]  

(5)

All quantum numbers characterizing individual asymptotic particles except the momenta
are just suppressed in eq. (5) for simplicity. They reside in the additional properties of the wave functions \( w_{r'}(r) (p, x \ldots) \).

\[
\hat{\tau} = \hat{S} - \mathbb{P} = i \hat{T} \; \rightarrow \\
\langle \text{out} ; p'_1 \cdots p'_m | \hat{S} | \text{out} ; p_1 \cdots p_n \rangle = S_{21} \\
\tau_{21} = \int \prod_{r', r} dy_{r'} dx_{r} (w_{r'}^*)(w_{r}) \times \\
[ K_{r'}(y_{r'}) K_{r}(x_{r}) ] \Theta_{21}
\]

(6)

In eq. (6) The quantities \( \hat{S}, \hat{\tau}, \hat{T} \) stand for qu.m. operators, while \((S, \tau, T, \Theta)_{21} \) denote
respective matrix elements. The factors of the products, defined through eq. (6) are (for the simplified case of spinless asymptotic particles)

$$w_r = e^{-i p_r x_r}, \quad w^*_r = e^{i p'_r y_r}$$

$$K_r = i \left( \Box x_r + m^2 r \right)$$

$$K^*_r = i \left( \Box y_r + m^2 r \right)$$

(7)

The quantities $K_r, K^*_r$, defined in eq. (7) are the inverse propagators with respect to the (assumed) spinless external particles or asymptotic states.
These states are described by associated free scalar fields, corresponding to the out states, denoted generically $\varphi^{ex}$

$$K_r = i \left( \Box_{x_r} + m^2 r \right) \leftrightarrow \varphi^{ex}_r(x_r)$$

$$(\Box_{x} + m^2 r) \varphi^{ex}_r(x) = 0 \quad (8)$$

$\varphi^{ex}_r(x) \leftrightarrow \varphi_r(x)$

$\varphi_r(x)$ denote the renormalized interacting fields.

The free field propagator is, dropping the suffix $r$
and assuming (without loss of generality) \( \varphi \) to be a complex field

\[
\langle \Omega | T ( \varphi^e x (x) \varphi^e x^* (y) ) | \Omega \rangle
\]

\[
= D (z) = \quad | z = x - y \quad (9)
\]

\[
\int d^4p \frac{1}{2\pi^4} \frac{e^{-ipz}}{p^2 - m^2 + i\varepsilon}
\]

The inverse relation between \( K \) and \( D \) is true in the operator-kernel sense
\[
K \rightarrow K ( x , y ) = \\
i \left( \Box_x + m^2 \right) \delta^{4 \rightarrow d} ( x - y )
\]

\[
D \rightarrow D ( x , y ) = D ( x - y )
\]

In the following we drop the dimensional extension
\[4 \rightarrow d = 4 - 2 \eta\]

The kernels in eq. (10) acting on test functions \( f ( x ) \) define linear transformations
\[ f \rightarrow (K f (x), D f (x)) \]

\[ K f (x) = \int d^4 y \, K (x, y) \, f (y) \text{ and } K \rightarrow D \]

\[ \rightarrow (K D) f = (D K) f = f \leftrightarrow K = D^{-1} \]

(11)
The time ordered product of two fields (eq. 9) is

\[
T \left( A \left( x \right) B \left( y \right) \right) =
\]

\[
\vartheta \left( x - y \right) A \left( x \right) B \left( y \right) +
\]

\[
+ \sigma_F \vartheta \left( y - x \right) B \left( x \right) A \left( y \right)
\]

(12)

\[
\sigma_F = \begin{cases} 
-1 \text{ for } A, B \text{ fermionic} \\
+1 \text{ else} 
\end{cases}
\]
Bogoliubov (- Feynman) parameters and Schläfli’s integral

We show the use of the Bogoliubov parameter yielding an integral representation for the free propagator (eq. 9), which we evaluate for arbitrary dimension \( d = 1 + (d - 1) \)

\((a) : \gamma \text{ Bogol. param. associated with (any) given propagator} \)

\[
\frac{i}{p^2 - m^2 + i \varepsilon} = \int_0^\infty d\gamma \exp i \gamma \left[ p^2 - m^2 + i \varepsilon \right]
\]

(13)
In $D(z)$ we encounter a generic Fresnel-Gauss type momentum integration

\[
D(z) = \int_0^\infty d\gamma E(z, \gamma)
\]

\[
E = e^{-i\gamma X} F
\]

\[
F(z, \gamma) = \int d^d p (2\pi)^{-d} \exp i Y
\]

\[
X = m^2 - i\epsilon; \quad Y = \gamma p^2 - p z
\]

\[
p \to l
\]

Because the above Fresnel-Gauss integrals occur in all loop diagrams in perturbation theory.
we shall rename the integration variable $p \rightarrow l$. For the same reason we generalize $F$, defined in eq. (14) $F \rightarrow F^n$

\[
F^n_{\mu_1 \ldots \mu_n} (z, \gamma) = \\
\int d^d l \ (2\pi)^{-d} (l_{\mu_1} \ldots l_{\mu_n}) e^{iY} \\
Y = \gamma l^2 - l z = \gamma \bar{l}^2 - \zeta
\]

(15)

\[
l = \bar{l} + z / (2\gamma) ; \ \zeta (z, \gamma) = \frac{z^2}{4\gamma}
\]
The next step in evaluating $F^n$ is to perform the substitution $l = \bar{l} + \Delta$

$$(b) : l = \bar{l} + \Delta \quad \text{shift of momentum int. variable}$$

$$\Delta = z / (2\gamma) \quad \text{here}$$

$$l_{\alpha_1} \cdots l_{\alpha_n} = \bar{l}_{\alpha_1} \cdots \bar{l}_{\alpha_n} + \cdots$$

$$F^n_{\mu_1 \cdots \mu_n}(z, \gamma) = \left[ \bar{F}^n_{\mu_1 \cdots \mu_n}(z, \gamma) + \right.$$

$$\left. \text{terms with } \bar{F}^{n'} < n \right]$$

$$(b) \rightarrow$$

$$ (16) $$
Thus modulo terms proportional to $\overline{F}^{n'} < n$ (there are no such for $n = 0$) we obtain

\[(b): \text{ continued ...} \]

\[\overline{F}_{\mu_1 \ldots \mu_n} (z, \gamma) = e^{-i\zeta} G_{\mu_1 \ldots \mu_n} (\gamma)\]

\[G_{\mu_1 \ldots \mu_n} (\gamma) = \int d^d \bar{l} \left( \frac{2\pi}{(2\pi)^d} \right)^{-d} (\bar{l}_{\mu_1} \ldots \bar{l}_{\mu_n}) e^{i\gamma(\bar{l}^2 + i\varepsilon)}\]

\[\zeta (z, \gamma) = z^2 / (4\gamma)\]

(17)

The functions $G^n$ in eq. (17) are standardized
integrals over momentum space, convergent upon the $i \varepsilon$ extension, which we will suppress unless essential.

\[ \text{ad (b) : reduction of the set } G^n (\gamma) \]

The functions $G^n (\gamma)$ (eq. 17) are reduced introducing a redundant shift in the integration momentum $\bar{l} \rightarrow \bar{l} - \kappa$ and associating an operator $\hat{l}_\alpha$ with every factor $\bar{l}_\alpha$ appearing under the integral, starting from $G \equiv G^0$
\[
G = \int d^d \bar{l} \left( 2\pi \right)^{-d} e^{i \gamma (\bar{l} - \kappa)^2} \rightarrow \\
\partial_{\kappa} \alpha \ G = (2 \gamma i) (\kappa_\alpha - \hat{l}_\alpha) \cdot G \\
\hat{l}_\alpha \cdot G = \int d^d \bar{l} \left( 2\pi \right)^{-d} \bar{l}_\alpha e^{i \gamma (\bar{l})^2} \rightarrow \\
\hat{l}_\alpha = \partial_{\kappa} \alpha - (2 \gamma i)^{-1} \kappa_\alpha \\
\partial_{\kappa} \alpha \kappa_\beta = g_{\alpha\beta} = diag (1, -1, -1, -1) \\

The resolution \( G \rightarrow G^n \) follows
\[ \text{ad (b)} : G^n \text{ reduction} \]

\[
G_{\mu_1 \ldots \mu_n} (\gamma) = \left( \hat{l}_{\alpha_1} \ldots \hat{l}_{\alpha_n} \right) \bigg|_{\kappa \to 0} G \\
\hat{l}_\alpha = \partial_{\kappa} \kappa_\alpha - (2\gamma i)^{-1} \kappa_\alpha
\]

(19)

It remains to determine \( G \) which is a product over all dimensions, distinguishing time from space

\[
G = i \left( 4\pi \gamma i \right)^{-\frac{1}{2}} d
\]

(20)

\( E : \text{Euclidean connection for } \gamma \)

\[
\gamma_E \equiv i \gamma \leftrightarrow \gamma \equiv \gamma_E / i
\]

(21)
back to Bogoliubov, Feynman and Schläfli

Collecting results from eq. (14 - 21) we obtain

\[
D ( z ) = \left[ ( 4 \pi )^{ - \frac{1}{2} d } \int_{0}^{\infty} \left( d ( i \gamma ) \right) \left( \gamma i \right)^{ - \frac{1}{2} d } \exp \left[ - ( i \gamma ) m^2 - i z^2 \left( 4 \gamma \right)^{-1} \right] \right] \\
m^2 \rightarrow m^2 - i \varepsilon , \ z^2 \rightarrow z^2 - i \varepsilon \\
0 \leq \gamma \leq + \infty
\]

(22)

Retaining (for the moment) the definite range of the variable \( \gamma \) in eq. (22) we substitute eq. (21) →
\[ \varrho = i \gamma = \gamma E \quad \varrho \text{ for räumlich} \]

\[ r^2 \equiv -z^2 \]

\[ D(z) = \left[ (4\pi)^{-\frac{1}{2}} d \int_C d\varrho \left( \varrho \right)^{-\frac{1}{2}} d \right. \]

\[ \exp \left[ -\varrho \frac{m^2}{2} - r^2 \left( 4\varrho \right)^{-1} \right] \]

\[ C : \varrho = i\gamma , \ 0 \leq \gamma \leq +\infty \]

Schläfli’s integral is of identical form [6] \[ \rightarrow \]

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\[ Z_n(x) = \]
\[ \frac{1}{2\pi i} \left[ \int_{C(Z)} d\, u \,(u)^{-n-1} \times \right. \]
\[ \left. \times \exp \left[ \frac{1}{2} x \,(u + u^{-1}) \right] \right] \]

\[ Z_n : \text{ generic Zylinder (Bessel) - function} \]

\[ C(Z) : \text{ path adapted to the specific cylinder - function} \]

We conclude the discussion of \( D(z) \) restricting to the spacelike region \( r^2 = -z^2 \geq 0 \).
In this case we can analytically deform the path $C$ in eq. (23) to real nonnegative values of $\varrho$

$$r^2 \equiv -z^2 \geq 0 \rightarrow$$

$$D ( z ) =$$

$$\begin{bmatrix}
(4\pi)^{-\frac{1}{2}} d \int_{0}^{+\infty} d \varrho ( \varrho )^{-\frac{1}{2}} d

\exp \left[ - \varrho m^2 - r^2 (4\varrho)^{-1} \right]
\end{bmatrix}
$$

$$0 \leq \varrho \leq +\infty$$

We can now perform a positive scala transformation on the integration variable $\varrho$
\[ \varrho = L^2 u ; \quad \text{dim } \varrho : \text{ length square} \]

\[ \varrho m^2 + r^2 (4 \varrho)^{-1} = \]

\[ L^2 m^2 u + \left[ r^2 / (4 L^2) \right] u^{-1} \text{ such that } \]

\[ L^2 m^2 = r^2 / (4 L^2) \rightarrow L^2 = r / (2 m) \]

\[ L^2 m^2 = r^2 / (4 L^2) = \frac{1}{2} (m \cdot r) \]

(26)

**With the substitutions in eq. (26) and**

\[ n = \frac{1}{2} (d - 2) \] eq. (25) becomes
in the spacelike region

\[
D(z) = \\
\left[ (4\pi)^{-1} \left( m^2 / (2\pi) \right)^n (\zeta) - n \right] \\
\int_0^{+\infty} du \left( u \right) - n - 1 \\
\times \exp - \left[ \frac{1}{2} \zeta \left( u + u^{-1} \right) \right] \\
\zeta = m r \ ; \ n = \frac{1}{2} \left( d - 2 \right)
\]  

(27)

This is to be compared with the integral representation
of the modified Bessel functions (of arbitrary order $n$) [7]

$$K_n (\zeta) =$$

$$\frac{1}{2} \int_0^{+\infty} d u \left( u \right) - n - 1$$

$$\times \exp - \left[ \frac{1}{2} \zeta \left( u + u^{-1} \right) \right]$$

(28)

For completeness we give also the differential equation for
Bessel- $Z_n$ and modified Bessel-functions $K_n$
below

\[ Z_n (\xi) : \left[ \frac{d}{d\xi} \right]^2 J + \left[ \frac{d}{\xi d\xi} \right] J + \left( 1 - \frac{n^2}{\xi^2} \right) J = 0 \]

(29)

\[ J (\xi) = (J_n, N_n) (\xi) \]
\[ K_n (\zeta) : \left[ \frac{d}{d\zeta} \right]^2 K + \left[ \frac{d}{\zeta \, d\zeta} \right] K - \left( 1 + \frac{n^2}{\zeta^2} \right) K = 0 \]

\[ K (\zeta) = (K_n, I_n) (\zeta) = J (\pm i \xi = \zeta) \]

(30)
Since \( K_n(\zeta) \) are standard, tabulated functions, we express the time ordered product (eq. 28) in their terms

\[
D(z) = \left( \frac{2\pi}{m^2} \right)^{-1} \left( \frac{m^2}{2\pi} \right)^n (\zeta)^{-n} K_n(\zeta)
\]

\[
K_n \sim \frac{1}{2} \Gamma(n) \left( \frac{1}{2} \zeta \right)^{-n} \text{ for } \zeta \to 0, \Re n > 0
\]

\[
\zeta = m r; \quad n = \frac{1}{2} (d - 2)
\]

(31)

We can use the asymptotic expansion for \( \zeta \to 0 \) in eq. (31) to derive
a) the limit for $m \to 0$, b) the spacelike leading short distance expansion

$$D ( z ) \sim$$

$$( 4 \pi )^{-1} \Gamma ( n ) ( \pi r^2 )^{-n} \text{ for } r^2 \to 0$$

(32)

We check eq. (32) using for $m = 0$ the representation in eq. (25)
\[ m = 0 : D(z) = \]
\[
= \left[ (4\pi)^{-n-1} \int_0^{+\infty} d\varrho \ (\varrho)^{-n-1} \right] \\
= \left[ \exp \left[ -r^2 \ (4\varrho)^{-1} \right] \right] \\
= \left[ (4\pi)^{-n-1} \int_0^{+\infty} d\sigma \ \sigma^{n-1} \right] \]
\[
= \left[ \exp \left[ -\sigma \ r^2 / 4 \right] \right] \quad \checkmark
\] (33)

We learn from eq. (33) that for \( m = 0 \) the representation of \( D(z) \) in eq. (32) is exact, whereas for finite \( m \) it represents...
the leading spacelike short distance expansion of the propagator.

The spacelike large distance expansion, for \( m \neq 0 \), first for \( K_n \) is

\[
K_n \sim \left[ \frac{\pi}{(2\zeta)} \right]^{\frac{1}{2}} \exp -\zeta \quad \text{for} \quad \zeta \rightarrow \infty \quad (34)
\]

which yields using eq. (31)

\[
D(z) =
\]

\[
\frac{1}{2} m^2 n \left( 2\pi m r \right)^{-n} - \frac{1}{2} \exp \left( -m r \right) \quad (35)
\]

\[
\text{for} \quad r \rightarrow +\infty ; \quad n = \frac{1}{2} \left( d - 2 \right)
\]

\[a^a \text{ Try to redo the formulae for } D(z) \text{ remembering the prescription } r^2 \rightarrow r^2 + i \varepsilon\]
Addenda to scalar propagator

Extended details on the scalar propagator discussed here (eqs. 22 - 35) can be found in the excellent book by Bogoliubov and Shirkov [8]. It has to be adapted for appropriate powers of $i$ multiplying the Green functions relative to our convention, as well as to the relatively negative choice of flat space metric

\[
\left( g_{\mu\nu} \right)_{\text{BogoShi}} = \text{diag} \left( -1, 1, 1, 1 \right)
\]  

(36)

The first systematic use of the time ordered Green function goes back to Ernest Stückelberg [9].
An incomplete list of textbooks on various aspects of field theory in flat dimensions is [10] - [15], but there are many more, e.g. about QCD: N. Muta, 'A field theory primer': P. Ramond ...

Addendum 1: dimensional extension (regularization)

We go back to the representation of the propagator in the spacelike region, in eq. (31),...
repeated below

\[
D ( z , m^2 , n ) = \left( \frac{2\pi}{m^2} \right)^{-\frac{1}{2}} \left( \frac{m^2}{2\pi} \right)^n (\zeta)^{-n} K_n (\zeta)
\]

\[K_n \sim \frac{1}{2} \Gamma(n) \left( \frac{1}{2} \zeta \right)^{-n} \text{ for } \zeta \to 0 , \Re n > 0\]

\[\zeta = m r ; n = \frac{1}{2} ( d - 2 )\]

(37)

As indicated in eq. (37) \( D ( z , m^2 , n ) \) can be extrapolated to arbitrary complex values of \( n \) or \( d \).
The propagator does not need any regularization and thus we can fix

\[ n = 1 \quad \text{for} \quad d = 4 \]  \hspace{1cm} (38)

Nevertheless it is instructive to consider dimensional extension and the associated (milder) short spacelike distance behaviour of the propagator, setting

\[ d = 4 - 2 \eta \; ; \; \eta \geq 0 \]  \hspace{1cm} (39)

\[ n = 1 - \eta \]

Then eq. (32) becomes

→
\[ D ( z, m^2, n ) \sim \]
\[ ( 4 \pi^2 r^2 )^{-1} \Gamma ( 1 - \eta ) ( \pi r^2 )^\eta \]

for \( r^2 \to 0 \); \( \eta \geq 0 \)

Addendum 2: the meaning of \( m \) at long range

We infer from eq. (35) for \( m > 0 \) that large (spacelike) distances are exponentially suppressed. This insures for actual physical asymptotic states with nonzero mass the proper separability, inherent to the existence of such scattering states.
In contrast to this, we can also attempt to use $m$ as a long range regulator, even though this is not viable directly for nonabelian gauge bosons.

This is illustrated by setting $m = 0$, whereby eq. (40) remains valid for all values of $r$

\[
D \left( z, m^2 = 0, n \right) \sim \\
(4 \pi^2 r^2)^{-1} \Gamma \left( 1 - \eta \right) \left( \pi r^2 \right)^\eta
\]  

(41)

for all $r^2 \geq 0$

Hence the short distance singularity becomes milder for $\eta > 0$, whereas (always for $m = 0$) the long range behaviour becomes more severe.
Fig. 1

The three spin \( \frac{1}{2} \) fermion families with helicity specified
Back to the LSZ reduction (eq. 6)

We reproduce eq. (6) below

\[ \hat{\tau} = \hat{S} - \mathbb{Q} = i \hat{T} ; \rightarrow \]

\[ \langle \text{out} ; \ p'_1 \ \cdots \ p'_m \ | \ \hat{S} \ | \ \text{out} ; \ p_1 \ \cdots \ p_n \rangle = S_{21} \]

\[ \tau_{21} = \int \prod \text{d}y_{r',} \text{d}x_{r} \left( w^*_{r'} \right) \left( w_{r} \right) \times \]

\[ \left[ K_{r'} \left( y_{r'} \right) K_{r} \left( x_{r} \right) \right] \Theta_{21} \]

\[ w_{r} \left( p_{r} ; \ x_{r}^0 , \ \vec{x}_{r} \right) = e^{-i p_{r} \cdot x_{r}} \]

\[ w_{r'} \left( p'_{r} ; \ y_{r'}^0 , \ \vec{y}_{r'} \right) = e^{-i p'_{r} \cdot y_{r'}} \]

(42)
Recalling eq. (3) we retain that with any incoming plane wave \( w_r = e^{-i \cdot p \cdot r \cdot x_r} \) is associated with an interacting scalar field, in creation mode, \( \Psi_r^* (x_r) \) and conversely any outgoing plane wave \( w_r^* = e^{+i \cdot p'_r \cdot y_{r'}} \) a corresponding interacting scalar field, in absorption mode, \( \Psi_{r'} (y_{r'}) \)\(^{a}\).

\[
\begin{align*}
    w_r &= e^{-i \cdot p \cdot r \cdot x_r} 
    \quad \Leftrightarrow \quad \Psi_r^* (x_r) \\
    w_r^* &= e^{+i \cdot p'_r \cdot y_{r'}} 
    \quad \Leftrightarrow \quad \Psi_{r'} (y_{r'})
\end{align*}
\]

\(^{a}\) The restriction to only scalar fields is chosen here for simplicity only. Yet let me emphasize that spin, particularly \( S \geq 1 \) is by no means an inessential complication.
We also recall eq. (7)

\[
K_r = i \left( \Box x_r + m^2 r \right)
\]

\[
K_{r'} = i \left( \Box y_{r'} + m^2 r' \right)
\]

(44)

So collecting eqs. (42 - 44) we obtain, introducing the Fourier transforms

\[
\tilde{K}_r = i \left( m^2 r - p^2 r \right)
\]

\[
\tilde{K}_{r'} = i \left( m^2 r', - p_r'^2 \right)
\]

(45)
\[ \hat{\tau} = \hat{S} - \mathbb{P} = i \hat{T} ; \rightarrow \]
\[ \langle \text{out} ; p'_1 \cdots p'_m \mid \hat{S} \mid \text{out} ; p_1 \cdots p_n \rangle = S_{21} \]
\[ \tau_{21} = \prod_{r'} \prod_r \left[ \tilde{K}_{r'}(p'_{r'}) \tilde{K}_r(p_r) \right] \times \]
\[ \int dy_{r'} dx_r \left( e^{+i p'_{r'}} y_{r'} e^{-i p_r x_r} \right) \times \Theta_{21} \]
\[ \Theta_{21} \left( y_{m'}, \cdots y_1, x_1, \cdots, x_m \right) = \]
\[ \langle \Omega \mid T \left\{ \Psi_{m'}(y_{m'}) \cdots \Psi_{1'}(y_1) \times \right. \]
\[ \left. \times \Psi_{1}^{*}(x_1) \cdots \Psi_{m}^{*}(x_m) \right\} \mid \Omega \rangle_{sc} \rightarrow \]
\[(46)\]
Eq. (46) represents the LSZ reduction, with the following specifications

1) The subscript \( s_c \), for 'scattering', denotes an appropriate prescription necessary to omit the contribution to the S matrix of the unit operator \( \mathbb{1} \).

2) \( T \{ \Psi_m', (y_m',) \cdots \Psi_1^*(x_1) \} \) denotes the time ordered product of the field operators in \( \{.\} \).
3) The derivation of eqs. (45 - 46) is non-perturbative. In order to describe scattering asymptotic long range relative to \( \bar{L}_{m_r} = 1 / m_r \) conditions on the representative scalar fields are implicitly imposed, called 'the Haag-Ruelle theory of asymptotic fields and particles' in ref. [11], [18]:

a) The spectrum of stable asymptotic states (or particles) defining the scattering situation exhibits a finite minimal mass among the \( m_r \), in short: a mass gap.
3) continued ...

b) The Green function involving the 'elementary' fields forming the time ordered product in 2):
\[ T \{ \Psi_{m'}(y_{m'}) \cdots \Psi^*_i(x_{1}) \} \]
upon Fourier transformation to on and off shell momenta, as in eq. (46) but omitting the external factors

\[ \Pi_{r', r} \left[ \tilde{K}_{r'}(p'_{r'}) \tilde{K}_r(p_r) \right] \tag{47} \]
are required to develop poles for each
3) continued ...

b) continued ...

'external leg' of the form

$$\prod_{r', r} \left[ \tilde{D}_{r'} \left( p'_{r'}, \right) \tilde{D}_r \left( p_r \right) \right]$$

$$\tilde{D} \left( p, m^2 \right) = \frac{i}{p^2 - m^2 + i \varepsilon} \quad (48)$$

Interestingly the converse of the necessity as additional 'axioms' of 3), a) and b) above, namely that there may be elementary fields which have long range behaviour not satisfying these
conditions, did not seem to occur to those discussing them in 1962. To myself, a student in the third year then, this seemed an obvious logical possibility.

Furthermore in the same year a paper by Julian Schwinger [19] appeared QED of a massless fermion in $d = 1 + 1$ dimensions, illustrating the alternative logical possibility (this is called the Schwinger model today).

As an illustration I include lecture notes on the extended form of this model from 1998 here.
Notes on Feynman Integrals

P. Minkowski


1. Bogoliubov parameters and momentum space integrals
   (left out)

2. The Schwinger model with $N_c$ photons and $n_{fl}$ charged
   fermions in 2 dimensions

The generalizations of the original model due to
Julian Schwinger [19], which deals with one U1
color (or charge) and one fermion flavor
($N_c = n_{fl} = 1$) with vanishing fermion mass,
is at the same time straightforward yet instructive, because of the striking at least apparent similarity with the fourdimensional associated case of a nonabelian gauge theory (like QCD in D = 4), based on the gauge group $SU(N_c)$ and $n_{fl}$ of massless quark (antiquark) flavors in the fundamental representations $(N_c \oplus \overline{N_c})$ of $SU(N_c)$.

Thus we take care to distinguish color labels ($\rightarrow A, B, C...$), which shall designate individual U1 gauge bosons ($V_\mu^A; A = 1,..,N_c$) from flavor labels ($\rightarrow s, t, u...$), which shall designate individual fermion flavors ($f_s; s = 1,..,n_{fl}$).
There are in general \( N_c \) \( n_{fl} \) charges \( e_{A,s} \) which control the coupling strength of flavor \( s \) to the gauge boson \( A \). We will only consider the case where all charges are equal (\( e_{A,s} = e \ \forall \ A, s \)).

The Lagrange density of the system thus is

\[
\mathcal{L} = \left[ \sum_A \left( -\frac{1}{4} F_{\mu\nu}^A F^{A\mu\nu} \right) + \sum_s \bar{f}_s \gamma^\mu \not\! \partial^\mu f_s - \sum_A \sum_s e_{A,s} \left( \bar{f}_s \gamma^\mu f_s \right) V_{\mu}^A \right]
\]

\( e_{A,s} \rightarrow e \ ; \ \forall \ \{ A = 1,..,N_c \ ; \ s = 1,..,n_{fl} \} \)

(49)
The Euler-Lagrange equations for the gauge fields are governed by the following relations

$$
\mathcal{L} \partial_\sigma V^A_\tau = F^A_\sigma\tau = \partial_\tau V^A_\sigma - \partial_\sigma V^A_\tau
$$

$$
F^A_{\mu\nu} = \partial_\nu V^A_\mu - \partial_\mu V^A_\nu = \varepsilon_{\mu\nu} B^A
$$

$$
B^A = -\varepsilon^A; \quad F^A_\sigma\tau = \varepsilon_\sigma\tau B^A;
$$

$$
\varepsilon_{\mu\nu} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \quad \varepsilon_\sigma\tau = -\varepsilon_{\sigma\tau} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
$$

(50)

In eq. 50 $B^A$ denotes the dimensionally reduced 'magnetic' field strength, which in $D = 2$ is
identical to the negative ’electric’ field strength $\mathcal{E}^A$. Now the equations for the gauge fields become

\[
\partial_\sigma \mathcal{L} \partial_\sigma V_\tau^A = \mathcal{L} V_\tau^A \rightarrow
\]

\[
\partial_\sigma F^A \sigma \tau = \partial_\sigma \varepsilon^{\sigma \tau} B^A =
\]

\[
= - \sum_s e_{A,s} \left( \bar{f}_s \gamma^\tau f_s \right) \rightarrow
\]

\[
\partial_\nu F^A \mu \nu = \varepsilon^{\mu \nu} \partial_\nu B^A = e \sum_s \left( \bar{f}_s \gamma^\mu f_s \right)
\]

(51)

The right hand side of the last relation in eq. 51 involves
the total flavor vector current

\[ j^\mu = \sum_s \left( \overline{f}_s \gamma^\mu f_s \right) \]  \hspace{1cm} (52)

Furthermore the two dimensional gamma matrices are chosen according to
the 'right chiral' convention adopted in four dimensions modulo redundancies:

\[ \gamma^0 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

\[ \gamma^1 = -i \sigma_y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (53) \]

\[ \gamma^0 \gamma^1 = \gamma^5 = \sigma_z; \quad \gamma_\mu \gamma^5 = \varepsilon_{\mu\nu} \gamma^\nu \]

We denote the pseudoscalar gamma matrix in two dimensions
as defined in eq. 53 $\gamma^5$, despite the 'more educated' assignment $5 \to 3$.

From the last relation in eq. 53 we deduce the vector - axial vector current Hodge duality characteristic of two dimensions

$$j^{(5)}_{\mu} = \sum_s \left( \bar{f}_s \gamma_{\mu} \gamma^5 f_s \right)$$

$$j^{(5)}_{\mu} = \varepsilon_{\mu\nu} j^\nu; \quad j^\mu = \varepsilon^{\mu\nu} j^{(5)}_{\nu}$$

The equations of motion (eq. 51) take the form

$$\varepsilon^{\mu\nu} \partial_\nu B^A = e j^\mu = e \varepsilon^{\mu\nu} j^{(5)}_{\nu} \rightarrow$$

$$\partial_\nu B^A = e j^{(5)}_{\nu}$$
Algebraic reduction - a posteriori - to the first
Chern character density

We have used the flat 2 dimensional metric
adapted to physical energy momentum vectors.

---

Note added in November 2005: The derivations here I first presented in a seminar at Caltech \( \sim 1997(-2) \) and I thank Edward Witten for a discussion. The subject was revived after a seminar in Bern in 1997 by Christian Lang from Graz, in presence of Rod Crewther visiting from Adelaide, whose remarks are gratefully acknowledged. Further discussion followed after these notes were written with Gabriele Veneziano at CERN in 1998, and finally during a hike in the bavarian alps in 2004(-1) with Valeri Khoze, Wolfgang Ochs and Yitzhak Frishman, the latter rendered me attentive to his published lectures in Mexico in 1973 [20].
Thus the Chern density relative to right chiral conventions inherits a minus sign:

\[
ch_1 = \frac{1}{2\pi} \text{tr} \left( \frac{1}{2} \varepsilon^{\mu\nu} F_{\mu\nu} \right)
\]

\[
F_{\mu\nu} = \sum_{c,s} e_c e_s F_{\mu\nu}^c
\]

\[
ch_1 = \frac{-e}{2\pi} \sum_c B^c
\]

(56)
The anomalous Ward identity for the axial current is generically equal to twice the Chern character appropriate for any given dimension. In two dimensions it follows

$$\partial_{\mu} j^{\mu} (5) = 2 \, c h_{1} \rightarrow$$

$$\partial_{\mu} j^{\mu} (5) = - n_{f l} \frac{e}{\pi} \sum_{C} B^{C}_{\pi}$$

Equations 55 and 57

Combining eqs. 55 and 57
the 'spontaneous' mass generation for the gauge degrees of freedom - due to the axial current anomaly - becomes manifest:

\[ \partial_\nu B^A = e j^{(5)}_\nu \rightarrow \Box B^A = e \partial_\mu j^\mu^{(5)} \]

\[ \Box B^A = -n_{fl} \frac{e^2}{\pi} \sum_C B^C ; \ A = 1, \ldots, N_c \]

(58)

From eq. 58 we infer that the gauge fields
split into one singlet combination and $N_c - 1$ linearly independent differences

$$
\mathcal{B}_\Sigma = \frac{1}{\sqrt{N_c}} \sum_C \mathcal{B}^C
$$

$$
\mathcal{B}_\Delta = \frac{1}{\sqrt{2}} \left( \mathcal{B}^A - \mathcal{B}^{A'} \right) , A \neq A'
$$

The normalization factors for the quantities $\mathcal{B}_\Sigma$, $\mathcal{B}_\Delta$ in eq. 31 maintain the orthogonal invariant of the sum over field strength squares in the Lagrangean density in eq. 49. Summing over the color index $A$ we obtain
the only nontrivial equation involving $\mathcal{B}_\Sigma$ - a pseudoscalar quantity -

$$(\Box + m_{\eta'}^2) \mathcal{B}_\Sigma = 0 ; \quad m_{\eta'}^2 = n_{fl} \frac{e^2 N_c}{\pi}$$

(60)

Nontrivial mass generation is exhausted by $m_{\eta'}$ according to eq. 60, which is thereby reduced to the case of one color and one flavor originally studied by Schwinger [19].
Despite the *rudimentary* role color plays in two dimensional U1 gauges the large $N_c$ limit - better known in four dimensions - is quite explicit. It is the mass of $\eta'$ which is asymptotically constant in the limit.

Nontrivial mass generation is exhausted by $m \eta'$ according to eq. 32 , which is thereby reduced to the case of one color and one flavor originally studied by Schwinger [19] . Despite the *rudimentary* role color plays in two dimensional U1 gauges the large $N_c$ limit - better known in four dimensions - is quite explicit.
It is the mass of $\eta'$ which is asymptotically constant in the limit
\[ e^2 \rightarrow 0 \; ; \; N_c \rightarrow \infty \]

\[ e^2 N_c \rightarrow \bar{e}^2 = \text{finite} \]

............... 

(Final) remarks on elementary fields ↔ scattering states

1) there is in general no connection between elementary fields and asymptotic scattering states, nor is there a mass gap in the complete picture of all known interactions.
2) In the restricted ’strong interactions only’
theory, outlined in eq. (1):

\[ QCD \text{ with } N_{fl} \leq 6 - \text{ and arbitrary color neutral masses} \rightarrow QCD_{N_{fl}} \]

\[ m_f; f = 1, \cdots, N_{fl} \left( m_f > 0 \right) \] \hspace{1cm} (62)

the mass gap exists and is determined by the quasi-Goldstone pseudoscalar mesons, in the \( B \rightarrow Q_j = 0 \) sectors. \( Q_j \) denotes the full set of charge-like quantum numbers.
2) continued

for $B = \pm 1$, ...

But Fields which do fulfil the associated asymptotic (scattering) conditoions are not elementary but have to be constructed as composite, gauge invariant and local fields.

3) $QCD_{N_{fl}} - QED$ extension:

It is useful to extend $QCD_{N_{fl}}$ to include electromagnetic interactions (QED).

Then the mass gap for $Q_j = 0$ disappears and

---

$^a$ Exercise 1: determine all exactly conserved quantum numbers (generalized charges) for $QCD_{N_{fl}}$.
3) continued
also for $Q_j \neq 0$ an infrared problem arises
with respect to the convergence to asymptotic
massive charged states.
If charged (anti-) leptons $e^\pm$, $\mu^\pm$, $\tau^\pm$ are
included beyond baryonic number(s) three
leptonic ones are also conserved $^a$.

4) $^b$ The real world shows – within a definite
treatment

---

$^a$ Exercise 2: Repeat exercise 1 for $QCD$ $N_{fL} - QED$.

$^b$ ”... trinkt oh Augen was die Wimper hält – von dem goldnen
Überfluss der Welt.”, Gottfried Keller [21].
4) continued

of neutrino mass and mixings only one exactly conserved charge: \( Q_{em} \) beyond the conserved nature of \( SU3_c \), which generates a dynamically complex (nonabelian) infrared problem [22], [23].

\[ \rightarrow \text{Elementary spin} \frac{1}{2} \text{fermion fields, substrates of } QCD_{N_{fl}} \text{ or } QCD_{N_{fl}} - QED \text{ and beyond are to be chosen out of the set displayed in Fig. 1.} \]

The gauge bosons associated with unbroken gauges – including gravity – are displayed in Fig. 2. \[ \rightarrow \]
Fig. 2

The gauge bosons
unbroken gauges
with helicity specified

a) $g_{\mu\nu}$

graviton, $|h| = 2$

b) $\gamma$

photon, $|h| = 1$

c) $G^1 \leftrightarrow G^0$

8 'gluons' $\leftrightarrow$ SU3_c, $|h| = 1$
Back to eq. (46), repeated below

\[ \hat{\tau} = \hat{S} - \mathbb{Q} = i \hat{T} ; \rightarrow \]

\[ \langle \text{out} ; p'_1 \cdots p'_m | \hat{S} | \text{out} ; p_1 \cdots p_n \rangle = S_{21} \]

\[ \tau_{21} = \prod_{r'} \tau_{r'} \left[ \tilde{K}_{r'}(p'_{r'}) \tilde{K}_r(p_r) \right] \times \int dy_{r'} dx_{r'} (e^{+i p'_{r'}} y_{r'} e^{-i p_r x_{r'}}) \times \Theta_{21} \]

\[ \Theta_{21}(y_{m'}, \cdots y_1, x_1, \cdots, x_m) = \]

\[ \langle \Omega | T \{ \Psi_{m'}(y_{m'}) \cdots \Psi_{1'}(y_1) \times \Psi^*_1(x_1) \cdots \Psi^*_m(x_m) \} | \Omega \rangle_{sc} \rightarrow \]

(63)
The time ordered product of fields in eq. (63) – in configuration space – is (in particular) translation invariant. We rearrange coordinates and fields in the following way:

\[ y_{m'}, \ldots, x_m \rightarrow \xi_1, \ldots, \xi_N \]

\[ \Psi_{m'}, \ldots, \Psi^*_m \rightarrow A_1, \ldots, A_N \]

\[ \Theta_{21} \rightarrow \Theta (\xi_1 \ldots \xi_N) = \]

\[ \langle \Omega | T \left\{ \prod_r^N A_r (\xi_r) \right\} | \Omega \rangle \]

Translation invariance thus implies
\[ \Theta \left( \xi_1 \cdots \xi_N \right) = \Theta \left( \xi_1 + a \cdots \xi_N + a \right) \]

\[ \xi_r, a : \text{4 (d) - vectors} \]  

Similarly to the coordinates we rearrange the momenta in eq. (63)

\[ p'_m, \cdots, p'_1 ; p_1, \cdots, p_m \rightarrow \]

\[ -q_1, \cdots, -q_m' ; q_{m'+1}, \cdots q_N \]

We note that the q - momenta are chosen all incoming in the assignment of eq. (66).
Thus it follows from eq. (65) using shorthand notation

\[ \tilde{\Theta} ( \underline{q} ) = \int d \underline{\xi} \, e^{-i \underline{q} \cdot \underline{\xi}} \Theta ( \underline{\xi} ) \]

\[ = ( 2\pi )^d \delta^d ( \sum_r q_r ) \tilde{\vartheta} ( \underline{q} ) \]

\[ \underline{\xi} = ( \xi_1 \ldots \xi_N ) , \underline{q} = ( q_1 \ldots q_N ) \]

(67)

Partial LSZ reduction for \( \pi^0 \to 2\gamma \)

Eqs. (46) and (63) describe the complete LSZ reduction of physical (in short 'on shell') scattering amplitudes, as well as possible extrapolations based on the
configuration space elements (eq. 64)

\[
\Theta ( \xi_1 \cdots \xi_N ) = \langle \Omega | T \{ \prod_{r}^{N} A_r (\xi_r) \} | \Omega \rangle
\]  

(68)

where \( A_r (\xi_r) \) here – for simplicity only – denote
gauge-invariant and composite yet local fields. The
latter incorporate the basic seed of extrapolation
away from asymptotic mass shell conditions.

We apply – without special derivation – the
implicit partial LSZ reduction to the transiton
amplitude pertinent to the decay \( \pi^0 \rightarrow 2 \gamma \)
Using the notation of eq. (63) we obtain

\[ \tau_{21} = \text{out} \langle 2 \, \gamma ; \, k_1, k_2 \mid \hat{T} \mid \pi^0 \, p \rangle \text{out} = \]

\[ = (2 \pi)^4 \, \delta^4 (k_1 + k_2 - p) \, i \, T \]

\[ T = \]

\[ (m_{\pi^0}^2 - p^2) \text{out} \langle 2 \, \gamma ; \, k_1, k_2 \mid \varphi(0) \mid \Omega \rangle \]

(69)

In eq. (69) \( \varphi(x) \) denotes the (fully interacting, pseudoscalar) local field, associated with the asymptotic pion state, with the normalization

\[ \langle \omega \mid \varphi(x) \mid \pi^0 \, p \rangle = e^{-i \, p \cdot x} \, 1 \]

(70)
For a one particle state the subscripts \( \text{out} \), \( \text{in} \) are redundant and we drop them hereafter.

The perturbative expansion with respect to \( e \) (QED) then yields

\[
T = ( m_{\pi 0}^2 - p^2 )_{\text{out}} \left\langle 2 \gamma ; \varepsilon_1, k_1, \varepsilon_2, k_2 \right| e^{-i e \int d^d x \ A^\mu(x) j_{\mu m}(x)} \varphi(0) \right\rangle | \Omega \rangle
\]

(71)

\(^a\)

---

\(^a\) In eq. (71) the strong interactions, which generate the pole amputated by the factor \( m_{\pi 0}^2 - p^2 \) is not included.
In eq. (71) $\varepsilon_{1(2)}$ stand for the polarization vectors of the outgoing photons, $A^\mu$ denote the fields of the electromagnetic potentials and $j_{\mu}^{em}$ the electromagnetic current of quarks (and antiquarks).

We expand the exponential in eq. (71)

$$T = - e^2 \left( m_\pi^2 - p^2 \right) \varepsilon^*_1 \mu^1 \varepsilon^*_2 \mu^2 t_{\mu_1 \mu_2}$$

$$t_{\mu_1 \mu_2} = \int dx_1 dx_2 e^{i \sum_{j}^2 k_j x_j} \times$$

$$\langle \Omega | T \left\{ j_{\mu_1}^{em}(x_1) j_{\mu_2}^{em}(x_2) \varphi(0) \right\} | \Omega \rangle$$

$$a$$

$$\text{In eq. (72) two photons are treated as distinct, to be compensated by a Bose factor. Ex. 3: Show eq. (72).}$$
The first (PCAC-) Ansatz for $\varphi$

We turn to the axial vector definition based on the right-chiral projection, defining the chiral basis for the (spin $\frac{1}{2}$) (lower index) gamma matrices

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \tilde{\sigma}_\mu & 0 \end{pmatrix} ; \quad \sigma_\mu = (\sigma_0, \sigma_k)$$

$$\tilde{\sigma}_\mu = (\sigma_0, -\sigma_k)$$

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \rightarrow (73)$$

---

\(^a\text{PCAC = partially conserved axial vector current.}\)
and

\[ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]  \hspace{1cm} (74)

It follows – recalling the \textit{base independent standard} conventions

\[ \sigma_{\mu\nu} = \frac{i}{2} \left[ \gamma_\mu, \gamma_\nu \right]; \quad \sigma_{jk} = \epsilon_{jkr} \sum_r \\
\alpha_k = \gamma^0 \gamma^k \rightarrow \begin{pmatrix} \sigma_k & 0 \\ 0 & -\sigma_k \end{pmatrix} \]  \hspace{1cm} (75)

\textit{a} $\sigma_{1,2,3}$ are the Pauli matrices.
and

\[
\sum_{r} \rightarrow \begin{pmatrix}
\sigma_r & 0 \\
0 & \sigma_r
\end{pmatrix} ; \quad \gamma_0 \gamma_1 \rightarrow \begin{pmatrix}
-\sigma_1 & 0 \\
0 & \sigma_1
\end{pmatrix}
\]

\[
\gamma_2 \gamma_3 \rightarrow \begin{pmatrix}
-\sigma_2 & \sigma_3 & 0 \\
0 & -\sigma_2 & \sigma_3
\end{pmatrix}
\]

\[
\gamma_0 \gamma_1 \gamma_2 \gamma_3 \rightarrow \begin{pmatrix}
\sigma_1 & \sigma_2 & \sigma_3 & 0 \\
0 & -\sigma_1 & \sigma_2 & \sigma_3
\end{pmatrix}
\]

\[
= i \gamma_5 R = - i \gamma_5 L
\]

(76)
The base free definition of $\gamma_5^R = - \gamma_5^L$ thus is

$$\gamma_5^R = - \gamma_5^L = \frac{1}{i} \gamma_0 \gamma_1 \gamma_2 \gamma_3$$

$$\gamma_5^R \rightarrow \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}$$

(77)

We thus start from the (right-chiral) axial currents for $N_{fl} = 2$

$$j_\mu^5 = \bar{q} \gamma_5^R \left( \frac{1}{2} \tau^a \right)_{ts} q^c$$

$$t, s : u, d ; \text{flavor}$$

$$c : \text{red, green, blue ; color}$$

(78)
Current algebra from $q$, $\bar{q}$ algebra at equal times $^a$

We note the algebraic triple (anti-) commutator relations

\[
\begin{align*}
[ X , Y ] _ \mp & = X Y \mp Y X
\end{align*}
\]

(79)

It follows for equal time

$^a$ For once.
\[
\left[ j_0^a(x), j_0^b(y) \right] \mid x^0 = y^0 = q^* \dot{c}(x) \gamma_5 R \left( \frac{1}{2} \tau^a \right)_{ts} \times
\]
\[
\times \left[ q^c_s(x), j_0^b(y) \right] \mid - -
\]
\[
- \left[ j_0^b(y), q^* \dot{c}(x) \right] \mid - \times
\]
\[
\times \gamma_5 R \left( \frac{1}{2} \tau^a \right)_{ts} q^c_s(x) \mid x^0 = y^0
\]

and using eq. (79) once more
\[
\left[ j_0^a(x), j_0^b(y) \right] - \left| x^0 = y^0 \right. = \\
q_t^* \dot{c}(x) \gamma_5 R \left( \frac{1}{2} \tau^a \right)_{ts} \times \\
\times \left[ q_s^c(x), q_u^d(y) \right] + \times \\
\times \gamma_5 R \left( \frac{1}{2} \tau^b \right)_{uv} q_v^d(y) - \\
- q_u^d(y) \gamma_5 R \left( \frac{1}{2} \tau^b \right)_{uv} \times \\
\times \left[ q_v^d(y), q_t^* \dot{c}(x) \right] + \times \\
\times \gamma_5 R \left( \frac{1}{2} \tau^a \right)_{ts} q_s^c(x) \left| x^0 = y^0 \right.
\]
\quad (81)
The only \(-4\) valued \(-\) index \((A)\) of quark fields \textit{not} displayed in eqs. \((80 - 81)\), acted upon by the \(\gamma_5 R\) matrices, appears on equal footing with flavor- and color- indices in the equal time anticommutation relations for (anti-) quark fields \(^a\)

\[
\left[ q^c_s A (x), q^* u^d_B (y) \right] + \quad | x^0 = y^0 = \\
= \delta_A \delta_B \delta^c_d \delta^s_u \delta^3 \left( \vec{x} - \vec{y} \right) \\
\left[ q^c_s A (x), q^d_t B (y) \right] + \quad | x^0 = y^0 = 0 \text{ and h.c.}
\]

\(^a\) These relations must be compensated for renormalization by associated Ward identities, including color.
Inserting eq. (82) in eq. (81) we obtain

\[
\left[ j_0^a \left( x \right), j_0^b \left( y \right) \right] = \delta^3 \left( \vec{x} - \vec{y} \right) i \varepsilon_{abg} j_0^g \left( x \right)
\]

\[
= \delta^3 \left( \vec{x} - \vec{y} \right) i \varepsilon_{abg} \frac{1}{2} \tau^g \left( x \right)
\]

\[
\dot{j}_\mu^g \left( x \right) = \overline{q}_t \gamma^\mu \left( \frac{1}{2} \tau^g \right)_t^s q_s \left( x \right)
\]

Throughout this \( N_{fl} = 2 \) section the matrices \( \tau^g ; g = 1, 2, 3 \) are identical to the Pauli matrices, but refer to isospin of 2 flavors.

In eq. (83) \( j_\mu^g \) denote the (vector-) currents pertaining to isospin.
We complete the full current algebra involving vector- and axialvector currents\(^a\)

\[
\left[ j^a_0 \left( x \right), j^b_0 \left( y \right) \right] - \bigg| _{x^0 = y^0} = \\
\left[ j^5_0 \left( x \right), j^b_0 \left( y \right) \right] - \bigg| _{x^0 = y^0} = \\
= \delta^3 \left( \mathbf{x} - \mathbf{y} \right) i \varepsilon_{abg} j^5_0 g \left( x \right)
\]

\[
\left[ j^a_0 \left( x \right), j^b_0 \left( y \right) \right] - \bigg| _{x^0 = y^0} = \\
= \delta^3 \left( \mathbf{x} - \mathbf{y} \right) i \varepsilon_{abg} j^g_0 \left( x \right)
\]

\(^a\) Currents without and with a \(^5\) superfix.
We extend the algebra to the $SU_2^L \times SU_2^R$ chiral isospin algebra, first adding (and subtracting) the two equations in eq. (84)

$$ j_0^g \pm j_0^g = j_0^g \pm j_0^g $$

$$ \left[ j_0^a (x), j_0^b (y) \right] - x_0 = y_0 = (85) $$

$$ = \delta^3 (\vec{x} - \vec{y}) i \varepsilon_{abg} j_0^g (x) $$

Next we add (and subtract) relations in eqs. (84 and 83)
\[
\left[ j^5_0 a(\mathbf{x}), j^\pm_0 b(\mathbf{y}) \right] - \bigg|_{x^0 = y^0} = \\
= \pm \delta^3(\mathbf{x} - \mathbf{y}) i \varepsilon_{abg} j^\pm_0 g(\mathbf{x})
\]

and combining with all four possible signs eqs. (85 and 86) we obtain

\[
\dot{j}^R_0 g = \frac{1}{2} \dot{j}^+_0 g, \quad \dot{j}^L_0 g = \frac{1}{2} \dot{j}^-_0 g
\]

\[
\left[ \dot{j}^R_0 a(\mathbf{x}), \dot{j}^R_0 b(\mathbf{y}) \right] - \bigg|_{x^0 = y^0} = \\
= \delta^3(\mathbf{x} - \mathbf{y}) i \varepsilon_{abg} \dot{j}^R_0 g(\mathbf{x}) \rightarrow
\]
\[
\left[ j^R_0 a(x), j^R_0 b(y) \right] - \bigg|_{x^0 = y^0} = \\
= \delta^3 (\vec{x} - \vec{y}) \ i \in a_{bg} \ j^R_0 g(x) \\
\left[ j^L_0 a(x), j^L_0 b(y) \right] - \bigg|_{x^0 = y^0} = \\
= \delta^3 (\vec{x} - \vec{y}) \ i \in a_{bg} \ j^L_0 g(x) \\
\]

\[\text{Exercise 4: Show that the current-associated charges (provided they exist) } Q^\bullet(t) = \int_t d^3x \ j^\bullet_0 \text{ satisfy the Lie-algebra of } SU2_L \times SU2_R.\]
The Chern characters [24] derive from the generating function $e^\lambda$

\[ \mathcal{F}^{(2)} = \frac{1}{2} F_{\mu\nu} \, dx^\mu \wedge dx^\nu \]

\[ \overline{\mathcal{F}}^{(2)} = \frac{1}{2\pi} \mathcal{F}^{(2)} \]

\[ tr \left[ \exp \left( \lambda \overline{\mathcal{F}}^{(2)} \right) \right] = \sum_n \lambda^n \, ch_n \left( \mathcal{F} \right) \]

\[ \rightarrow ch_2 \left( \mathcal{F} \right) = \frac{1}{8\pi^2} \, tr \left( \mathcal{F}^{(2)} \right)^2 = \]

\[ \frac{1}{32\pi^2} tr \left( \mathcal{F}_{\mu\nu} \mathcal{F}_{\sigma\tau} \right) \, dx^\mu \wedge dx^\nu \wedge dx^\sigma \wedge dx^\tau \]

\[ (89) \]

\[ a \quad ch_2 \text{ here}. \]
The 'curvature' $F$ in eq. (89) is for the electromagnetic application over the square of a diagonal matrix with entires given by the charges of u and d quarks respectively, normalized here to the current $^{a}$

$$j_{\mu}^{\ 5\ \pi^{0}} = \frac{1}{2} \left( \bar{u}^{\ \dot{c}} \gamma_{\mu} \gamma_{5\ R} \ u^{\ c} - \bar{d}^{\ \dot{c}} \gamma_{\mu} \gamma_{5\ R} \ d^{\ c} \right)$$

(90)

Thus in analogy with the 2-dimensional case ( eq. 57 ) we make the Ansatz ( to be verified )

$^{a}$ In its 'base'-form the Chern characters deal with one single fermion at a time. Here we have included the differentials in the deefinition of $ch_{n}$.
\[
\begin{align*}
&d^4 x \ \partial \mu \ j^5_{\mu} \pi^0 \bigg|_{an} = \\
&= \frac{1}{2} 2 \left( \text{ch}_{\frac{1}{2}}^{(u)} (F) - \text{ch}_{\frac{1}{2}}^{(d)} (F) \right) \\
&\text{ch}_{\frac{1}{2}}^{(q)} = \frac{1}{32\pi^2} \left( \sum (e_q)^2 \right) \times \\
&\times (F_{\mu\nu} F_{\sigma\tau}) \ dx^{\mu} \wedge dx^{\nu} \wedge dx^{\sigma} \wedge dx^{\tau}
\end{align*}
\]

(91)

In eq. (91) \( F_{\mu\nu} \) denote the conventional electromagnetic field strengths.

We shall define the anomalous divergence pertaining to \( j^5_{\mu} \pi^0 \) (eq. 91) by a special symbol \( An \pi^0 \).
\[ d^4 x \partial^\mu j^5_\mu \pi^0 \bigg|_{\alpha_n} = A_n \pi^0 = ch^{(u)}_2 - ch^{(d)}_2 \]

\[
\sum (e_u)^2 = e^2 \frac{4}{9} N_c , \quad \sum (e_u)^2 = e^2 \frac{1}{9} N_c
\]

Thus eq. (91) takes the form

\[
A_n \pi^0 = \left( \frac{N_c}{3} \right) \frac{e^2}{16\pi^2} \times \\
\times \left( \frac{1}{2} F_{\mu\nu} F_{\sigma\tau} \right) dx^\mu \wedge dx^\nu \wedge dx^\sigma \wedge dx^\tau
\]

\[
N_c / 3 \rightarrow 1
\]

There is a sign uncertainty before the definition...
of $\pi^0$ to be remembered, since we are not working in Euclidean space.

In physical space-time we have

$$dx^\mu \wedge dx^\nu \wedge dx^\sigma \wedge dx^\tau = - \varepsilon^{\mu\nu\sigma\tau} d^4 x \quad (94)$$

whereby eqs. (91 - 93) become

$$\partial^\mu j^\mu_{\pi^0} \bigg|_{an} = an \pi^0 = - \left( N_c / 3 \right) \frac{e^2}{16\pi^2} \times$$

$$\times F^{\mu\nu} \tilde{F}_{\mu\nu}$$

$$\tilde{F}_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\sigma\tau} F^{\sigma\tau} ; \varepsilon_{0123} = 1 \quad (95)$$
Chern characters, back to eqs. 69 and 70

We reproduce eqs. (69 and 70) below

\[ T = \]

\[( m^2 - p^2 )_{out} \langle 2 \gamma ; k_1 , k_2 | \varphi (0) | \Omega \rangle \]

\[(96)\]

\[\langle \omega | \varphi (x) | \pi^0 p \rangle = e^{-ipx} \]

\[(97)\]

Neglecting e at first and using the Ward identity for the divergence of the axial current

\[ j^\mu_{a=3} = j^\pi^0_{\mu} \] we obtain

\[ \langle \Omega | j^\pi^0 (0) | \pi^0 ; p \rangle = i f \pi^0 p_{\mu} \]

\[(98)\]
It follows

\[ \delta \sim \hat{m} \left( \bar{u} i \gamma_5 R u - \bar{d} i \gamma_5 R d \right) \]

\[ \langle \Omega | \delta(0) | \pi^0 ; p \rangle = f \pi^0 m_\pi^2 \rightarrow \]

\[ \varphi = \frac{1}{\delta} \delta \rightarrow \]

\[ f \pi^0 m_\pi^2 \]

\[ \sim \frac{1}{f \pi^0 m_\pi^2} \left( \partial_\mu j^{\mu} \pi^0 - an \pi^0 \right) \]

\[ \hat{m} = \frac{1}{2} (m_u + m_d) ; f \pi^0 \sim 92.4 \text{ MeV} \]
Thus – extrapolating to the soft point $p \to 0$ – we anticipate

$$T \sim$$

$$- \left( \frac{1}{f \pi^0} \right) \langle 2 \gamma ; k_1, k_2 | an \pi^0 (0) | \Omega \rangle$$

$$- an \pi^0 = \left( \frac{N_c}{3} \right) \frac{e^2}{16\pi^2} F^\mu\nu \tilde{F}_\mu\nu$$

Thus, denoting the polarisation vectors
of the outgoing photons \( \varepsilon_{1,2} \) we obtain \(^a\)

\[
T \rightarrow ( N_c / 3 ) \int_0^1 \frac{e^2}{4\pi^2} \varepsilon_{\mu
u\sigma\tau} \varepsilon_2^* \mu \varepsilon_1^* \nu \ k_2 \sigma \ k_1 \tau \\
= N_c / 3 \int_0^1 \frac{e^2}{4\pi^2} \varepsilon_{\tau\mu\nu\sigma} \varepsilon_2^* \mu \varepsilon_1^* \nu \ \Delta \sigma \ p^\tau \\
\Delta = \frac{1}{2} ( k_1 - k_2 )
\]

(101)

In the rest system of \( \pi^0 \) eq. (101) becomes

\[
T = ( N_c / 3 ) \int_0^1 \frac{e^2}{8\pi^2} m_{\pi^0}^2 \varepsilon_{rst} \varepsilon_2^* \tau \varepsilon_1^* \sigma \varepsilon^t \\
\vec{e} = \frac{\vec{k}_1}{| \vec{k}_1 |}
\]

(102)

\(^a\) Exercise 5: Verify step by step.
Summing over the final photon polarisations and including a Bose factor $\frac{1}{2}$ we obtain

$$X = \frac{1}{2} \sum_{\text{pol}} |T|^2 =$$

$$\left( \frac{N_c}{3} \right)^2 \frac{1}{f^2 \pi^0} \left( \frac{e^2}{8 \pi^2} \right)^2 m^4 \pi^0$$

(103)
Finally we turn to invariant phase space

\[ \Phi ( \pi^0 \rightarrow 2\gamma) = \]

\[ = (2\pi)^{-6} (2\pi)^4 \int d^3 k_1 d^3 k_2 \times \]

\[ \times (4 | \vec{k}_1 | | \vec{k}_2 |)^{-1} \delta^4 (k_1 + k_2 - p) \]

\[ \frac{1}{8\pi} \]

(104)

and include the initial state normalization factor

\[ (2 m_{\pi^0})^{-1} \]
to obtain the width (at rest)

\[
\Gamma \left( \pi^0 \rightarrow 2 \gamma \right) = \\
\Phi \left( \pi^0 \rightarrow 2 \gamma \right) \left( 2 \, m_{\pi^0} \right)^{-1} X = \\
= \left( \frac{N_c}{3} \right)^2 \frac{1}{16 \pi} \frac{m^{3}_{\pi^0}}{f^{2}_{\pi^0}} \left( \frac{e^2}{8 \pi^2} \right)^2
\]

(105)

\[
= \left( \frac{N_c}{3} \right)^2 \frac{1}{4 \pi} \frac{m^{3}_{\pi^0}}{f^{2}_{\pi^0}} \left( \frac{e^2}{16 \pi^2} \right)^2
\]
The numeric evaluation yields

\[ \Gamma ( \pi^0 \rightarrow 2\gamma ) \rightarrow \left( \frac{N_c}{3} \right)^2 (7.73 \text{ eV}) \]

\[ (7.74 \pm 0.55) \text{ eV}_{PDG} \]

(106)

1b) The axial vector current anomaly of \( j^5_{\mu} \pi^0 \) (eq. 95).

From eqs. (96 - 99) we retain

---

\[ a \] The best value of the PDG [1] agrees even too well with the calculated one, for \( N_c = 3 \).
remembering the original remark on the surprising situation witnessed by the decay $\pi^0 \rightarrow 2 \gamma$, by David Sutherland [25]

$$T = \left( m_{\pi^0}^2 - p^2 \right)_{\text{out}} \langle 2 \gamma ; k_1 , k_2 | \varphi (0) | \Omega \rangle$$

$$\varphi (x) \rightarrow$$

$$\rightarrow \frac{1}{f \pi^0 \ m_{\pi^0}^2} \left( \partial \mu \ j_\mu^5 \pi^0 - an \pi^0 \right) (x)$$

(107)
which gives rise to the anomalous Ward identity

\[ \partial \mu \, j_\mu^5 \pi^0 = \]
\[ = an \pi^0 + \hat{m} \left( \bar{u} \, i \, \gamma_5 R \, u - \bar{d} \, i \, \gamma_5 R \, d \right) \]

(108)

We further recall, that the anomalous part corresponds to the doubly counted Chern characters (eqs. 90 - 95)

\[ j_\mu^5 \pi^0 = \frac{1}{2} \left( \bar{u} \, \dot{c} \, \gamma_\mu \, \gamma_5 R \, u^c - \bar{d} \, \dot{c} \, \gamma_\mu \, \gamma_5 R \, d^c \right) \]

(109)
\[
\begin{align*}
\int d^4 x \partial \mu j_\mu^{\pi^0} |_{an} &= An \pi^0 \\
&= \frac{1}{2} 2 \left( ch^{(u)}_{2} ( F ) - ch^{(d)}_{2} ( F ) \right) \\
ch^{(q)}_{2} &= \frac{1}{32\pi^2} \left( \sum (e_q)^2 \right) \times \nonumber \\
&\times (F_{\mu\nu} F_{\sigma\tau}) dx^\mu \wedge dx^\nu \wedge dx^\sigma \wedge dx^\tau
\end{align*}
\]

(110)

\[
\begin{align*}
An \pi^0 &= \int d^4 x an \pi^0 \\
an \pi^0 &= -\left( N_c / 3 \right) \frac{e^2}{16\pi^2} F_{\mu\nu} \tilde{F}_{\mu\nu} \\
\tilde{F}_{\mu\nu} &= \frac{1}{2} \varepsilon_{\mu\nu\sigma\tau} F^{\sigma\tau} ; \varepsilon_{0123} = 1
\end{align*}
\]

(111)
Fig. 3

The gauge bosons broken gauges:

$W^\pm$ and $Z$

with spins specified
The four scalar fields, leaving one independent mode:

\[ H : \text{with momentum specified} \]
Figures 3 and 4 [26] complete the SO10 extended elementary fields (standard model +).

The charged W-s open a probe for the pion decay constants (eq. 99) [27], [28]

\[ F_\pi^\pm \sim 130.7 \pm 0.46 \text{ MeV} \]  

\[ f_{\pi^0} \sim \frac{1}{\sqrt{2}} F_\pi^\pm \sim 92.42 \pm 0.33 \text{ MeV} \]  

(112)

Back to eq. (107).

Remembering eq. (72) we cast eq. (107) to the form

\[ \rightarrow \]
\[ \mathcal{T} = f_{\pi_0} T \left( 1 - p^2 / m_{\pi_0}^2 \right)^{-1} \]

\[ \mathcal{T} \to - e^2 \varepsilon_1^{* \mu_1} \varepsilon_2^{* \mu_2} \tau_{\mu_1 \mu_2} (k_1, k_2) \]

\[ \tau_{\mu_1 \mu_2} = \int dx_1 dx_2 e^{+ i \sum_j k_j x_j} \times \]

\[ \langle \Omega | T \{ j_{\mu_1}^{em}(x_1) j_{\mu_2}^{em}(x_2) \tilde{\varphi}(0) \} | \Omega \rangle \]

\[ \tilde{\varphi} = \left( \partial_{\mu} j_5^\mu \pi^0 - a n \pi^0 \right) = \]

\[ = \hat{m} \left( \bar{u} i \gamma_5 R u - \bar{d} i \gamma_5 R d \right) \]

(113)

Hence the strategy is
to compare the following Green functions

$$
\left( 
\begin{array}{cc}
  t_{\mu_1 \mu_2} & q \\
  t_{\mu_1 \mu_2}
\end{array}
\right)
= \\
- e^2 \int dx_1 dx_2 e + i \sum_{k=1}^{2} \gamma_{\mu_k} x_{i_k} \times

\langle \Omega \mid T \left\{ j_{\mu_1}^\text{em}(x_1) j_{\mu_2}^\text{em}(x_2) j_{\not{q}}^5 \pi^0 \phi \left( 0 \right) \right\} \right.

j_{\not{q}}^5 \pi^0 = \frac{1}{2} \left( \overline{u} \gamma_{\ell} \gamma_5 R u - \overline{d} \gamma_{\ell} \gamma_5 R d \right)

\phi = \frac{m}{e} \left( \overline{u} i \gamma_5 R u - \overline{d} i \gamma_5 R d \right)

(114)
The divergence of the axial $\pi^0$ associated current results from the substitution $^a$

\[ j_\varrho \, \pi^0 \rightarrow j_\varrho \, \pi^0 (z) \, e^{-i \, p \, z} \rightarrow \]

\[ \partial \frac{\varrho}{z} \cdot j_\varrho \, \pi^0 (z) \leftrightarrow i \, p^\varrho \cdot j_\varrho \, \pi^0 (z) \]

Hence the anomaly will result in a mismatch between

\[ i \, p^\varrho \, t_{\mu_1 \, \mu_2, \varrho} - t_{\mu_1 \, \mu_2} \neq 0 \]

as defined in eq. (114) $^a$

---

$^a$Exercise 6: Show that the substitution in eq. (115) is correct.
Thus we denote the two amplitudes corresponding to a) and b) in Fig. 5.

Fig. 5
The triangular diagrams for $t_{\mu_1 \mu_2 ; \varrho}$ in eq. (114).

... Kapitel ... WS 2005/6
\[
\begin{bmatrix}
A \\
B
\end{bmatrix}
\left( \mu_1, k_1; \mu_2, k_2, \varrho \right)
\]  \hspace{1cm} (117)

respectively.

First we note the Bose-symmetry restoring relation

\[
B\left( \mu_1, k_1; \mu_2, k_2, \varrho \right) = A\left( \mu_2, k_2; \mu_1, k_1, \varrho \right)
\]  \hspace{1cm} (118)

So we abbreviate

\[
a = A\left( \mu_1, k_1; \mu_2, k_2, \varrho \right)
\]  \hspace{1cm} (119)
and introduce three Bogoliubov parameters

\[
\begin{align*}
\gamma_1 &= \gamma \beta_1 \leftrightarrow L - k_2 \\
\gamma_2 &= \gamma \beta_2 \leftrightarrow L + k_1 \\
\gamma_3 &= \gamma \beta_3 \leftrightarrow L \\
\beta_k &\geq 0 ; \sum_{j}^{3} \beta_j = 1 \\
\end{align*}
\tag{120}
\]

with the integration measure

\[
\begin{align*}
\prod_j^3 d\gamma_j &= \gamma^2 d\gamma d\Omega_3 (\beta) \\
 d\Omega_3 &= \prod_j^3 d\beta_j \delta (1 - \sum_k^3 \beta_k) \\
\end{align*}
\tag{121}
\]
The triangle loop expression for \( a \) becomes

\[
a = e^2 \int \gamma^2 d\gamma d\Omega_3 d^d \mathcal{L} \left( 2\pi \right)^{-d} e^{i\gamma E F}
\]

\[
E = B - \hat{m}^2; \quad B = \sum_{j}^{3} \beta_j l_j^2
\]

\[
l_j = \overline{L} + \Delta_j; \quad l_1 = L - k_2
\]

\[
l_2 = L + k_1; \quad l_3 = L = \overline{L} + \Delta_3
\]

\[
\Delta_3 = \beta_1 k_2 - \beta_2 k_1
\]

\[
\Delta_1 = \Delta_3 - k_2; \quad \Delta_2 = \Delta_3 + k_1
\]

(122)

The quadratic separation of variables in \( E \) yields
\[ E = \overline{L}^2 - X; \quad X = \hat{m}^2 - \delta \]
\[ \delta = \beta_1 \beta_2 \rho^2 + \beta_1 \beta_3 k^2_2 + \beta_2 \beta_3 k^2_1 \]
\[ \rightarrow a = \]
\[ e^2 \int \gamma^2 d \gamma d \Omega_3 d^d \overline{L} (2\pi)^{-d} e^{i \gamma \overline{L}^2} \times \]
\[ \times e^{-i \gamma X F} \]

In the definition of F in eqs. (122 - 123), to which we next turn, a factor -1 from the one fermion loop has been included.
\[ F = S f ; \quad S = \frac{1}{2} \left( N_c / 3 \right) \]

\[ f = \text{tr} \gamma \gamma \mu_1 \left( \overline{L} + \Delta_2 + \hat{m} \right) \gamma_\rho \gamma_5 R \times \]

\[ \times \left( \overline{L} + \Delta_1 + \hat{m} \right) \gamma_{\mu_2} \left( \overline{L} + \Delta_3 + \hat{m} \right) \]

\[ \overline{L}_\alpha \rightarrow \hat{l}_\alpha = \partial_{\kappa} \alpha - \left( 2 \gamma_i \right)^{-1} \kappa_\alpha \]

\[ \underbrace{\psi} = \psi^\kappa \gamma_\kappa \]

(124)

In the substitution in eq. (124) we use eq. (18).
With this substitution (eq. 124) we can perform the $\bar{L}$ integration (dimensionally, eq. 20)

\[ \rightarrow a = \]
\[ e^2 S \int \gamma^2 d\gamma d\Omega_3 i (4\pi\gamma i)^{-\frac{1}{2}} d\times \]
\[ \times e^{-i\gamma X} \hat{f} \]
\[ \hat{f} = tr \gamma\gamma_{\mu_1} \left( \hat{l} + \Delta_2 + \hat{m} \right) \gamma_\rho \gamma_5 R \times \]
\[ \times \left( \hat{l} + \Delta_1 + \hat{m} \right) \gamma_{\mu_2} \left( \hat{l} + \Delta_3 + \hat{m} \right) \]
\[ \hat{l}_\alpha = \partial_{\kappa} \alpha - (2\gamma i)^{-1} \kappa_\alpha \]

(125)
The limits of dimensional regularization

We expand \( \hat{f} \) defined in eq. (125) in inverse powers of \( \gamma \), yielding precisely two terms

\[
\hat{f} = - \left( 2 \gamma i \right)^{-1} f_1 + f_0
\]

\[
f_1 =
\]

\[
\text{tr} \gamma \gamma_{\mu_1} \Gamma^A \gamma_\rho \gamma_5 R \Gamma_A \gamma_{\mu_2} \Delta_3 +
\]

\[
+ \text{tr} \gamma \gamma_{\mu_1} \Delta_2 \gamma_\rho \gamma_5 R \Gamma^A \gamma_{\mu_2} \Gamma_A +
\]

\[
+ \text{tr} \gamma \Gamma^A \gamma_{\mu_1} \Gamma_A \gamma_\rho \gamma_5 R \Delta_1 \gamma_{\mu_2}
\]

(126)
We will consider $f_0$ defined together with $f_1$ in eq. (126) subsequently, but turn first to evaluate $f_1$.

In eq. (126) the d-dimensional gamma matrices are denoted $\Gamma_A$, whereby we extend dimensions – by steps of 2 – to values beyond four. This procedure is also applicable for the study of infrared divergencies [29], but shall be extrapolated to $d = 4 - 2 \varepsilon$ below four.

We proceed by dimensional reduction, keeping $\gamma_5$ as product of the first four gamma matrices $(\times i^{-1})$. 
Thus the following 3 expression can be reduced as follows

\[ \Gamma^A \gamma_\rho \gamma_5 R \Gamma_A = r_3^{(5)} \]

\[ \Gamma^A \gamma_{\mu_2} \Gamma_A = r_2 \]

\[ \Gamma^A \gamma_{\mu_1} \Gamma_A = r_1 \]

\[ r_3^{(5)} = (2 - (d - 4)) \gamma_\rho \gamma_5 R \]

\[ r_{j} = - (2 + (d - 4)) \gamma_{\mu_j}; \ j = 1, 2 \]

and substituting \( d = 4 - 2 \varepsilon \)

\[ \rightarrow (127) \]
\[ r_3^{(5)} = 2 \left( 1 + \varepsilon \right) \gamma_q \gamma_5 R \]

\[ r_j = -2 \left( 1 - \varepsilon \right) \gamma_\mu_j ; \ j = 1, 2 \]

We note that by the dimensional reduction the corner of the triangle where \( \gamma_5 R \) resides is singled out.

So much for dimensional reduction of \( \gamma_5 R \), yet had we generalised to d dimension also \( \gamma_5 R \)

\[ \gamma_5 R \rightarrow \Gamma_{d+1} R \]  

\[ ^a \text{This implies generalising at the same time the triangle to a higher polygon} \]
such that we would be led to conclude
\[
\{ \gamma_5 R, \Gamma_A \} = 0
\]  \hspace{1cm} (130)

then instead of the relations in eq. (128) we would have found, denoting the second type of quantities through the substitution \( r \rightarrow rr \)
\[
rr_3^{(5)} = 2 \left( 1 - \varepsilon \right) \gamma_\rho \gamma_5 R
\]  \hspace{1cm} (131)
\[
rr_{j} = -2 \left( 1 - \varepsilon \right) \gamma_{\mu j}; \ j = 1, 2
\]

Let's continue evaluating \( f_1 \) using the quantities \( r \) in eq. (128).
\[ f_1 = \]
\[ 2 \left( 1 + \varepsilon \right) tr \gamma \gamma_{\mu_1} \gamma_{\rho} \gamma_{5} R \gamma_{\mu_2} \Delta_3 - \]
\[ - 2 \left( 1 - \varepsilon \right) tr \gamma \gamma_{\mu_1} \Delta_2 \gamma_{\rho} \gamma_{5} R \gamma_{\mu_2} - \]
\[ - 2 \left( 1 - \varepsilon \right) tr \gamma \gamma_{\mu_1} \gamma_{\rho} \gamma_{5} R \Delta_1 \gamma_{\mu_2} \]

\[ (132) \]

The traces involving a product of 4 \( \gamma \) matrices with \( \gamma_{5} R \) are totally antisymmetric with respect to the 4 , which yields
\[ f_1 = 2 \left( \left( 1 - \varepsilon \right) \Delta_3 + \left( 1 - \varepsilon \right) (\Delta_1 + \Delta_2) \right)^\sigma \times \]
\[ \times \text{tr} \gamma \gamma \mu_1 \gamma \varrho \gamma \mu_2 \gamma \sigma \gamma 5 R = \]
\[ = 8 i \varepsilon \mu_1 \varrho \mu_2 \sigma \times \]
\[ \left( \left( 1 - \varepsilon \right) \Delta_3 + \left( 1 - \varepsilon \right) (\Delta_1 + \Delta_2) \right)^\sigma \]

(133)

\[ a \]

\[ a \text{ Exercise 7: verify eq. (133), the lower sign in eqs. 132 and 133 corresponds to the all anticommuting choice of } \gamma 5 \rightarrow d + 1 R. \]
Remembering the definition $r = i \gamma = \gamma E$ in eq. (23) we substitute $f_1$ from eq. (133) into the corresponding decomposition of $a = a_1 + a_0$ (eqs. 125, 126)

$$a = a_1 (f_1) + a_0 (f_0)$$

$$e^2 S \int \gamma^2 d\gamma d\Omega_3 i (4\pi \gamma i)^{-\frac{1}{2}} d \times$$

$$\times e^{-i\gamma x} \hat{f}$$

$$\hat{f} = - (2\gamma i)^{-1} f_1 + f_0$$

We will compute $a_0$ later and turn to $a_1$ first.
\[ a_1 = \]
\[ e^2 S \int r^2 d\vec{r} d\Omega_3 (4\pi r)^{\varepsilon - 2} \times \]
\[ \times e^{-r X (2r)^{-1}} f_1 \]
\[ f_1 = 8i \varepsilon_{\mu_1 \varepsilon \mu_2 \sigma} \times \]
\[ \times \left( (\varepsilon^+ \Delta_3 + (1-\varepsilon)(\Delta_1 + \Delta_2)) \right)^{\sigma} \]
\[ \Delta_3 = \beta_1 k_2 - \beta_2 k_1 \]
\[ \Delta_1 = \Delta_3 - k_2 ; \Delta_2 = \Delta_3 + k_1 \]
\[ \rightarrow \]
\[ (135) \]
Remembering eqs. (122 and 123) we see, that the quantity $f_1$ does not depend on the loop variable $r$, so that we can perform the $r$ integration. We introduce the rationalized electromagnetic square coupling constant $\kappa = \kappa_e = e^2 / (16 \pi^2)$ →

$$a_1 = \kappa S \int r^{-1} d\Omega_3 \left( 4 \pi r \right)^\varepsilon \times \left( f_1 / 2 \right)$$

$$X = \hat{m}^2 - \delta$$

$$\delta = \beta_1 \beta_2 p^2 + \beta_1 \beta_3 k_2^2 + \beta_2 \beta_3 k_1^2$$
The \( r \) integration is not (unconditionally) convergent for \( \varepsilon \to 0 \). Thus we introduce the decomposition

\[
\begin{align*}
a_1 &= \left( b_1 \left( \mu^2 \right) + \delta b_1 \right) \left( f_1 / 2 \right) \\
b_1 \left( \mu^2 \right) &= \\
\kappa \mathcal{S} \int r^{-1} d\mathbf{r} d \Omega_{3} \left( 4 \pi r \right)^\varepsilon \times \\
&\quad \times \left( e - \frac{r X}{12} - e - r \mu^2 \right) \\
\delta b_1 &= \\
\kappa \mathcal{S} \int r^{-1} d\mathbf{r} d \Omega_{3} \left( 4 \pi r \right)^\varepsilon e - r \mu^2 \rightarrow
\end{align*}
\]

(137)
For the quantity $b_1 (\mu^2)$ in eq. (137) we can set $\varepsilon = 0$ rightaway, whereas $\delta b_1$ is independent of the Bogoliubov parameters. It follows

$$b_1 (\mu^2) = \kappa \, S \, \log \left( \frac{\mu^2}{X} \right) \, d \Omega_3$$

$$\delta b_1 = \kappa \, S \, \left( \frac{4\pi}{\mu^2} \right)^\varepsilon \, \Gamma (\varepsilon) \, d \Omega_3$$

(138)

The regularization sensitive term $\delta b_1$ in eq. (138) has a simple structure with respect to $\beta$ and the momenta.
We introduce the momentum variables

\[ q = k_1 + k_2 , \quad d = k_1 - k_2 \]

\[ \int d \Omega_3 \begin{bmatrix}
1 \\
\Delta_1 \\
\Delta_2 \\
\Delta_3
\end{bmatrix} = \begin{bmatrix}
\frac{1}{2} \\
- \frac{1}{6} d - \frac{1}{2} k_2 \\
- \frac{1}{6} d + \frac{1}{2} k_1 \\
- \frac{1}{6} d
\end{bmatrix} \quad (139) \]

Thus we obtain
using the decomposition

\[
\begin{align*}
    a_1 &= a_1 \left( \mu^2 \right) + \delta a_1 \\
    b_1 &= b_1 \left( \mu^2 \right) + \delta b_1 \\
    \delta a_1 &= \frac{2}{3} \kappa S \left( \frac{4 \pi}{\mu^2} \right)^\varepsilon \Gamma(\varepsilon) \times \\
    &\times i \varepsilon \mu_1 \varepsilon \mu_2 \sigma d^\sigma C \\
    C &= \left( \left( 1 - \varepsilon \right) - \left( 1^{+}_+ \varepsilon \right) \right) \\
    &= -2 \varepsilon \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\end{align*}
\]
Thus we learn that the ambiguous term in the dimensional extension of $\gamma_5 \to d + 1$ yields an ambiguous but finite contribution for $\varepsilon \to 0$

$$\delta a_1 = \frac{4}{3} \kappa S i \varepsilon_{\mu_1 \mu_2 \rho \sigma} d^\sigma K$$

$$K = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad d = k_1 - k_2$$

(141)

Next we turn to $a_1 (\mu^2)$ as defined in eqs. (135 - 140)
\[ a_1 ( \mu^2 ) = a_1 \log = \]
\[ = \kappa \int S \log ( \mu^2 / X ) \, d \Omega_3 ( f_1 / 2 ) \]
\[ f_1 / 2 = 4 i \varepsilon_{\mu_1 \varepsilon} \mu_2 \sigma ( 3 \Delta_3 + d )^\sigma \]  \hspace{1cm} (142)
\[ \Delta_3 = \beta_1 k_2 - \beta_2 k_1 ; \quad X = \tilde{m}^2 - \delta \]
\[ \delta = \beta_1 \beta_2 p^2 + \beta_1 \beta_3 k_2^2 + \beta_2 \beta_3 k_1^2 \]

We have redefined \( a_1 ( \mu^2 ) \rightarrow a_1 \log \) in eq. (142) such that ( eqs. 141 - 142 )
\[ a_1 = a_1 \log + \delta a_1 \]  \hspace{1cm} (143)
We remark that the quantity \( a_1 \log \) in eq. (142) does not depend on \( \mu^2 \) because

\[
\int d \Omega_3 \left( 3 \Delta_3 + d \right) = 0
\]  

(144)

The remaining term \( f_0 \) in eqs. 125, 126

The decomposition \( \hat{f} = - (2 \gamma i)^{-1} f_1 + f_0 \) in eq. (126) yields

\[
f_0 = tr \gamma_\gamma \mu_1 \left( \Delta_2 + \hat{m} \right) \gamma_\rho \gamma_5 R \times \\
\times \left( \Delta_1 + \hat{m} \right) \gamma_\mu_2 \left( \Delta_3 + \hat{m} \right)
\]  

(145)
This gives in analogy with the term \( a_1 \leftrightarrow f_1 \) (eq. 125) upon performing the \( r = i \gamma \) - integration

\[
a_0 = - \kappa S \int d \Omega \, X^{-1} f_0
\]

\[
f_0 = g + \hat{m}^2 h
\]

\[
g = tr \gamma \gamma \mu_1 \Delta_2 \gamma_\rho \gamma_5 R \times \Delta_1 \gamma \mu_2 \Delta_3
\]

The term \( h \) defined in eq. (146) we reduce step by step
\[ h = \text{tr}\, \gamma \left[ \gamma_{\mu_1} \triangleleft 2 \gamma_{\varrho} \gamma_{\mu_2} + \gamma_{\mu_1} \gamma_{\varrho} \gamma_{5} R \gamma_{\mu_2} + \gamma_{\mu_1} \gamma_{\varrho} \gamma_{5} R \gamma_{\mu_2} \triangleleft 3 \right] \]

\[ = \text{tr}\, \gamma \gamma_{\mu_1} \gamma_{\varrho} \gamma_{\mu_2} \left( \triangleleft 3 - \triangleleft 1 - \triangleleft 2 \right) \times \gamma_{5} R \]

(147)

For the following I shall collect a few 'kinematical' \( \gamma \) - relations.
\[
\text{tr} \gamma \gamma_a \gamma_b \gamma_c \gamma_d \gamma_5 R = 4 i \varepsilon_{a b c d}
\]
\[
\gamma_a \gamma_b \gamma_c = \left[ \eta_{a b} \gamma_c - \eta_{a c} \gamma_b + \eta_{b c} \gamma_a - \right. \\
\left. - i \varepsilon_{a b c d} \gamma^d \gamma_5 R \right]
\]

In eq. (148) \( \eta_{a b} = \text{diag} \left( 1, -1, -1, -1 \right) \) denotes the flat space metric.

Using eq. (148) we obtain for \( h \)
\[ h = 4 i \varepsilon_{\mu_1 \rho \mu_2 \sigma} \left( \Delta_3 - \Delta_1 - \Delta_2 \right)^\sigma \]
\[ \Delta_3 = \beta_1 k_2 - \beta_2 k_1, \quad d = k_1 - k_2 \]
\[ \Delta_1 = \Delta_3 - k_2; \quad \Delta_2 = \Delta_3 + k_1 \]
\[ \Delta_3 - \Delta_1 - \Delta_2 = - \left( \Delta_3 + d \right) \quad \rightarrow \]
\[ h = - 4 i \varepsilon_{\mu_1 \rho \mu_2 \sigma} \left( \Delta_3 + d \right)^\sigma \]

and for g (eq. 146)
\[ g = - \text{tr} \gamma \gamma \mu_1 \Delta_2 \gamma_\varrho \Delta_1 \times \]

\[ \times \gamma \mu_2 \Delta_3 \gamma 5 R \]

\[ = \left[ \begin{array} \text{(} \Delta_1 \Delta_2 \text{)} \text{tr} \gamma \gamma \mu_1 \gamma_\varrho \gamma \mu_2 \Delta_3 \gamma 5 R \quad - \\
- \Delta_2 \varrho \text{tr} \gamma \gamma \mu_1 \Delta_1 \gamma \mu_2 \Delta_3 \gamma 5 R \quad - \\
- \Delta_1 \varrho \text{tr} \gamma \gamma \mu_1 \Delta_2 \gamma \mu_2 \Delta_3 \gamma 5 R \quad + \\
+ i \in \Delta_2 \Delta_1 \Delta_3 \gamma 5 R \end{array} \right] \times \]

\[ \times \text{tr} \gamma \gamma \mu_1 \gamma^d \gamma 5 R \gamma \mu_2 \Delta_3 \gamma 5 R \]

(150)
The two $\gamma_5 R$ factors in the last line of eq. (150) combine to the unit ($\gamma$-) matrix

$$g = 
\begin{bmatrix}
(\Delta_1 \Delta_2) \, tr \, \gamma \gamma \mu_1 \gamma \rho \gamma \mu_2 \Delta_3 \, \gamma_5 R - \\
- \Delta_2 \rho \, tr \, \gamma \gamma \mu_1 \, \Delta_1 \gamma \mu_2 \, \Delta_3 \, \gamma_5 R - \\
- \Delta_1 \rho \, tr \, \gamma \gamma \mu_1 \, \Delta_2 \gamma \mu_2 \, \Delta_3 \, \gamma_5 R + \\
+ i \varepsilon_{a \rho c d} \, \Delta_2 a \, \Delta_1 c \, \Delta_3 \sigma \times \\
\times tr \, \gamma \gamma \mu_1 \gamma^d \gamma \mu_2 \gamma \sigma
\end{bmatrix}
$$

(151)
The trace over 4 $\gamma$ - matrices in the last line of eq. (151) is obtained by using eq. (148)

\[ tr \gamma \gamma \gamma_{\mu_1} \gamma_{d} \gamma_{\mu_2} \gamma_{\sigma} = \]

\[ = 4 \left[ \delta_{\mu_1}^{d} \eta_{\mu_2} \sigma - \delta_{\sigma}^{d} \eta_{\mu_1} \mu_2 + \delta_{\mu_2}^{d} \eta_{\mu_1} \sigma \right] \]  

(152)

The middle term in brackets in eq. (152) does not contribute \(^a\) and we obtain

\(^a\) Exercise . Verify the relations used for $g$ and $h$ .
\[ g = 4 i \times \left[ \begin{array}{c}
( \Delta_1 \Delta_2 ) \varepsilon_{\mu_1 \rho \mu_2 \sigma} \Delta_3^\sigma - \\
- \Delta_2 \rho \varepsilon_{\mu_1 \tau \mu_2 \sigma} \Delta_1^\tau \Delta_3^\sigma - \\
- \Delta_1 \rho \varepsilon_{\mu_1 \tau \mu_2 \sigma} \Delta_2^\tau \Delta_3^\sigma + \\
+ \Delta_3 \mu_2 \varepsilon_{a \rho \varepsilon \mu_1} \Delta_2^a \Delta_1^c + \\
+ \Delta_3 \mu_1 \varepsilon_{a \rho \varepsilon \mu_2} \Delta_2^a \Delta_1^c \\
\end{array} \right] \]

\[ \Delta_1 = \Delta_3 - k_2 ; \quad \Delta_2 = \Delta_3 + k_1 \]

\[ \Delta_3 = \beta_1 k_2 - \beta_2 k_1 \]
The last 4 terms in brackets in eq. (153) can be reduced to involve only the antisymmetric product
\[ \cdots k_1 \land k_2. \]
This we will do in due time (8. February 2006: end of the WS 2005/2006).
Classification of spin (pseudo-) tensors

We take the contributions to the quantity \( g \) in eq. (153) as representing a collection of characteristic (pseudo-) tensors with 3 spin polarization indices: \( \mu_1 \not\in \mu_2 \). Thus we aim to establish an irreducible basis for these (pseudo-) tensorial objects.

To this end we introduce the following notation

\[
I^{(0, 3)} = \varepsilon_{\mu_1 \not\in \mu_2 \tau} \nu^\tau \\
I^{(1, 2)} = \left( I^{(1, 2)}_{\eta}, I^{(1, 2)}_{\mu_1}, I^{(1, 2)}_{\mu_2} \right)
\]

(154)
In eq. (154) \( \nu^\tau \) denotes a generic four momentum, which is decomposed into the base momenta \( k_1, k_2 \). Thus the \( I^{(0, 3)} \) basis consists of two terms

\[
\begin{align*}
(1) \quad I^{(0, 3)}_1 &= \varepsilon_{\mu_1 \sigma_1 \mu_2 \rho} k_{1}^{\sigma_1} \\
&= \varepsilon_{\mu_1 \rho \mu_2 \tau} ( - k_{1}^{\tau} ) \\
(2) \quad I^{(0, 3)}_2 &= \varepsilon_{\mu_1 \rho \mu_2 \sigma_2} k_{2}^{\sigma_2} \\
&= \varepsilon_{\mu_1 \rho \mu_2 \tau} k_{2}^{\tau}
\end{align*}
\]

(155)
In each of the three \((1, 2)\) sectors there are two basis elements. We give them in sequence below

\[
\begin{align*}
(3) \quad I_{\varrho 1}^{(1, 2)} &= k_{1 \varrho} \varepsilon \mu_1 \sigma_1 \mu_2 \sigma_2 k_{1}^{\sigma_1} k_{2}^{\sigma_2} \\
(4) \quad I_{\varrho 2}^{(1, 2)} &= k_{2 \varrho} \varepsilon \mu_1 \sigma_1 \mu_2 \sigma_2 k_{1}^{\sigma_1} k_{2}^{\sigma_2} \\
(5) \quad I_{\mu_1 1}^{(1, 2)} &= k_{1 \mu_1} \varepsilon \varrho \tau \mu_2 \sigma_2 q^{\tau} k_{2}^{\sigma_2} \\
(6) \quad I_{\mu_1 2}^{(1, 2)} &= k_{2 \mu_1} \varepsilon \varrho \tau \mu_2 \sigma_2 q^{\tau} k_{2}^{\sigma_2} \\
(7) \quad I_{\mu_2 1}^{(1, 2)} &= k_{1 \mu_2} \varepsilon \mu_1 \sigma_1 \varrho \tau k_{1}^{\sigma_1} q^{\tau} \\
(8) \quad I_{\mu_2 2}^{(1, 2)} &= k_{2 \mu_2} \varepsilon \mu_1 \sigma_1 \varrho \tau k_{1}^{\sigma_1} q^{\tau}
\end{align*}
\]
We note that the six basis elements defined in eq. (156) ( (3) - (8) ) only occur for the quantity $g$ defined in eq. (150) .

We shall denote the basis (pseudo-) tensors ( eqs. 155 - 156 ) , $\mathcal{B}_\alpha$ as they are numbered $\alpha = (1) \cdots (8)$ .

In order to safeguard electromagnetic gauge invariance or equivalently e.m. current conservation, we give the divergences associated with each base element $\mathcal{B}_\alpha = \mathcal{B}_\alpha \mu_1 \& \mu_2$ below.
To this end we introduce three two component invariants

\[ C_{12} = \varepsilon_{\mu_1 \sigma_1 \mu_2 \sigma_2} k_1^{\sigma_1} k_2^{\sigma_2} \]
\[ C_{31} = \varepsilon_{\mu_1 \sigma_1 \varrho \sigma_3} k_1^{\sigma_1} ( - q )^{\sigma_3} \]
\[ C_{23} = \varepsilon_{\varrho \sigma_3 \mu_2 \sigma_2} ( - q )^{\sigma_3} k_2^{\sigma_2} \]
\[ \mu_3 = \varrho ; \quad q = k_1 + k_2 = - k_3 \]

With the substitutions specified in eq. (157) the two component (pseudo-) tensors

\[ C_{m n} = C_{n m} ; \quad m \neq n = 1, 2, 3 \] defined there
satisfy the cyclic identities \(^a\)

\[
\mathcal{C}_{mn} = \mathcal{C}_{\mu m \mu n} (k_m, k_n) \\
= \varepsilon_{\mu m \sigma m \mu n \sigma n} k_m^m k_n^n
\]  

\(^a\) We are running into a kinematically induced 2 out of 3 situation. This means that maximally 2 out of the 3 currents 
\((V_1 V_2 A_3)\) at the vertices of the triangular diagrams can be conserved, even for the regulator mass \(\overline{m} \to 0\). This is similar to the corresponding 1 out of 2 \((V_1 A_2)\) situation pertinent to two dimensions.
'Mein Hut der hat drei Ecken ...'

We proceed to characterize the three divergences

\[ D_{1}^{\alpha} = D_{1}^{\alpha \mu_{3} \mu_{2}} = k_{1}^{\mu_{1}^{1}} B_{\alpha \mu_{1} \mu_{3} \mu_{2}} \]
\[ D_{2}^{\alpha} = D_{2}^{\alpha \mu_{1} \mu_{3}} = k_{2}^{\mu_{2}^{2}} B_{\alpha \mu_{1} \mu_{3} \mu_{2}} \]
\[ D_{3}^{\alpha} = D_{2}^{\alpha \mu_{1} \mu_{2}} = k_{3}^{\mu_{3}^{3}} B_{\alpha \mu_{1} \mu_{3} \mu_{2}} \]

\[ k_{1} + k_{2} + k_{3} = 0 \]

We begin with \( D_{m}^{1,2} \) ( eq. 155 ) and then add \( D_{m}^{3} - 8 \) ( eq. 156 ) at

\[ \rightarrow \]

\[ \text{Check the divergence relations in the table of eq. (160).} \]
<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$D_1^\alpha$</th>
<th>$D_2^\alpha$</th>
<th>$D_3^\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>0</td>
<td>$C_{31}$</td>
<td>$-C_{12}$</td>
</tr>
<tr>
<td>(2)</td>
<td>$C_{23}$</td>
<td>0</td>
<td>$-C_{12}$</td>
</tr>
<tr>
<td>(3)</td>
<td>0</td>
<td>0</td>
<td>$k_1 k_3 C_{12}$</td>
</tr>
<tr>
<td>(4)</td>
<td>0</td>
<td>0</td>
<td>$k_2 k_3 C_{12}$</td>
</tr>
<tr>
<td>(5)</td>
<td>$k_1 k_1 C_{23}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(6)</td>
<td>$k_1 k_2 C_{23}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(7)</td>
<td>0</td>
<td>$k_1 k_2 C_{31}$</td>
<td>0</td>
</tr>
<tr>
<td>(8)</td>
<td>0</td>
<td>$k_2 k_2 C_{31}$</td>
<td>0</td>
</tr>
</tbody>
</table>
We end this section by establishing the relations among $\mathcal{B}_\alpha$ with respect to the Bose substitution

\[ k_1, \mu_1 \leftrightarrow k_2, \mu_2 : \]

\[ \mathcal{B}_1 \leftrightarrow \mathcal{B}_2, \mathcal{B}_3 \leftrightarrow \mathcal{B}_4 \]

\[ \mathcal{B}_5 \leftrightarrow \mathcal{B}_8, \mathcal{B}_6 \leftrightarrow \mathcal{B}_7 \]

\[ \mathcal{B}_\alpha \text{ decomposition of } \tilde{g} = g / (4i) \]

We go back to eq. (153) where the quantity $g = (4i) \tilde{g}$ is defined.
\[ \tilde{g} = \begin{bmatrix}
(\Delta_1 \Delta_2) \varepsilon_{\mu_1 \varrho \mu_2 \sigma} \Delta_{\frac{\sigma}{3}} - 1 \\
- \Delta_2 \varrho \varepsilon_{\mu_1 \tau \mu_2 \sigma} \Delta_{\frac{\tau}{1}} \Delta_{\frac{\sigma}{3}} - 2 \\
- \Delta_1 \varrho \varepsilon_{\mu_1 \tau \mu_2 \sigma} \Delta_{\frac{\tau}{2}} \Delta_{\frac{\sigma}{3}} + 3 \\
+ \Delta_3 \mu_2 \varepsilon_{a \varrho c \mu_1} \Delta_{\frac{a}{2}} \Delta_{\frac{c}{1}} + 4 \\
+ \Delta_3 \mu_1 \varepsilon_{a \varrho c \mu_2} \Delta_{\frac{a}{2}} \Delta_{\frac{c}{1}} + 5
\end{bmatrix} \]

\[ \Delta_1 = - \left( \beta_2 k_1 + \left( 1 - \beta_1 \right) k_2 \right) \]

\[ \Delta_2 = \left( \left( 1 - \beta_2 \right) k_1 + \beta_1 k_2 \right) \]

\[ \Delta_3 = \beta_1 k_2 - \beta_2 k_1 \]

... Kapitel ... WS 2005/6
We perform evaluations of the te labeled $1 - 5$ in eq. (162) one by on.

$\mathcal{B}_\alpha$ decomposition of $\tilde{g} 1 - 5$

\[
\Delta_1 \Delta_2 = 
\begin{align*}
\beta_2 (1 - \beta_2) k_1^2 + \\
+ \beta_1 (1 - \beta_1) k_2^2 + \\
+ (\beta_3 + 2\beta_1\beta_2) k_1 k_2
\end{align*}
\]

\[= (X - \tilde{m}^2) - \beta_3 k_1 k_2 \]
and using eq. (155)

\[
\tilde{g}^1 = \beta_2 \Delta_1 \Delta_2 B_1 + \beta_1 \Delta_1 \Delta_2 B_2
\]

\[
\Delta_1 \Delta_2 = (X - \hat{m}^2) - \beta_3 k_1 k_2
\]

\[
\tilde{g}^2 = -\Delta_2 \varepsilon \epsilon_{\mu_1 \tau \mu_2 \sigma} \Delta_{\tau_1} \Delta_{\sigma_3}
\]

\[
\epsilon_{\mu_1 \tau \mu_2 \sigma} \Delta_{\tau_1} \Delta_{\sigma_3} =
\]

\[
= \epsilon_{\mu_1 \tau \mu_2 \sigma} k_{1 \tau} k_{2 \sigma} \times
\]

\[
\times (-\beta_1 \beta_2 - (1 - \beta_1) \beta_2)
\]

\[
= (-\beta_2) \epsilon_{\mu_1 \tau \mu_2 \sigma} k_{1 \tau} k_{2 \sigma} \rightarrow
\]
$$\beta_2 \Delta_2 = \beta_2 (1 - \beta_2) k_1 + \beta_1 \beta_2 k_2$$

$$\tilde{g}_2 = \beta_2 (1 - \beta_2) \mathcal{B}_3 + \beta_1 \beta_2 \mathcal{B}_4$$

$$\tilde{g}_3 = -\Delta_1 \varepsilon_{\mu_1 \tau \mu_2 \sigma} \Delta_{\tau} \Delta_{\sigma}$$

$$\varepsilon_{\mu_1 \tau \mu_2 \sigma} \Delta_{\tau} \Delta_{\sigma} =$$

$$= \varepsilon_{\mu_1 \tau \mu_2 \sigma} k_{\tau} k_{\sigma} \times$$

$$\times (\beta_1 (1 - \beta_2) + \beta_1 \beta_2)$$

$$= \beta_1 \varepsilon_{\mu_1 \tau \mu_2 \sigma} k_{\tau} k_{\sigma}$$

$$- \beta_1 \Delta_1 = \beta_1 \beta_2 k_1 + \beta_1 (1 - \beta_1) k_2 \rightarrow$$

$$\qquad (167)$$
\[ \tilde{g}_3 = \beta_1 \beta_2 \mathcal{B}_3 + \beta_1 (1 - \beta_1) \mathcal{B}_4 \]  

(168)

We continue with \( \tilde{g}_4 - 5 \) defined in eq. (162)

\[ \tilde{g}_4 = \Delta_3 \mu_2 \varepsilon_{abc} \mu_1 \Delta_{\frac{a}{2}} \Delta_{\frac{c}{1}} \]

\[ \varepsilon_{abc} \mu_1 \Delta_{\frac{a}{2}} \Delta_{\frac{c}{1}} = \]

\[ = \varepsilon_{abc} \mu_1 k_{\frac{a}{1}} k_{\frac{c}{2}} \times \]

\[ \times \left( - (1 - \beta_1) (1 - \beta_2) + \beta_1 \beta_2 \right) \]

\[ = - \beta_3 \varepsilon_{abc} \mu_1 k_{\frac{a}{1}} k_{\frac{c}{2}} \rightarrow \]  

(169)
and we continue, substituting the bilinear $C_{31}$ from eq. (157)

\[
\varepsilon_{\alpha \varrho \varepsilon \mu_1} k_{\alpha_1} k_{\alpha_2} = - \varepsilon_{\varrho \alpha \mu_1 \varepsilon} k_{\alpha_2} k_{\alpha_1} = \varepsilon_{\varrho \alpha \mu_1 \varepsilon} k_{\alpha_3} k_{\alpha_1} = C_{31}
\]

\[- \beta_3 \Delta_3 = \beta_2 \beta_3 k_1 - \beta_1 \beta_3 k_2 \rightarrow \tilde{g}_4 = \beta_2 \beta_3 B_7 - \beta_1 \beta_3 B_8\]

(170)

Finally we turn to $\tilde{g}_5$
\[ \tilde{g} \, 5 \quad = \quad \Delta \, _3 \, \mu \, _1 \quad \varepsilon \, _a \, q \, c \, \mu \, _2 \quad \Delta \, _a \, _2 \quad \Delta \, _c \, _1 \]

\[ \varepsilon \, _a \, q \, c \, \mu \, _2 \quad \Delta \, _a \, _2 \quad \Delta \, _c \, _1 \quad = \]

\[ = \quad \varepsilon \, _a \, q \, c \, \mu \, _2 \quad k \, _a \, _1 \quad k \, _c \, _2 \quad \times \]

\[ \times \quad ( \quad - \quad ( \quad 1 \quad - \quad \beta \, _1 \quad ) \quad ( \quad 1 \quad - \quad \beta \, _2 \quad ) \quad + \quad \beta \, _1 \quad \beta \, _2 \quad ) \quad ] \]

\[ \quad = \quad - \quad \beta \, _3 \quad \varepsilon \, _a \, q \, c \, \mu \, _2 \quad k \, _a \, _1 \quad k \, _c \, _2 \quad ] \]

\[ \beta \, _3 \quad \Delta \, _3 \quad = \quad - \quad \beta \, _2 \quad \beta \, _3 \quad k \, _1 \quad + \quad \beta \, _1 \quad \beta \, _3 \quad k \, _2 \quad ] \]

(171)

and substituting \( C \, _2 \, _3 \) from eq. (157)
\[
\varepsilon_{\alpha \rho \sigma \mu_2} k_{a_1} k_{c_2} = \varepsilon_{\alpha \rho \sigma \mu_2} k_{c_2} k_{a_1} = -\varepsilon_{\rho \sigma \alpha \mu_2} k_{a_3} k_{c_2} = -C_{23}
\]

\[\tilde{g} \, 5 = -\beta_2 \beta_3 B_5 + \beta_1 \beta_3 B_6\]

(172)

We collect the expressions for \( \tilde{g} \, 1 \to 5 \) in one table

\[\rightarrow\]

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\[ \tilde{g} 1 = \beta_2 \Delta_1 \Delta_2 \mathcal{B}_1 + \beta_1 \Delta_1 \Delta_2 \mathcal{B}_2 \]
\[ \tilde{g} 2 = \beta_2 (1 - \beta_2) \mathcal{B}_3 + \beta_1 \beta_2 \mathcal{B}_4 \]
\[ \tilde{g} 3 = \beta_1 \beta_2 \mathcal{B}_3 + \beta_1 (1 - \beta_1) \mathcal{B}_4 \]
\[ \tilde{g} 4 = \beta_2 \beta_3 \mathcal{B}_7 - \beta_1 \beta_3 \mathcal{B}_8 \]
\[ \tilde{g} 5 = -\beta_2 \beta_3 \mathcal{B}_5 + \beta_1 \beta_3 \mathcal{B}_6 \]

\[ \Delta_1 \Delta_2 = (X - \tilde{m}^2) - \beta_3 k_1 k_2 \]

\[ \text{(173)} \]

\[ ^a \text{Exercise: check all expressions in eq. (173).} \]
Reconstructing the amplitude $a_0$ in eqs. (125, 126, 146)

We resume the amplitude construction repeating eq. (146) below

$$a_0 = - \kappa S \int d \Omega_3 X^{-1} f_0$$

$$f_0 = g + \hat{m}^2 h$$

$$g = tr \gamma \gamma_\mu_1 \Delta_2 \gamma_\rho \gamma_5 R \times$$

$$\times \Delta_1 \gamma_\mu_2 \Delta_3$$

The associated amplitudes following the decomposition in eq. (174) shall be denoted \( \rightarrow \)
\[ a_0 = a_0^g + a_0^h \]  \hspace{1cm} (175)

We first turn to \( a_0^g \) defined in eq. (175):

\[ a_0^g = - (4i \kappa S) \int d\Omega_3 X^{-1} \times \]

\[ \times \sum_{n=1}^{5} \tilde{g}_n \]

with the quantities \( \tilde{g}_n \) to be substituted according to eq. (173).

For \( a_0^h \) it follows
\[ a^h_0 = - \kappa S \int d \Omega_3 X^{-1} \hat{m}^2 h \]

\[ h = -4 i \varepsilon_{\mu_1 \varepsilon \mu_2 \sigma} (\Delta_3 + d)^\sigma \]

\[ d = k_1 - k_2, \quad \Delta_3 = \beta_1 k_2 - \beta_2 k_1 \]

\[ \Delta_3 + d = (1 - \beta_2) k_1 - (1 - \beta_1) k_2 \]

(177)

Substituting \( h \), \( \Delta_3 \) and \( d \) from eq. (149).

\( \mathcal{B}_\alpha \) decomposition of \( \tilde{h} = \frac{1}{4i} h \)

We complete the systematic decomposition of amplitudes, step by step
\[
\tilde{h} = (1 - \beta_1) \varepsilon_{\mu_1 \rho \mu_2 \sigma} k^\rho \sigma - \\
- (1 - \beta_2) \varepsilon_{\mu_1 \rho \mu_2 \sigma} k_1^\rho \sigma = (1 - \beta_2) B_1 + (1 - \beta_1) B_2
\]

(178)

using the definitions in eq. (155).

The relation analogous to eq. (176) for \( a_0^h \) thus becomes

\[
a_0^h = - (4 i \kappa S) \int d\Omega_3 X^{-1} \tilde{m}^2 \tilde{h}
\]

(179)

\[
\tilde{h} = (1 - \beta_2) B_1 + (1 - \beta_1) B_2
\]
This completes the 'deconstruction' of the $a_0$ amplitudes and I repeat eqs. (176 and 179) below

\[
\begin{align*}
    a^g_0 &= - (4 \, i \, \kappa \, \mathcal{S}) \int d \Omega_3 \, X^{-1} \times \\
        &\quad \times \sum_{n=1}^{5} \tilde{g}_n \\
    a^h_0 &= - (4 \, i \, \kappa \, \mathcal{S}) \int d \Omega_3 \, X^{-1} \tilde{m}^2 \tilde{h} \\
    \tilde{h} &= (1 - \beta_2) \mathcal{B}_1 + (1 - \beta_1) \mathcal{B}_2
\end{align*}
\]

(180)

Exercise: check whether the so decomposed $a_0$ amplitudes by themselves are Bose symmetric with respect to the two photons.
Reconstructing the amplitude $a_1$ in eqs. (137 - 142)

We go back to eqs. (137 - 142) from which we retain the composition of the $a_1$ amplitudes →
\[ a_1 = a_1 (\mu^2) + \delta a_1 \]

\[ \delta a_1 = \frac{4}{3} \kappa S i \varepsilon_{\mu_1 \mu_2 \varrho \sigma} d^\sigma K \]

\[ K = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]

\[ a_1 (\mu^2) = a_1 \log = \]

\[ = \kappa S \int \log (\mu^2 / X) d \Omega_3 (f_1 / 2) \]

\[ f_1 / 2 = 4 i \varepsilon_{\mu_1 \varrho \mu_2 \sigma} (3 \Delta_3 + d)^\sigma \]
Again proceeding step by step we decompose first \( \delta a_1 \) from eq. (181) using eq. (155)

\[
\delta a_1 = \frac{4}{3} i \kappa S K ( B_1 + B_2 )
\]  

(182)

Because of the ambiguous value of the constant \( K \) in eq. (181) we leave it here as indefinite.

Next we turn to \( a_1 \) \( \log \)

\[
a_1 \log = 4 i \kappa S \int \log ( \mu^2 / X ) \, d \Omega_3 \times
\]

\[
\times \varepsilon_{\mu_1 \mu_2 \sigma} ( 3 \Delta_3 + d )^\sigma \rightarrow
\]

(183)
and substitute the expressions

\[ \Delta_3 = \beta_1 k_2 - \beta_2 k_1, \quad d = k_1 - k_2 \]

\[ 3 \Delta_3 + d = \begin{bmatrix} (1 - 3 \beta_2) k_1 - \\ - (1 - 3 \beta_1) k_2 \end{bmatrix} \] (184)

and thus reach the decomposition for \( a_1 \)_{log}

\[ a_1 \text{}_{log} = -4 i \kappa S \int \log \left( \mu^2 / X \right) d \Omega_3 \times \]

\[ \times \left( (1 - 3 \beta_2) B_1 + (1 - 3 \beta_1) B_2 \right) \] (185)

\[ a \quad \text{As for} \ a_0 \ \text{we note the Bose symmetry of} \ a_1, \ \text{check!} \]
The decomposition of the $a_1$ amplitudes thus yields

$$a_1 = a_1 \log + \delta a_1$$

$$a_1 \log = -4i \kappa S \int \log \left( \frac{\mu^2}{X} \right) d\Omega_3 \times$$

$$\times ( (1 - 3\beta_2) B_1 + (1 - 3\beta_1) B_2 )$$

$$\delta a_1 = \frac{4}{3} i \kappa S K (B_1 + B_2)$$

(186)
Invariant amplitudes \( 4 i \kappa S \sum_\alpha \mathcal{M}_\alpha \mathcal{B}_\alpha \)

Factoring out a common constant \( P = 4 i \kappa S \) we collect the contributions to invariant multipliers denoted \( \mathcal{M}_\alpha \)

\[
\mathcal{M}_1 = \\
= \left[ \frac{1}{3} K + \\
+ \int \log \left( \frac{X}{\mu^2} \right) d \Omega_3 \left( 1 - 3 \beta_2 \right) - \\
- \int d \Omega_3 X^{-1} \left( \beta_2 \Delta_1 \Delta_2 + \right. \\
\left. + \right. \\
\left. \hat{m}^2 \left( 1 - \beta_2 \right) \right) \right]
\]

(187)
\[ \mathcal{M}_2 = \]
\[
\begin{bmatrix}
\frac{1}{3} K + & \\
+ \int \log \left( \frac{X}{\mu^2} \right) d\Omega_3 & (1 - 3\beta_1) - \\
- \int d\Omega_3 X^{-1} & \left( \beta_1 \Delta_1 \Delta_2 + \
+ \tilde{m}^2 \left( 1 - \beta_1 \right) \right)
\end{bmatrix}
\]

(188)

For the amplitudes \( \mathcal{M}_{1,2} \) in eqs. (187, 188) the following equations enter: eq. 186 for \( a_1 \), eq. 177 for \( \tilde{m}^2 \tilde{h} \) and eq. 173 for \( \tilde{g}_1 \).
The remaining 6 amplitudes $M_{3-8}$ come from $a_{0g}$ (eq. 176) through the quantities $\tilde{g}^{2-5}$ defined in eq. 173.

$$M_3 = - \int d\Omega_3 X^{-1} \left[ \begin{array}{c} \beta_1 \beta_2 + \\ + \beta_2 (1 - \beta_2) \end{array} \right]$$

$$M_4 = - \int d\Omega_3 X^{-1} \left[ \begin{array}{c} \beta_1 \beta_2 + \\ + \beta_1 (1 - \beta_1) \end{array} \right]$$

(189)
\[ M_5 = \int d\Omega_3 X^{-1} \beta_2 \beta_3 \]
\[ M_8 = \int d\Omega_3 X^{-1} \beta_1 \beta_3 \]
\[ M_6 = -\int d\Omega_3 X^{-1} \beta_1 \beta_3 \]
\[ M_7 = -\int d\Omega_3 X^{-1} \beta_2 \beta_3 \]  

This completes the calculation of the amplitudes \( M_1 - 8 \) (eqs. 188 - 190) showing the separate Bose symmetry of both diagrams in Fig. 5. This implies a multiplicative factor of 2 accounting for the crossed diagram.
Implementing electromagnetic gauge invariance

The divergencies were compiled in eqs. 159, 160, reproduced below

\[ D^\alpha_1 = D^\alpha_1 \mu_3 \mu_2 = k^\mu_1 \beta_{\alpha \mu_1 \mu_3 \mu_2} \]

\[ D^\alpha_2 = D^\alpha_2 \mu_1 \mu_3 = k^\mu_2 \beta_{\alpha \mu_1 \mu_3 \mu_2} \]

\[ D^\alpha_3 = D^\alpha_2 \mu_1 \mu_2 = k^\mu_3 \beta_{\alpha \mu_1 \mu_3 \mu_2} \]

\[ k_1 + k_2 + k_3 = 0 \]
<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$D_1^\alpha$</th>
<th>$D_2^\alpha$</th>
<th>$D_3^\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>0</td>
<td>$C_{31}$</td>
<td>$-C_{12}$</td>
</tr>
<tr>
<td>(2)</td>
<td>$C_{23}$</td>
<td>0</td>
<td>$-C_{12}$</td>
</tr>
<tr>
<td>(3)</td>
<td>0</td>
<td>0</td>
<td>$k_1 k_3 C_{12}$</td>
</tr>
<tr>
<td>(4)</td>
<td>0</td>
<td>0</td>
<td>$k_2 k_3 C_{12}$</td>
</tr>
<tr>
<td>(5)</td>
<td>$k_1 k_1 C_{23}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(6)</td>
<td>$k_1 k_2 C_{23}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(7)</td>
<td>0</td>
<td>$k_1 k_2 C_{31}$</td>
<td>0</td>
</tr>
<tr>
<td>(8)</td>
<td>0</td>
<td>$k_2 k_2 C_{31}$</td>
<td>0</td>
</tr>
</tbody>
</table>
Electromagnetic gauge invariance requires

\[ \sum_{\alpha} M_{\alpha} D_{n}^{\alpha} = 0 \quad \text{for} \quad n = 1, 2 \rightarrow \]

\[ M_{2} = -k_{1}^{2} M_{5} - k_{1} k_{2} M_{6} \]

\[ M_{1} = -k_{2}^{2} M_{8} - k_{1} k_{2} M_{7} \]

(193)

Bose symmetry implies \( M_{1} \leftrightarrow M_{2} \), i.e. the consistency of the two last equations remembering eq. 161. This resolves the ambiguity of the amplitudes \( a_{1} \) as e.m. gauge invariance is required for the renormalization of QED.
The divergence of the axial current

Now we go back to eqs. 114 - 116 and Fig. 5 and calculate the divergence of the pion associated actual current from the order $e^2$ vertex function

\[
i \partial \frac{\rho}{z} \cdot j_5 \pi^0(z) \leftrightarrow k \frac{\rho}{3} \cdot j_5 \pi^0(z)
\]

\[
k \frac{\rho}{3} \cdot \equiv 2 \ P \ \sum_{\alpha} M_{\alpha} D_{3}^{\alpha}
\]

\[
= 2 \ P \ C_{12} D_{5}
\]

\[
D_{5} = \left[ - M_{1} - M_{2} +
\right. \\
\left. + k_{1} k_{3} M_{3} + k_{2} k_{3} M_{4} \right]
\]

\[
\rightarrow
\]
Substituting $\mathcal{M}_{1,2}$ using eq. 193 we obtain

$$D_5 = \left[ k_1^2 \mathcal{M}_5 + k_2^2 \mathcal{M}_8 + 
+ k_1 k_2 (\mathcal{M}_6 + \mathcal{M}_7) + 
+ k_1 k_3 \mathcal{M}_3 + k_2 k_3 \mathcal{M}_4 \right]$$

(195)

"Do not believe just what I say
but do conceive it your own way."

... Kapitel ... WS 2005/6
The partial divergence insertion

In the assignment of eq. (194) we are led to associate with the quantity

\[ k \frac{\partial}{\partial z} \leftrightarrow i \partial \frac{\partial}{\partial z} j \pi^0 (z) \]

\[ = - \frac{1}{2} (2 \hat{m}) \begin{bmatrix} \bar{u}^c \gamma_5 R u^c - \\
-\bar{d}^c \gamma_5 R d^c \end{bmatrix} \]

(196)

This is tantamount to the Green function, as expressed in eq. 114, tracing however the correct prefactors of i and \( \frac{1}{2} \).
\[ \kappa \frac{\phi}{3} \rightarrow \tau_{\mu_1 \mu_2} = \]

\[ = \frac{1}{2} \left( 2 \hat{m} \right) e^2 \int dx_1 \, dx_2 \, e + i \sum_j^2 k_j x_j \times \]

\[ \times \langle \Omega | \, T \left\{ j_{\mu_1}^{em}(x_1) \, j_{\mu_2}^{em}(x_2) \, j_5^m(0) \right\} | \Omega \rangle \]

\[ j_5^m = \begin{bmatrix} \bar{u}^c \gamma_5 R \, u^c - \\
- \bar{d}^c \gamma_5 R \, d^c \end{bmatrix} \]

(197)

The decomposition of \( \tau_{\mu_1 \mu_2} \) follows closely the vertex Green function.
\[ \tau_{\mu_1 \mu_2} = \]
\[= -2 \hat{m} S e^2 \int d^4 L (2\pi)^{-4} \left[ A_m + B_m \right] \]
\[A_m = a_m \prod_{r=1}^{3} \left[ i / \left( L^2_r - \hat{m}^2 \right) \right] \]
\[a_m = tr \gamma \left\{ \gamma_{\mu_1} \left( \frac{L}{\hat{m}} + \frac{k_1}{\hat{m}} \right) \gamma_5 R \times \right. \]
\[\times \left( \frac{L - k_2}{\hat{m}} + \frac{k_1}{\hat{m}} \right) \gamma_{\mu_2} \times \]
\[\left. \times \left( \frac{L}{\hat{m}} + \frac{k_1}{\hat{m}} \right) \right\} \]
\[L_1 = L - k_2, \quad L_2 = L + k_1, \quad L_3 = L \rightarrow \]
\[\text{(198)} \]
As in eq. 118 the crossed amplitude $B_m$ defined in eq. 198 is obtained from $A_m$ by Bose transposition

$$B_m(\mu_1, k_1; \mu_2, k_2) =$$

$$A_m(\mu_2, k_2; \mu_1, k_1)$$

The Bogoliubov parameters are the same as in eq. 120
\[ \gamma_1 = \gamma \beta_1 \leftrightarrow L - k_2 \]
\[ \gamma_2 = \gamma \beta_2 \leftrightarrow L + k_1 \]
\[ \gamma_3 = \gamma \beta_3 \leftrightarrow L \]
\[ \beta_k \geq 0 \; ; \; \sum_j^3 \beta_j = 1 \]

with the integration measure (eq. 121)

\[ \prod_j^3 d \gamma_j = \gamma^2 d \gamma d \Omega_3 (\beta) \]

\[ d \Omega_3 = \prod_j^3 d \beta_j \delta (1 - \sum_k^3 \beta_k) \]

Here the steps in eq. 122 are identical
\[ A_m = -2 \widehat{m} S e^2 \int \gamma^2 \, d\gamma \, d\Omega \times \]
\[ \times d^4 \overline{L} \cdot (2\pi)^{-4} \, e^{i \gamma E} a_m \]
\[ E = B - \widehat{m}^2 ; \quad B = \sum \beta_j L_j^2 \]
\[ L_j = \overline{L} + \Delta_j ; \quad L_1 = L - k_2 \]
\[ L_2 = L + k_1 ; \quad L_3 = L = \overline{L} + \Delta_3 \]
\[ \Delta_3 = \beta_1 k_2 - \beta_2 k_1 \]
\[ \Delta_1 = \Delta_3 - k_2 ; \quad \Delta_2 = \Delta_3 + k_1 \]

\[ \text{Exercise: check for the fermion loop -1 factor.} \]
Using eqs. 123 and 125 we obtain

\[ A_m = 2 \hat{m} S \kappa \int d \varrho d \Omega_3 e^{-\varrho X} \hat{a}_m \]

\[ \varrho = i \gamma = \gamma_E \]

\[ \hat{a}_m = tr \gamma \gamma \mu_1 \left( \hat{l} + \Delta_2 + \hat{m} \right) \gamma_5 R \times \]

\[ \times \left( \hat{l} + \Delta_1 + \hat{m} \right) \gamma \mu_2 \left( \hat{l} + \Delta_3 + \hat{m} \right) \]

\[ \hat{l}_\alpha = \partial_\kappa \alpha - \left( 2 \gamma i \right)^{-1} \kappa_\alpha \]

(203)

The quantity X in eq. 203 is identical to its definition in eq. 123.
The subsequent evaluation of $A_m$ proceeds on the same lines as the $a_0$ amplitudes in eq. 146

$$A_m = 2 \hat{m} S \kappa \int d \Omega_3 X^{-1} a_{m0}$$

$$a_{m0} = tr \gamma \gamma_{\mu_1} \left( \Delta_2 + \hat{m} \right) \gamma_5 R \times$$

$$\times \left( \Delta_1 + \hat{m} \right) \gamma_{\mu_2} \left( \Delta_3 + \hat{m} \right)$$

The differential operators $\hat{l}$ do not contribute.

For $a_{m0}$ we obtain
\[ a_{m_0} = \tilde{m} \text{ tr } \gamma \left( \begin{array}{c} \gamma \mu_1 \Delta_2 \gamma 5 R \Delta_1 \gamma \mu_2 + \\ + \gamma \mu_1 \gamma 5 R \Delta_1 \gamma \mu_2 \Delta_3 + \\ + \gamma \mu_1 \Delta_2 \gamma 5 R \gamma \mu_2 \Delta_3 \end{array} \right) \]

\[ = \tilde{m} \text{ tr } \gamma \left( \begin{array}{c} \gamma \mu_1 \Delta_1 \gamma \mu_2 \Delta_2 \gamma 5 R - \\ - \gamma \mu_1 \Delta_1 \gamma \mu_2 \Delta_3 \gamma 5 R + \\ + \gamma \mu_1 \Delta_2 \gamma \mu_2 \Delta_3 \gamma 5 R \end{array} \right) \]

(205)
and using the relations among $\Delta_n$ in eq. 202 and eq. 157

\[
\begin{align*}
    a_{m0} &= \hat{m} \text{ tr } \gamma \left( -\gamma \mu_1 \, k_1 \, \gamma \mu_2 \, \Delta_1 \, \gamma 5 \, R + \right.
    \left. + \gamma \mu_1 \, k_1 \, \gamma \mu_2 \, \Delta_3 \, \gamma 5 \, R \right) \\
    &= \hat{m} \text{ tr } \gamma \left( \gamma \mu_1 \, k_1 \, \gamma \mu_2 \, k_2 \, \gamma 5 \, R \right) \\
    &= 4 \, i \, \hat{m} \, C_{12}
\end{align*}
\]

(206)
Invariant amplitude \( A_m = 4i \kappa S M_m C_{12} \)

Factoring out the common constant \( P = 4i \kappa S \) we associate the invariant function \( M_m \) with \( A_m \)

\[
A_m = P M_m C_{12}
\]

\[
P = 4i \kappa S
\]

\[
M_m = 2 \tilde{m}^2 \int d \Omega_3 X^{-1}
\]

\[
B_m = A_m
\]

Again Bose symmetry is manifest.
Invariant amplitudes for soft and hard parts of the divergence of the axial current

We subtract the soft part of the axial current divergence $A_m + B_m$ as given in eq. 207 from the full divergence defined in eqs. 194 and 195

$$k \frac{\partial}{\partial z} j^5 \pi^0 (z) \bigg|_{soft} = 2 P \mathcal{M}_m \mathcal{C}_{12}$$

$$\Delta k \frac{\partial}{\partial z} = k \frac{\partial}{\partial z} - k \frac{\partial}{\partial z} \bigg|_{soft}$$

$$= 2 P \mathcal{C}_{12} \Delta D^5$$

$$\Delta D^5 = D^5 - \mathcal{M}_m$$
\[ i \partial_{\frac{q}{z}} j_{\frac{q}{z}} \pi^0(z) \leftrightarrow k_{\frac{q}{3}} j_{\frac{q}{z}} \pi^0(z) \]

\[ k_{\frac{q}{3}} \equiv 2 P \sum_{\alpha} M_{\alpha} D_{\frac{3}{3}} \]

\[ = 2 P C_{12} D_{5} \]

\[ D_{5} = \begin{bmatrix}
  k_{\frac{2}{1}} M_{5} + k_{\frac{2}{2}} M_{8} + \\
  + k_{1} k_{2} (M_{6} + M_{7}) + \\
  + k_{1} k_{3} M_{3} + k_{2} k_{3} M_{4}
\end{bmatrix} \]

We collect the invariant functions \( M_{3-8} \) and \( M_{m} \) (eqs. 189, 190 and 207) below.
\[ \mathcal{M}_\kappa = \int d\Omega_3 \ X^{-1} \ \Phi_\kappa \]

\[ \Phi_3 = - \left[ \beta_1 \beta_2 + \right. \left. + \beta_2 (1 - \beta_2) \right] \]

\[ \Phi_4 = - \left[ \beta_1 \beta_2 + \right. \left. + \beta_1 (1 - \beta_1) \right] \]

\[ \Phi_5 = \beta_2 \beta_3, \quad \Phi_8 = \beta_1 \beta_3 \]

\[ \Phi_6 = - \beta_1 \beta_3, \quad \Phi_7 = - \beta_2 \beta_3 \]

\[ \Phi_m = 2 \ \widehat{m}^2 \]
We rearrange the scalar products in the expression for $D_5$ in eq. 209

$$2 \, k_1 \, k_3 = k_2^2 - k_1^2 - k_3^2$$

$$2 \, k_2 \, k_3 = k_1^2 - k_2^2 - k_3^2$$

$$2 \, k_1 \, k_2 = k_3^2 - k_1^2 - k_2^2$$

$D_5$ transforms into

$\rightarrow$
\[ D_5 \rightarrow \]
\[ k \frac{2}{1} \left( 5 + \frac{1}{2} \left( 4 - 3 - 6 - 7 \right) \right) \]
\[ + \quad k \frac{2}{2} \left( 8 + \frac{1}{2} \left( 3 - 4 - 6 - 7 \right) \right) \]
\[ + \quad k \frac{2}{3} \left( 6 + 7 - 3 - 4 \right) \frac{1}{2} \]

We cast eq. 210 into the form
\[ \Phi_3 = - \left[ 2 \beta_1 \beta_2 + \beta_2 \beta_3 \right] \]
\[ \Phi_4 = - \left[ 2 \beta_1 \beta_2 + \beta_1 \beta_3 \right] \]
\[ \Phi_5 = \beta_2 \beta_3 \quad , \quad \Phi_8 = \beta_1 \beta_3 \]
\[ \Phi_6 = - \beta_1 \beta_3 \quad , \quad \Phi_7 = - \beta_2 \beta_3 \]
\[ \Phi_m = 2 \hat{m}^2 \]

Next, substituting \( \Phi \rightarrow \kappa \) as in eq. 212, we obtain
\[-(3 + 4) = 4\beta_1\beta_2 + \beta_2\beta_3 + \beta_1\beta_3\]
\[(6 + 7) = -(\beta_2\beta_3 + \beta_1\beta_3)\]
\[(4 - 3) = (\beta_2\beta_3 - \beta_1\beta_3)\]

We go step by step, determining eq. 212

\[4 - 3 - 6 - 7 = 2\beta_2\beta_3\]
\[3 - 4 - 6 - 7 = 2\beta_1\beta_3\]
\[6 + 7 - 3 - 4 = 4\beta_1\beta_2\]
Thus eq. 212 becomes

\[
D_5 \rightarrow 2 \left( \begin{array}{c}
\beta_2 \beta_3 k_1^2 + \\
\beta_1 \beta_3 k_2^2 + \\
\beta_1 \beta_2 k_3^2 \\
\end{array} \right)
\]

\[
= 2 \left( \tilde{m}^2 - X \right)
\]

Next we repeat eq. 208 below
\[ k \frac{e}{3} \cdot j \frac{5^\pi}{3} (z) \bigg|_{soft} = 2 P M_m \mathcal{C}_{12} \]
\[ \Delta k \frac{e}{3} . = k \frac{e}{3} . - k \frac{e}{3} . \bigg|_{soft} \]
\[ = 2 P \mathcal{C}_{12} \Delta D_5 \]
\[ \Delta D_5 = D_5 - M_m \]
\[ M_m \rightarrow \Phi_m = 2 \widehat{m}^2 \]

We recall that the mnemonic \( \rightarrow \) sign in eqs. 212 and 216 is defined in eq. 210
\[ \mathcal{M}_\kappa \rightarrow \Phi_\kappa \equiv \mathcal{M}_\kappa = \int d\Omega_3 \ X^{-1} \Phi_\kappa \] (218)

Thus it follows from eq. 217

\[ \Delta D_5 \rightarrow -2X \rightarrow \]

\[ \Delta D_5 = \int d\Omega_3 \ X^{-1} (-2X) = -1 \] (219)
From invariant amplitudes to the local operator relation for soft and hard parts of the divergence of the axial current

We go back to eqs. 208 and 209 reproduced below

\[
\frac{k_3^0 \cdot j_5^q \pi^0 (z)}{\text{soft}} = 2 P \, M_m^C \, C_{12}
\]

\[
\frac{1}{i} \Delta k_3^0 = \frac{1}{i} \left( k_3^0 - k_3^0 \right) \bigg|_{\text{soft}}
\]

\[
= R \, C_{12} \, \Delta D_5 = - R \, C_{12}
\]

\[
\Delta D_5 = D_5 - M_m = -1
\]

\[
2P = i R; \quad R = 8 \kappa S; \quad S = \frac{1}{2} \left( N_c / 3 \right)
\]

(220)
\[ \partial_z \frac{\rho}{q} \cdot j_{q}^{5 \pi^0} (z) \Leftrightarrow \frac{1}{i} k_{\frac{q}{3}} \cdot j_{q}^{5 \pi^0} (z) \]

\[ \frac{1}{i} k_{\frac{q}{3}} \equiv R \sum_{\alpha} M_{\alpha} D_{3}^{\alpha} \]

\[ = R C_{12} D_{5} \]

(221)

\[ D_{5} = \begin{bmatrix} k_{1}^{2} M_{5} + k_{2}^{2} M_{8} + \\
+ k_{1} k_{2} (M_{6} + M_{7}) + \\
+ k_{1} k_{3} M_{3} + k_{2} k_{3} M_{4} \end{bmatrix} \]

In eqs. 220, 221 I have substituted the result in eq. 219 and adapted a factor \( i \).
We also recall the form of the current and its soft divergence from eq. 196

\[
\begin{align*}
\dot{j}_\pi^5 \pi^0 &= \frac{1}{2} \left[ \begin{array}{c}
\bar{u}^c \gamma_\rho \gamma_5 R u^c \\
- \bar{d}^c \gamma_\rho \gamma_5 R d^c
\end{array} \right] \\
\frac{1}{i} k \frac{g}{3} \bigg|_{\text{soft}} &\leftrightarrow \partial_\rho \cdot j_\pi^5 \pi^0 (z) \bigg|_{\text{soft}} \\
&= \frac{1}{2} \left( 2 \hat{m} \right) \left[ \begin{array}{c}
\bar{u}^c i \gamma_5 R u^c \\
- \bar{d}^c i \gamma_5 R d^c
\end{array} \right]
\end{align*}
\]

The clear consequence of eq. 219 – at least to the order in e considered – is, that the $\pi^0$ associated
axial current possesses a local 'hard' divergence, which I shall call \( \pi^0 \) in the following

\[
\partial_{\bar{z}} \cdot j_{\bar{q}} \pi^0 (z) = \left( \partial_{\bar{z}} \cdot j_{\bar{q}} \pi^0 (z) \bigg|_{soft} + \right) + an \pi^0 (z)
\]

(223)

From eq. 220 we infer

\[
\langle \gamma_2 , \gamma_1 | an \pi^0 (0) | \Omega \rangle = - R \varepsilon^*_{\mu_1} k_{\sigma_1} \varepsilon^*_{\mu_2} k_{\sigma_2} \varepsilon_{\mu_1 \sigma_1 \mu_2 \sigma_2}
\]

(224)
remembering the definition of the tensors $C_{mn}$ in eq. 157.\(^a\)

The local operator $an^{\pi^0(z)}$ defined through eq. 223 is – again to the order in $\epsilon$ considered – bilinear in the electromagnetic field strengths

$$an^{\pi^0(z)} =$$

$$= k_{an} \left( \frac{1}{2} \right) \left[ F^{\varrho 1 \sigma 1}(z) F^{\varrho 2 \sigma 2}(z) \times \right.$$

$$\left. \times \varepsilon^{\varrho 1 \sigma 1 \varrho 2 \sigma 2} \right] ^{225}$$

\(^a\) Exercise: show the validity of the deduction leading to eqs. 223 and 224.
In order to determine the (real) proportionality constant $k_{an}$ defined in eq. 225 we first determine the one photon matrix element

$$\langle \gamma \mid F^{\sigma} (z) \mid \Omega \rangle =$$

$$= \left[ \partial_{z}^{\sigma} \left( e^{i k z} \langle \gamma \mid A^{\sigma} (0) \mid \Omega \rangle \right) - \partial_{z}^{\sigma} \left( e^{i k z} \langle \gamma \mid A^{\sigma} (0) \mid \Omega \rangle \right) \right]$$

$$= e^{i k z} i \left( \varepsilon^{*} \sigma k \varepsilon - \varepsilon^{*} \sigma k \varepsilon \right)$$

(226)

It follows from eq. 226 and our convention on the $2 \gamma$ state \(^a\)

\(^a\) "Please show".

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\[ \langle \gamma_2, \gamma_1 | \left[ F_{\ell_1 \sigma_1} (0) F_{\ell_2 \sigma_2} (0) \times \varepsilon_{\ell_1 \sigma_1 \ell_2 \sigma_2} \right] | \Omega \rangle = -8 \varepsilon^* \mu_1 k_1^{\sigma_1} \varepsilon^* \mu_2 k_2^{\sigma_2} \varepsilon_{\mu_1 \sigma_1 \mu_2 \sigma_2} \]

Comparing eqs. 224, 225 and 227 we obtain
\[ k_{an} = \frac{1}{4} R = \kappa (2S) \]
\[ \kappa = e^2 / (16 \pi^2) \]

\[ a \]

\[ a \text{ This point was reached 12. March 2006.} \]
The connection between $\pi^0$ and $ch_2(M_4)$

We complement the form of the axial current anomaly in eqs. 223 and 225 using eq. 228. The expressions for $j_{\ell}^5 \pi^0$ and $\partial_{\ell}^\rho \cdot j_{\ell}^5 \pi^0(\bar{z})|_{soft}$ are given in eq. 222.
\[ \partial \frac{\rho}{z} \cdot j^{5 \pi^0}(z) = \left( \partial \frac{\rho}{z} \cdot j^{5 \pi^0}(z) \Big|_{\text{soft}} + \right) \]

\[ \text{an} \pi^0(z) = \]

\[ = k \text{an} \frac{1}{2} \left[ F_{\rho 1 \sigma 1}(z) F_{\rho 2 \sigma 2}(z) \times \varepsilon_{\rho 1 \sigma 1 \rho 2 \sigma 2} \right] \]

\[ k \text{an} = \kappa \left( 2 S \right) ; \quad S = \frac{1}{2} \left( S_u - S_d \right) \]

\[ S_q = N_c \left( \frac{e_q}{e} \right)^2 ; \quad q = u, d \]

(229)
Chern characters in physical, uncurved space-time \( ch_2 ( M_d ) \)

We give here a definition of chern characters in \( d \) dimensions with Lorentzian metric
\[ g_{\mu \nu} = \text{diag} ( 1 , - 1 , \cdots , -1 ) . \] The arguments are the hermitian charge and color weighted electromagnetic field strengths and associated two forms

\[
\mathcal{F}^{\prime \prime}_{\mu \nu} = e_q \delta^{\prime \prime}^{c} \left( \frac{1}{2 \pi} F_{\mu \nu} \right)
\]

\[
F_{\mu \nu} = \partial_{\nu} A_{\mu} - \partial_{\mu} A_{\nu} = F^{\ast}_{\mu \nu} \rightarrow (230)
\]

\[
\left( \mathcal{F}^{(2)} \right)^{\prime \prime^{c}} = \frac{1}{2} \mathcal{F}^{\prime \prime}_{\mu \nu} e_q d x^\mu \wedge d x^\nu
\]
and consider $\mathcal{F}_q^{(2)}$ as a (diagonal) matrix in color space.

$e_u = \frac{2}{3} e$ and $e_d = -\frac{1}{3} e$ in eq. 230 denote the absolute electromagnetic charges of up ($u$) and down ($d$) quark flavors respectively.

We consider the generic association of hermitian 2-form with the exponential function $e^\lambda$

$$e^\lambda \iff \text{ch}_q(\lambda) = -\text{tr}_{\text{color}} \left[ \exp \left( \lambda \mathcal{F}_q^{(2)} \right) \right]$$

$$\text{ch}_q(\lambda) = \sum_{n=0}^{\infty} \lambda^n \text{ch}_{n,q} \quad \rightarrow \quad (231)$$
For \( d = 2 \) \( n = 4 \) we obtain

\[
ch_{2 \ q} = \left( -\frac{1}{2} \kappa \ S_q \left[ \begin{array}{c}
F_{\mu \ 1 \ \sigma \ 1} \ F_{\mu \ 2 \ \sigma \ 2} \\
x \ d \ x^{\mu \ 1} \ \wedge \ d \ x^{\mu \ 2} \ \wedge \ d \ x^{\mu \ 3} \ \wedge \ d \ x^{\mu \ 4}
\end{array} \right] \right)
\]

\[
S_q = \begin{cases} 
\frac{4}{3} \ N_c / 3 & \text{for } q = u \\
-\frac{1}{3} \ N_c / 3 & \text{for } q = d 
\end{cases}
\]

(232)

The sign(s) is (are) here adapted to the convention

\[
\varepsilon_{0\ 1\ 2\ 3} = 1 = -\varepsilon^{0\ 1\ 2\ 3}
\]
\[ d \ x^\mu_1 \wedge d \ x^\mu_2 \wedge d \ x^\mu_3 \wedge d \ x^\mu_4 = \]
\[ = - d^4 x \times \varepsilon^{\mu_1 \mu_2 \mu_3 \mu_4} \tag{233} \]

**Substituting eq. 233 in eq. 232 we obtain**

\[ ch_{2q} (x) = \]
\[ = d^4 x \ \kappa S_q \frac{1}{2} \left[ F \varrho_1 \sigma_1 (x) F \varrho_2 \sigma_2 (x) \times \right. \]
\[ \left. \times \varepsilon_{\varrho_1 \sigma_1 \varrho_2 \sigma_2} \right] \tag{234} \]

**We substitute eq. 231 in eq. 229 multiplied with**

\[ d^4 z \]
\[ d^4 \pi^0 (z) = \]
\[ = 2 \left( \frac{1}{2} (ch_2 u(z) - ch_2 d(z)) \right) \big|_{e.m.} \] (235)

The qualifying subscript \( e.m. \) in eq. 235 serves to restrict the Chern characters – Chern character densities – to exclusively electromagnetic gauge fields.

**Insertion concerning the abbreviation** \( F \tilde{F} \)

I note here that the quantity in eq. 234 →
\[
\frac{1}{2} \left[ F_{\ell_1 \sigma_1} (x) F_{\ell_2 \sigma_2} (x) \times \varepsilon_{\ell_1 \sigma_1 \ell_2 \sigma_2} \right] = F^{\mu \nu} \tilde{F}_{\mu \nu} \equiv F \tilde{F} \\
\tilde{F}_{\mu \nu} = \frac{1}{2} \varepsilon_{\mu \nu \sigma \tau} F^{\sigma \tau}
\]

end of insertion about \( F \tilde{F} \)

Extension to include QCD

It is straightforward to extend the axial current

\[
j_{\ell_5}^{\pi^0} \quad (\text{eq. 222}) \text{ to the flavor basis currents}
\]
\[
\begin{align*}
\jmath_{\bar{q}} \gamma_5 \pi^0 &= \frac{1}{2} \left[ \overline{u}^c \gamma_\bar{q} \gamma_5 R u^c - \
- \overline{d}^c \gamma_\bar{q} \gamma_5 R d^c \right] \\
\jmath_{\bar{q}} \gamma_5 q &= \overline{q}^c \gamma_\bar{q} \gamma_5 R q^c; \quad q = u, d, \ldots t \\
\partial_z \jmath_{\bar{q}} \gamma_5 q (z) \bigg|_{\text{soft}} &= 2 \hat{m}_q \left[ \overline{q}^c i \gamma_5 R q^c \right] (z) \\
\text{Eq. 229 becomes} \\
\partial_z \jmath_{\bar{q}} \gamma_5 q (z) &= \left( \partial_z \jmath_{\bar{q}} \gamma_5 q (z) \bigg|_{\text{soft}} + \right) \\
&+ \alpha n^q (z) \\
\end{align*}
\]
The curvature 2-form \((\mathcal{F}^{(2)}_q)^{c'}_c\) in eq. 230 now is extended to include the QCD field strengths

\[
\mathcal{F}^{c'}_\mu^{c}_\nu_q = \mathcal{F}^{c'}_\mu^{c}_\nu_q (QED) + \mathcal{F}^{c'}_\mu^{c}_\nu_q (QCD)
\]

\[
\mathcal{F}^{c'}_\mu^{c}_\nu_q (QED) = e_q \delta^{c'}_c \left( \frac{1}{2\pi} F^{\mu\nu}_q \right)
\]

\[
\mathcal{F}^{c'}_\mu^{c}_\nu_q (QCD) = g_s \left( \frac{1}{2} \lambda^A \right)^{c'}_c \left( \frac{1}{2\pi} G^{A}_{\mu\nu} \right)
\]

\[
A = 1, \ldots, 8
\]

(239)

In eq. 239 \(g_s\) is the strong coupling constant, \(A\) numbers the adjoint basis of \(SU3_c\), \(\lambda^A\) denote the associated Gell-Mann matrices,
defining the triplet color representation, and $G^A_{\mu \nu}$ are the (nonabelian) field strengths

$$G^A_{\mu \nu} = \left[ \partial_\nu V^A_\mu - \partial_\mu V^A_\nu - g_s f_{ABC} V^B_\nu V^C_\mu \right]$$

$$\left[ \frac{1}{2} \lambda^A , \frac{1}{2} \lambda^B \right] = i f_{ABC} \left( \frac{1}{2} \lambda^C \right)$$

$$tr_{\text{color}} \left( \frac{1}{2} \lambda^A \right) \left( \frac{1}{2} \lambda^B \right) = \frac{1}{2} \delta^{AB}$$

Eq. 231 defining the Chern characters (character densities) remains unchanged
\[ e^\lambda \leftrightarrow ch_q(\lambda) = -tr_{color}\left[ \exp(\lambda F_q^{(2)}) \right] \]

\[ ch_q(\lambda) = \sum_{n=0}^{\infty} \lambda^n ch_{n\,q} \]

(241)

Eq. 234 then becomes

\[ ch_{2\,q}(x) = \]

\[ = d^4 x \left( ch_{2\,q}(x; F) + ch_{2\,q}(x; G) \right) \]

\[ ch_{2\,q}(x; F) = \kappa S_q F \tilde{F}(x) \]

\[ ch_{2\,q}(x; G) = \frac{1}{2} \kappa s G^A \tilde{G}^A(x) \]

(242)
In eq. 242 $\kappa$ and $\kappa_s$ denote the rationalized square coupling constants of QED and QCD respectively

$$\kappa = e^2 / (16\pi^2), \quad \kappa_s = g_s^2 / (16\pi^2) \quad (243)$$

whereas (for $N_c = 3$) the quantities $S_q$ are (eq. 232)

$$S_q = \begin{cases} \frac{4}{3} & \text{for } q = u, c, t \\ -\frac{1}{3} & \text{for } q = d, s, b \end{cases} \quad (244)$$
Eqs. 235 and 238 merge into

\[ d^4 z \ an^q(z) = 2 \ ch_2 q(z) \]

\[ d^4 z \ an^{\pi^0}(z) = \]

\[ = 2 \ \frac{1}{2} \ (ch_2 u(z) - ch_2 d(z)) \]

\[ = 2 \ \frac{1}{2} \ (ch_2 u(z) - ch_2 d(z)) |_{e.m.} \]

The QED and QCD Chern densities are defined in eq. 242 repeated below
\[ \text{ch}_2(q)(x) = \]
\[ = d^4 x \left( \text{ch}_2(q)(x; F) + \text{ch}_2(q)(x; G) \right) \]
\[ \text{ch}_2(q)(x; F) = \kappa S_q F \tilde{F}(x) \]
\[ \text{ch}_2(q)(x; G) = \frac{1}{2} \kappa_s G^A \tilde{G}^A(x) \]

(246)
The (conditional) quantization of the integrals over Chern densities

Preamble

The underlying ’just’ perturbative method – within quantized field theory – of deriving the axial current ’anomaly’ is discussed here in full detail with my specific intention to take care in distinguishing the consequences – in and for local field theory – from the perfectly rigorous framework of algebraic topology alone. →

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A remarkable fact
– including correctly the nonrenormalization properties, as they apply to renormalized composite operators –
is the emergence of the Chern densities in the local relations summarized in eqs. 245 and 246 and the factor 2 to be discussed below.

This emergence implies specific (boundary- as well as continuity-) conditions on the associated composite operators, which cannot be restricted to perturbative expansions.
These conditions can – at least it appears so – readily be formulated using the path integral representation. Yet the appropriate measure as well as the boundary conditions on the paths not only undergo full nonperturbative renormalization but also depend on all spontaneous parameters of the ground state of the field theory.

This framework is not at present reducible to a mathematical formulation (not even in $M_4$ uncurved space-time).
Hence loose connections with well defined mathematical frameworks are to serve at best as guidelines of how to achieve this, but by no means are to be considered superior structures of local field theory.

\[
\text{end of preamble}
\]

In the sense of the preceding preamble we shall assume \textit{conditionally} the following hypotheses\(^a\) concerning the quantities in eqs. 237 and 245, but considering the limit \(\widehat{m}_q = 0\) at least for a restricted set of flavors.

\(^a\) ‘Conditional’ is here synonymous with eventually incorrect.
1) The integral over $M_4$ of the divergence of chiral quark-flavor currents

\[ j_{\bar{q} q}^5 = \bar{q}^c \gamma \gamma \gamma 5 R q^c \text{ with } \widehat{m}_q = 0 \]

\[ d^4 x \partial \frac{\partial}{\partial \vec{z}} j_{\bar{q} q}^5 (x) = 2 ch_2 q (x) \]

(247)

The integral over the left hand side of the anomalous divergence relation in eq. 247 shall be evaluated as follows

\[ \rightarrow \]
\[
\int d^4 x \, \partial_z j^5_\varrho q (x) =
\]
\[
= \left[ \int dt \left( \frac{d}{dt} I^5 q(t) + \right) \right] + \int dt \int^t d^3 x \, \text{div} \, \vec{j}^5 q
\]
\[
\int^t d^3 x \, \text{div} \, \vec{j}^5 q \rightarrow 0
\]
\[
I^5 q(t) = \int^t d^3 x \, j^5_0 q
\]

and we assume that the integral over the space divergence \( \int^t d^3 x \, \text{div} \, \vec{j}^5 q \) vanishes.

The integral relation from eq. 247 then becomes

\[ \Delta I^5 q = 2 \Delta ch_2 q \]

\[ \Delta ch_2 q = \int d^4 x \ ch_2 q (x) \]  

(249)

2) Free massless fermion interpretation of the axial charge \( I^5 q (t) \)

We assume the purely intuitive operator relation

\[ I^5 q (t) = \left[ \hat{N}_R (q) + \hat{N}_R (\bar{q}) - \hat{N}_L (q) - \hat{N}_L (\bar{q}) \right] (t) \]

\[ \rightarrow \]

(250)
where $\hat{N}_R^{(L)}$ are number operators with quantized eigenvalues $N_R^{(L)} = \{ 0, \pm 1, \pm 2 \ldots \}$

3) Quantized transition eigenvalues of the operator $\Delta ch_{2q}$

We further assume that the transition operator (from $t = -\infty$ to $t = \infty$) $\Delta ch_{2q}$, defined in eq. 249 has quantized (transition-) eigenvalues $\nu = \{ 0, \pm 1, \pm 2 \ldots \}$.
Given hypotheses 1) - 3) discussed above, the factor 2 in eq. 249 assures conservation of fermion parity (for each q).

This property is necessary to ensure causal properties of Green functions and also the validity of CPT invariance.
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References


[26] Figures 1-4 were drawn by my daughter Claudia Minkowski.

[27] P. Minkowski, lunch seminar 24.11.05.