The trace anomaly in QCD and search for the associated canonical structure

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Abstract

The derivation of the trace anomaly in QCD is reviewed and its consistent embedding into a canonical structure of field variables is formulated. While a complete solution requires full control of the infrared instabilities , I will sketch the path required by consistency of renormalized field equations and trace anomaly.

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1 - Introduction

1-1 – Assembling elements of the Lagrangean density – premises

We face the theoretical abstraction of QCD with $N_{fl} = 6$, representing strong interactions – adaptable to two or three light flavors (u , d , s) of quarks and antiquarks. $\leftrightarrow$

quarks : color is counted in $\pi^0 \to \gamma\gamma$ (assuming global color- and flavor-projections to commute) yet see ref. [1-2001]

spin and flavor are clearly seen in $q\overline{q}$ and $3q$, $3\overline{q}$ spectroscopy (a pre-condition to count color) .

$$\mathcal{L} = \left[\begin{array}{c} \overline{q}^{\,\dot{c}'}_{\dot{\mathcal{S}}'\,\dot{f}} \left\{ \begin{array}{c} \frac{i}{2} \overleftrightarrow{\partial}_{\mu}\ \delta_{c'\dot{c}} \\ + W^{\,r}_{\mu} \left(\frac{1}{2} \lambda_r \right)_{c'\dot{c}} \end{array} \right\} \gamma^{\mu}_{\dot{\mathcal{S}}'\mathcal{S}}\, q^{\,c}_{\mathcal{S}\,f} \\ - m_f\, \overline{q}^{\,\dot{c}}_{\dot{\mathcal{S}}\,\dot{f}}\, q^{\,c}_{\mathcal{S}\,f} \end{array} \right]$$

(1)
$$- \frac{1}{4\,g^2} B^{\mu\nu\,r} B^{\,r}_{\mu\nu} + \Delta\mathcal{L}$$

$W^{\,r}_{\mu} \equiv - v^{\,r}_{\mu}$: for identification of convention for potentials

quarks : c' , $c = 1, 2, 3$ color , $f = 1, \cdots, 6$ flavor

$\mathcal{S}', \mathcal{S} = 1, \cdots, 4$ spin , m_f mass

$\longrightarrow$

In eq. 1 , the $\mathcal{D}$ associated gauge connection fields – where $\mathcal{D} = \mathcal{D}\,(\,\mathcal{G}\,)$ denotes a general , irreducible representation of the local gauge group $\mathcal{G} = SU3_{\,c}$ – appear in the form appropriate for quarks : $\mathcal{D} = \{\,3\,\}$, and antiquarks : $\mathcal{D} = \left\{\,\overline{3}\,\right\}$ respectively

(2)
$$\boxed{\begin{array}{l} (\,\mathcal{W}_{\,\mu}\,(\,\mathcal{D}\,)\,)_{\,\alpha\beta}\,(\,x\,) \,=\, W_{\,\mu}^{\,r}\,(\,x\,)\,(\,d_{\,r}\,)_{\,\alpha\beta} \;\;\leftrightarrow\;\; \mathcal{W}_{\,\mu}\,(\,\mathcal{D}\,) \,=\, -\,\mathcal{W}_{\,\mu}\,(\,\mathcal{D}\,)^{\,\dagger} \\ d_{\,r} \,=\, -\,d_{\,r}^{\,\dagger} \,=\, \frac{1}{i}\,J_{\,r} \,\in\, Lie\,(\,\mathcal{D}\,) \;\;;\;\; [\,d_{\,p}\,,\,d_{\,q}\,] \,=\, f_{\,pqr}\,d_{\,r} \\ r\,,\,p\,,\,q \,=\, 1,\,\cdots,\,dim\,\mathcal{G} \;\;;\;\; \alpha\,,\,\beta \,=\, 1,\,\cdots,\,dim\,\mathcal{D} \end{array}}$$

For $\mathcal{D}\,(\,SU3_{\,c}\,) = \left\{\,3\,(\overline{3})\,\right\}$ the representation matrices become (the Gell-Mann matrices [2-1964])

(3)
$$\begin{array}{l} \left(\,d_{\,r}\,(3) \,=\, \frac{1}{i}\,\frac{1}{2}\,\lambda_{\,r}\,\right)_{\,\alpha\beta} \;;\; r \,=\, 1,\,\cdots,\,8\,;\; (\,\alpha\,,\,\beta\,) \leftrightarrow \left(\,c^{\prime}\,,\,\dot{c}\,\right) = 1,\,\cdots,\,3 \\ d_{\,r}\,(\overline{3}) \,=\, \overline{d}_{\,r}\,(3) \end{array}$$

with the *conventional* normalization conditions : $-\;tr\,d^{\,r}\,d^{\,s} \;=\; \frac{1}{2}\,\delta^{\,rs}$

The quantity proportional to the gauge potentials $W_{\,\mu}^{\,r}$ for the $\overline{q}\,q$ in eq. 1 is thus identified as

(4)
$$\left[\,W_{\,\mu}^{\,r}\,\left(\,\tfrac{1}{2}\lambda_{\,r}\,\right)_{\,c'\dot{c}} \,=\, i\,(\,\mathcal{W}_{\,\mu}\,(\,\mathcal{D} \,=\, \{\,3\,\}\,)\,)_{\,c'\dot{c}}\,\right](\,x\,)$$

Here we postpone the discussion of complete connections and extend the QCD Lagrangean density to include the term quadratic in the field strengths $B_{\,\mu\nu}^{\,r}$ and $\Delta\,\mathcal{L}$ in eq. 1, pertinent to Fermi gauges. $\longrightarrow$

gauge bosons : $\mathcal{L}_B = - \frac{1}{4\, g^2} B^{\mu\nu\, r} B^{\, r}_{\mu\nu}$

$$B^{\, r}_{\mu\nu} = \partial_\mu W^{\, r}_\nu - \partial_\nu W^{\, r}_\mu + f_{rst} W^{\, s}_\mu W^{\, t}_\nu \longleftarrow \left(W^{\, r}_\mu \equiv - v^{\, r}_\mu \right)$$

(5) $$r, s, t = 1, \cdots, dim \left(G = SU3_c \right) = 8$$

Lie algebra labels, $\left[\frac{1}{2} \lambda^r , \frac{1}{2} \lambda^s \right] = i\, f_{rst} \frac{1}{2} \lambda^t$

perturbative rescaling :

$$W^{\, r}_\mu = g\, W^{\, r}_{\mu\, pert} , \; B^{\, r}_{\mu\nu} = g\, B^{\, r}_{\mu\nu\, pert}$$

Degrees of freedom are seen in jets , in (e.g.) the energy momentum sum rule in deep inelastic scattering but not clearly in spectroscopy.

Completing $\Delta\, \mathcal{L}$ in Fermi gauges

$$\Delta\, \mathcal{L} = \left\{ \begin{array}{l} - \frac{1}{2\, \eta\, g^2} \left(\partial_\mu W^{\mu\, r} \right)^2 \\ + \partial^\mu \overline{c}^{\, r} \left(D_\mu c \right)^r \end{array} \right\} \; ; \quad \eta \; : \; \textbf{gauge parameter}$$

(6)

ghost fermion fields : $c , \overline{c} \; ; \; \left(D_\mu c \right)^r = \partial_\mu c^r + f_{rst} W^{\, s}_\mu c^t$

gauge fixing constraint : $C^{\, r} = \partial_\mu W^{\mu\, r}$

$\longrightarrow$

1-2 – The two central anomalies alongside : scale- or trace- and U1-axial anomaly and the renormalization group invariant scale Λ

We display the sructure of the two 'central' anomalies in eq. 7 below

$$\left\{ \vartheta^{\mu}{}_{\mu} = \sum_{f} m_{f} S_{\dot{f}\, f} + \delta_{0} \right\} (x)$$
$$\left\{ \partial^{\mu} \left(j^{5}_{\mu} \right)^{S} = 2 \langle m \rangle \, i P^{S} + \delta_{5} \right\} (x)$$

(7)

$$\delta_{0} = \left\{ - \left(-2 \beta(g) / g^{3} \right) \left[\tfrac{1}{4} (:) B^{t}_{\mu \nu} B^{\mu \nu t} (:) \right] \right\}_{\to ren.gr.inv}$$
$$\delta_{5} = \left\{ \left(2 N_{fl} \right) \tfrac{1}{8 \pi^{2}} \left[\tfrac{1}{4} (:) B^{t}_{\mu \nu} \widetilde{B}^{\mu \nu t} (:) \right] \right\}_{\to ren.gr.inv}$$

$$- \beta / g^{3} = b_{0} / (16 \pi^{2}) + O(\kappa) \; ; \; \kappa = g^{2} / (16 \pi^{2})$$

The predicate 'central' for the anomalies in eq. 7 stands for the property that in rendering the square coupling constant and the associated $\vartheta -$ parameter in the gauge boson *naively renormalized* **Lagrangean density x dependent**

(8)

$$\mathcal{L}_{g.b.} = - \tfrac{1}{g^{2}} \tfrac{1}{4} (:) B^{t}_{\mu \nu} B^{\mu \nu t} (:) + \vartheta \tfrac{1}{8 \pi^{2}} \tfrac{1}{4} (:) B^{t}_{\mu \nu} \widetilde{B}^{\mu \nu t} \longrightarrow$$
$$g^{2} \to g^{2}(x) \; ; \; \vartheta \to \vartheta(x)$$

maintains perturbative renormalizability and acts together with suitable boundary- – more generally – regularity conditions $\longrightarrow$

as external sources for the scalar and pseudoscalar local field strength bilinears

(9) $$\frac{1}{4}\,(:)\,B^{\,t}_{\mu\,\nu}\,B^{\,\mu\,\nu\,t}\,(:) \quad , \quad \frac{1}{4}\,(:)\,B^{\,t}_{\mu\,\nu}\,\widetilde{B}^{\,\mu\,\nu\,t}\,(:)$$

We will use the following definitions relative to the rescaling function β

$$-\,\beta\,/\,g \;=\; \kappa\,B\,(\,\kappa\,) \;\;;\;\; B\,(\,\kappa\,) \;=\; b_{\,0}\,A\,(\kappa\,)$$

$$B\,(\,\kappa\,) \;\sim\; \sum_{n=0}^{\infty} b_{\,n}\,\kappa^{\,n} \quad , \quad A\,(\,\kappa\,) \;\sim\; \sum_{n=0}^{\infty} a_{\,n}\,\kappa^{\,n}$$

$$\kappa \;=\; g^{\,2}\,/\,(\,16\,\pi^{\,2}\,) \;\; \text{generic} \;\; \longrightarrow \;\; X\,,\,Y$$

(10) $$b_{\,0} \;=\; \frac{1}{3}\,\left(\,33\,-\,2\,N_{\,fl}\,\right) \quad , \quad a_{\,0} \;=\; 1 \quad , \quad a_{\,n} \;=\; b_{\,n}\,/\,b_{\,0}$$

$$b_{\,1} \;=\; \frac{2}{3}\,\left(\,153\,-\,19\,N_{\,fl}\,\right)$$

$$b_{\,2} \;=\; \frac{1}{54}\,\left(\,77139\,-\,15099\,N_{\,fl}\,+\,325\,N^{\,2}_{\,fl}\,\right)$$

$$b_{\,3} \;\sim\; 29243\,-\,6946.3\,N_{\,fl}\,+\,405.089\,N^{\,2}_{\,fl}\,+\,1.49931\,N^{\,3}_{\,fl}$$

References in conjunction with this section ('1-1') are presented in five (partial) collections :

1 : (R) directly related to the two central anomalies

2 : (rBsquare) establishing the one renormalization group invariant quantity of dimension $[\,M^{\,4}\,]$

3 : (r-sp-1) a recent paper by Guido Altarelli and references cited therein

4 : (r-A2x) a selection of papers and textbooks for the entire realm of QCD

5 : (r-condx) : Condensation phenomena and field theory realizations

→

This concludes this introdctory layout of premises . The extended discssion of the sliding scale coupling constant and its relation to the renormalization group invariant scale Λ – as seen from the ultraviolet asymptotic region – is subsummed in eqs. 11 and 12 below

(11)
$$t = Z^{-1} - a \log \left(Z^{-1} \right) - G(Z) \;\; ; \;\; a = b_1 / b_0^2$$

$$G(Z) = Z \sum_{n=0}^{\infty} G_n Z^n \; ; \; Z = b_0 \overline{\kappa} (\mu^2)$$

$$t = \log (\mu^2 / \Lambda^2) \longrightarrow$$

$$\Lambda^2 = \mu^2 \left[\exp \left(- \frac{1}{b_0 \overline{\kappa}} \right) \right] \left[\frac{1}{b_0 \overline{\kappa}} \right]^a \exp \left[G (b_0 \overline{\kappa}) \right]$$

The inverted asymptotic expansion for $\zeta = \left(b_0 \overline{\kappa} (\mu^2) \right)^{-1}$ for $t = e^L \to \infty$ is

(12)
$$\zeta(L) \sim e^L + a L + [0] \text{ for } L = \log \left[\log \left(\mu^2 / \Lambda^2 \right) \right] \to \infty$$

→

2 – Direct consequences from the central anomalies

2-1 – Renormalizing composite local operators at sliding scale $\mu = \infty$

2-1-1 – rescaling equations in the $\overline{\text{MS}}$ scheme for coupling constant g and quark masses m_f

$\beta(g)$ and $\gamma_m(\kappa)$, $\kappa = g^2 / (16\pi^2)$

First we choose the (renormalization group invariant) scale Λ, which lies outside the perturbatively accessible region, and the sliding scale μ in the form shown in eq. 11 together with the rescaling functions β, γ_m governing the universal rescaling of coupling constant and quark masses $\overline{g}$, $\overline{m}_q$ respectively

(13)

$$t = \log\left(\mu^2 / \Lambda^2\right) \; ; \; \overline{\kappa} = \overline{g}^2 / \left(16\pi^2\right) \rightarrow$$

$$\left\{ \widehat{\beta} = (\beta / \overline{g})\, \overline{\kappa} = \widehat{\beta}(\overline{\kappa}), \; \overline{m}_q, \; \gamma_m(\overline{\kappa}) \right\}$$

In eq. 13 the overlined quantities refer to the sliding scale μ, which shall be chosen $\mu \gg \Lambda$, such as to ensure perturbative accessibility

(14)

$\overline{\kappa} = \overline{\kappa}(\mu) \; ; \; \overline{m}_q = \overline{m}_q(\mu)$ for quark flavor q

with initial values for $\mu = \Lambda$:

$$\overline{\kappa}(\Lambda) = \kappa^*, \; \overline{m}_q(\Lambda) = m_q^*$$

→

The rescaling equations can be cast into the form

$$\bullet = d/dt \; ; \; t = \log \left(\mu^2 / \Lambda^2 \right)$$

(15)
$$\frac{1}{\overline{\kappa}} \; \overset{\bullet}{\overline{\kappa}} = - \left[- \widehat{\beta} / \overline{\kappa} \right] (\overline{\kappa}) \quad ; \quad \frac{1}{\overline{m}_q} \; \overset{\bullet}{\overline{m}}_q = - \left[- \gamma_m \right] (\overline{\kappa})$$

Two remarks concerning the structure of the relations highlighted in eq. 15 are in order here :

1) the rescaling function γ_m

does not depend on the values of the quark masses, i.e. on flavor q , be these sliding scale masses $\overline{m}_q$ or renormalization group invariant ones , as e.g. $m_q^{N*} = \overline{m}_q (\mu = N \Lambda)$ for any fixed multiple $N \Lambda$. N can perfectly well be chosen large enough such that $N \Lambda$ lies within the perturbatively accessible region .

2) both sliding scale functions $\widehat{\beta} / \overline{\kappa}$ and γ_m

are negative at least for μ chosen large enough and dominated by the lowest term in the sliding scale expansion as made explicit below . The first four relevant powers in the expansion of the rescaling functions are given in Appendix 1 .

$\longrightarrow$

The sliding scale quark mass, integrating eq. 15 within the limits $t_> \geq t' \geq t_<$, becomes

$$\log\left(\frac{m_{>,q}}{\overline{m}_{<,q}}\right) = -\int_{t_<}^{t_>} dt'\,\gamma_m\left(\overline{\kappa}\left(t'\right)\right) \quad ; \quad dt' = -\left(\widehat{\beta}\left(\overline{\kappa}'\right)\right)^{-1} d\overline{\kappa}' \tag{16}$$

$$t_< \leftrightarrow m_{>,q}\,;\,\overline{\kappa}_> \quad , \quad t_> \leftrightarrow \overline{m}_{<,q}\,;\,\overline{\kappa}_<$$

Performing the substitution of integration variables $t' \to \overline{\kappa}'$ it follows

$$\log\left(\frac{m_{>,q}}{\overline{m}_{<,q}}\right) = \int_{\overline{\kappa}_<}^{\overline{\kappa}_>} d\overline{\kappa}'\,\frac{\gamma_m\left(\overline{\kappa}'\right)}{\widehat{\beta}\left(\overline{\kappa}'\right)} = (4/b_0)\int_{\overline{\kappa}_<}^{\overline{\kappa}_>} dY\,\frac{\Gamma(Y)}{Y\,A(Y)}$$

$$\gamma_m(Y) \simeq \gamma_0\;Y\;\Gamma(Y) \qquad \gamma_0 = -4$$

$$\widehat{\beta}(Y) \simeq \beta_0\,Y^2\,A(Y) \qquad \beta_0 = -b_0 = -\left(11 - 2N_{fl}\right) \tag{17}$$

$$\gamma_m(Y) \simeq \sum_{n=0}^{\infty}\gamma_n\,Y^{n+1} = \gamma_0\,Y\,\Gamma(Y) \qquad ; \qquad \Gamma(Y) \simeq \sum_{n=0}^{\infty}\Gamma_n\,Y^n$$

$$\Gamma_n = \gamma_n/\gamma_0$$

$$\widehat{\beta}(Y) \simeq \sum_{n=0}^{\infty}\beta_n\,Y^{n+2} = -Y^2\,B(Y) \quad ; \quad B(Y) \simeq \sum_{n=0}^{\infty} b_n\,Y^n$$

$$B(Y) = b_0\,A(Y) \qquad\qquad b_n = -\beta_n$$

→

The integration region $\overline{\kappa} \geq \overline{\kappa}' \geq \overline{\kappa}_-$, specified in eqs. 16 and 17, is restricted to contain no nontrivial zero of $\widehat{\beta}(\overline{\kappa}')$, but can well extend to the perturbatively inaccessible region towards the upper value $\overline{\kappa}_>$.

In an abbreviated way performing the steps leading to the asymptotic relations for the sliding scale coupling constant for $\overline{\kappa}_> \rightarrow \infty$, given in eq. 11 it follows from eq. 17

$$\overline{m}_{<,q} \rightarrow 0 \ : \ \log\left(\frac{m_{>,q}}{\overline{m}_{<,q}}\right) \sim (4/b_0)\log\left(\frac{1}{b_0\overline{\kappa}_<}\right) + \Phi(\overline{\kappa}_>)$$

(18)

$$\frac{1}{b_0\overline{\kappa}_<} \sim t_> = \log(\mu^2/\Lambda^2)$$

The key point in the asymptotic form of eq. 18 is that the function $\Phi(\overline{\kappa}_>)$ does not depend on the quark flavor q , nor on the sliding scale $t_> \leftrightarrow \overline{\kappa}_<$, in particular in the limit $\mu \rightarrow \infty$, equivalent to $t_> \rightarrow \infty \leftrightarrow \overline{\kappa}_< \rightarrow 0$.

We choose a renormalization group invariant set of quark masses denoted m_q^{**} in the following, $\longrightarrow$

such that eq. 18 asymptotically takes the form

(19)
$$\overline{m}_{<,q} \to 0 \ : \ \log\left(\frac{m_q^{**}}{\overline{m}_{<,q}}\right) \sim \left\{ \begin{array}{l} (4/b_0)\log\left(\dfrac{1}{b_0\,\overline{\kappa}_<}\right) + \\ + \left[\Phi(\overline{\kappa}_>) - \log\left(\dfrac{m_{>,q}}{m_q^{**}}\right)\right] \end{array} \right\}$$

$$\frac{1}{b_0\,\overline{\kappa}_<} \sim t_> = \log(\mu^2/\Lambda^2)$$

imposing the condition on the lower term in curly brackets in eq. 19 to vanish

(20)
$$\left[\Phi(\overline{\kappa}_>) - \log\left(\frac{m_{>,q}}{m_q^{**}}\right)\right] = 0$$

→

Then eq. 19 takes the form as shown in eq. 21 together with the structure of the renormalization group invariant scale Λ as derived from the asymptotic sliding coupling constant behaviour in eq. 11

$$\overline{m}_{<,q} \rightarrow 0 \quad : \quad \log\left(\frac{m_q^{**}}{\overline{m}_{<,q}}\right) \sim (4/b_0)\log\left(\frac{1}{b_0\,\overline{\kappa}_<}\right)$$

$$m_q^{**} = \lim_{t\rightarrow\infty} \overline{m}_q(t)\left(\frac{1}{b_0\,\overline{\kappa}(t)}\right)^{4/b_0} \quad ; \quad \overline{\kappa}(t) \rightarrow \overline{\kappa} \searrow$$

(21)

$$\Lambda^2 = \mu^2\left[\exp\left(-\frac{1}{b_0\,\overline{\kappa}}\right)\right]\left[\frac{1}{b_0\,\overline{\kappa}}\right]^a \exp\left[\,G(b_0\,\overline{\kappa})\,\right]$$

$$t = \log(\mu^2/\Lambda^2) \sim Z^{-1} - a\log\left(Z^{-1}\right) \; ; \; Z = b_0\,\overline{\kappa}(t) \; ; \; a = b_1/b_0^2$$

$$b_0 = 11 - \tfrac{2}{3}N_{fl} \; ; \; b_1 = \tfrac{2}{3}\left(153 - 19\,N_{fl}\right)$$

This concludes main derivations in this subsection, subject to the following remarks $\rightarrow$

1) nonuniversality of the quantities Λ . m_q^{**}

whence extracted from the limiting values of $t \sim \infty$. The reason is illustrated in figure 1 below, which shows the flavor dependence of Λ as a function of N_{fl} contained in the coefiicients b_0 , b_1 determining this scale as defined through the above limit. Intuitively this can be traced back to the threshold behaviour of heavy quark (antiquark) flavors. The latter are represented also by the nonanomalous quark mass part of the QCD Lagrangean, and its renormalized counterpart in the trace of the energy momentum density tensor, dependent linearly on the quark mass parameters $\overline{m}_q^{**}$. The physical properties of these threshod effects however are by no means linear in these parameters.

2) The renormalization of composite operators at zero distance

is a useful tool especially for defining gauge field strength bilinears forming central anomalies. It however becomes mandatory to include all interactions up to infinite energies , not only unified gauge and associated interactions but ultimately also gravity. This is illustrated in Fig. 3 below including standard model interactions in the three sliding scale coupling constant rescaling equations to three loops by Mathias Steinhauser et al. in ref. [2-2012] .

3) Segregating the pure gauge degrees of freedom

in their combined infrared and ultraviolet role from quark flavors. This shall be the substrate of the next subsection. $\longrightarrow$

2-1-Fig. 1

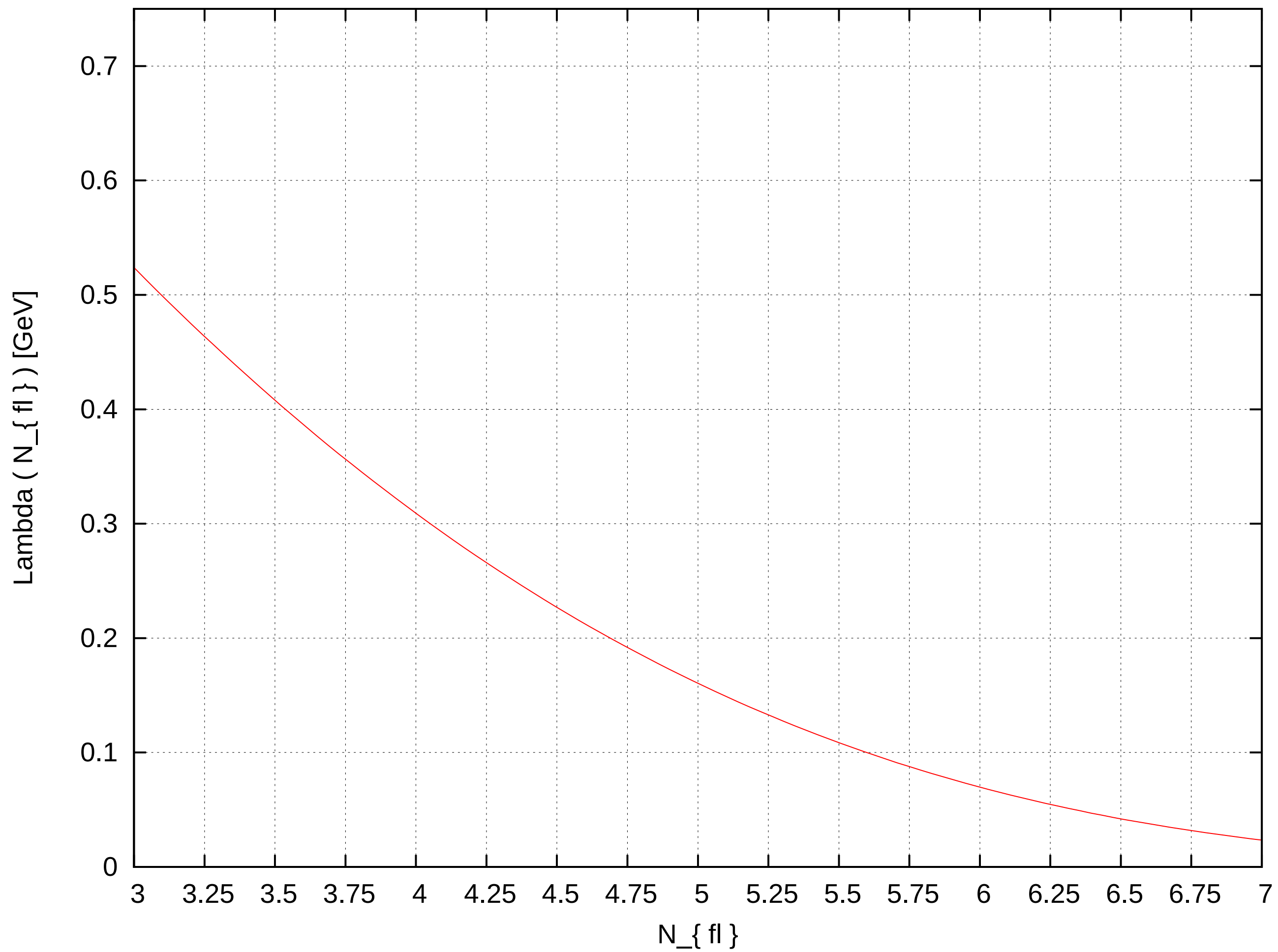

Fig. 1 : Scale Λ for fixed $\overline{\kappa}\,(\,m_{\,Z}\,)$ for continuous values of $N_{\,fl}$.

The input value $\alpha_{\,s}\,(\,m_{\,Z}\,)\;=\;0.1135\;(\,\pm\;\sim\;0.0009\,)$

is taken from André Hoang et al. , ref. [1-2010] .

$\longrightarrow$

2-1-Fig. 3

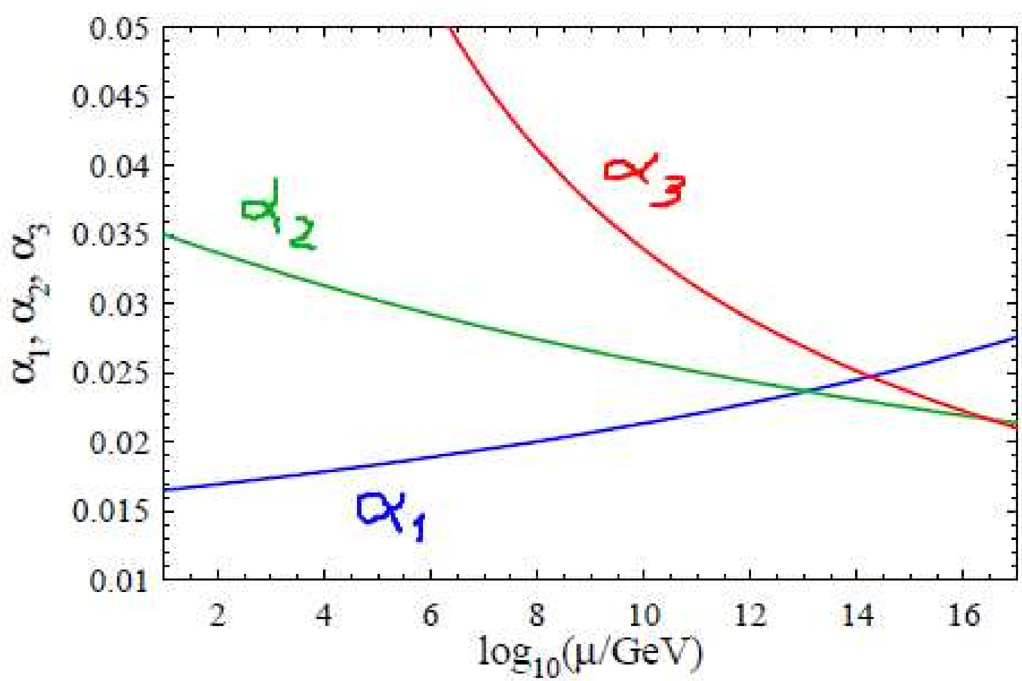

FIG. 3. The running of the gauge couplings at three loops. The curve with the smallest initial value corresponds to α_1, the middle curve to α_2, and the curve with the highest initial value to α_3.

to Fig. 3 : The figure is taken from Mathias Steinhauser et al. in ref. [2-2012] .

It is based as far as $\alpha_3 = \alpha_s = 4\pi\overline{\kappa}(m_Z)$ **is concerned**

on the value $\alpha_s(m_Z) = 0.1173 \, (\pm \sim 0.00069)$

→

2-1-2 – reinterpreting the central anomalies

$\frac{1}{4} \left[B^{\ r}_{\mu\nu} B^{\mu\nu\ r} \right]_{\infty} = \mathcal{B}^{\ +}_{\infty}$ & $\frac{1}{4} \left[B^{\ r}_{\mu\nu} \widetilde{B}^{\mu\nu\ r} \right]_{\infty} = \mathcal{B}^{\ -}_{\infty}$

The central anomalies (eq. 7) become, using the field strength bilinears (re-)normalized at $\mu = \infty$, and separating gauge- and quark antiquatrk parts in the lowest order constant $b_0 = b_{0\ g.b.} - \frac{2}{3} N_{fl}$ of the coupling constant rescaling function $\beta(g)$

$$\left\{ \begin{array}{ll} \vartheta^{\mu}_{\ \ \mu} & = \sum_q \left[m_q(\mu) \overline{q} q(\mu) + \frac{1}{12\pi^2} \mathcal{B}^{\ +}_{\infty} \right] - b_{0\ g.b.} \frac{1}{8\pi^2} \mathcal{B}^{\ +}_{\infty} \\ \partial^{\nu} \left(\overline{q} \gamma_{\nu} \gamma_5 q \right) = 2\ m_q(\mu) \overline{q}\, i \gamma_5 q(\mu) & \qquad + \quad 2 \quad \frac{1}{8\pi^2} \mathcal{B}^{\ -}_{\infty} \end{array} \right\} (x)$$

$$-\beta / g^3 = b_0 / (16\pi^2) + O(\kappa) \ ; \ \kappa = g^2 / (16\pi^2)$$

$$b_0 = b_{0\ g.b.} - \frac{2}{3} N_{fl} \ ; \ b_{0\ g.b.} = \frac{11}{3} C_2(adj(Lie - SU3_c)) = 3$$

(22)

In eq. 22 normal ordering symbols are omitted for brevity of notation. Furthermore the renormalization scale for the field strengths bilinears

$$\frac{1}{4} \left[B^{\ r}_{\mu\nu} B^{\mu\nu\ r} \right]_{\infty} = \mathcal{B}^{\ +}_{\infty} \ ; \ \frac{1}{4} \left[B^{\ r}_{\mu\nu} \widetilde{B}^{\mu\nu\ r} \right]_{\infty} = \mathcal{B}^{\ -}_{\infty}$$

is set to $\mu = \infty$ which does lead to a conceptual clarification of these quantities most importantly for a selfdual or anti-selfdual classical background field configuration. $\longrightarrow$

The sliding scale quark masses $m_q(\mu)$ and contragredient $\overline{q}\,q$ scalar and pseudoscalar bilinears, for individual color triplet-antitriplet flavors q have been kept in eq. 22 , i.e. no attempt is made to fix μ in favor of a renormalization group invaraint scale.

2-1-2a – the catalytic effect of a quark triplet-antitriplet flavor with $\lim m_{q\,kat.}(\mu) = \infty\,;\,\mu$ fixed

We enlarge the physical substrate of 6 quark-antiquark flavors u, d, s, c, b, t by an additional flavor $q\,catalytic$ which shall have an initially assigned finite but very heavy quark mass, subsequently increased to become infinite, as indicated in the title of this subsection. The anomaly relations in eq. 22 then take the form

$$\vartheta^{\mu}{}_{\mu} = \left[\begin{array}{l} \sum_{q=1}^{6} \left[m_q(\mu)\,\overline{q}\,q(\mu) + \frac{1}{12\pi^2}\,\mathcal{B}_{\infty}^{+} \right] - b_{0\,g.b.}\,\frac{1}{8\pi^2} \\ + \left[m_{q\,kat.}(\mu)\,\overline{q}_{kat.}\,q_{kat.}(\mu) + \frac{1}{12\pi^2}\,\mathcal{B}_{\infty}^{+} \right] \end{array} \right] \tag{23}$$

$$\partial^{\nu}\left(\overline{q}_{kat}\,\gamma_{\nu}\,\gamma_5\,q_{kat.}\right) = 2\,m_{q\,kat.}(\mu)\,\overline{q}_{kat.}\,i\gamma_5\,q_{kat.}(\mu) + 2\,\frac{1}{8\,\pi^2}\,\mathcal{B}_{\infty}^{-}$$

$$\lim m_{q\,kat.}(\mu) \rightarrow \infty \quad ; \quad \mu \text{ fixed}$$

It follows that the catalytic flavors $q_{kat.}, \overline{q}_{kat.}$ decouple in the limit considered in a nontrivial way. $\longrightarrow$

$$\lim_{m_{q\,kat.}(\mu)\to\infty}\begin{bmatrix} m_{q\,kat.}(\mu)\,\overline{q}_{kat.}\,q_{kat.}(\mu) \\ 2\,m_{q\,kat.}(\mu)\,\overline{q}_{kat.}\,i\gamma_5\,q_{kat.}(\mu)\end{bmatrix} = -\begin{bmatrix}\frac{1}{12\pi^2}\mathcal{B}^{+}_{\infty} \\ \frac{1}{4\pi^2}\mathcal{B}^{-}_{\infty}\end{bmatrix}$$

(24) μ **fixed**

The space-time argument x ,of the composite operators in eq. 24 are suppressed for simplicity of notation.

As limiting operator identies the chiral properties of $\overline{q}_{kat.}\,\frac{1}{2}\left(¶ \pm \gamma_5\right))$ become entangled with the positive and negative parity gauge boson field strength bilinears.

From the 'catalytic' limiting relations in eq. 24 taking vacuum expected values on both sides of it, choosing the positive parity relation we obatain

(25)
$$\lim_{m_{q\,kat.}(\mu)\to\infty} m_{q\,kat.}(\mu)\,\langle\Omega|\,\overline{q}_{kat.}\,q_{kat.}(\mu)\,|\Omega\rangle = -\frac{1}{12\pi^2}\,\langle\Omega|\,\mathcal{B}^{+}_{\infty}\,|\Omega\rangle$$

could *perfectly well* **vanish. This is a dynamical question, brought forth by Shifman, Vainshtein and Zakharov in 1979 [7-1979] in conjunction with the charmonium spectroscopy . We will take up this discussion in the next subsection.** →

2-1-Fig. 4

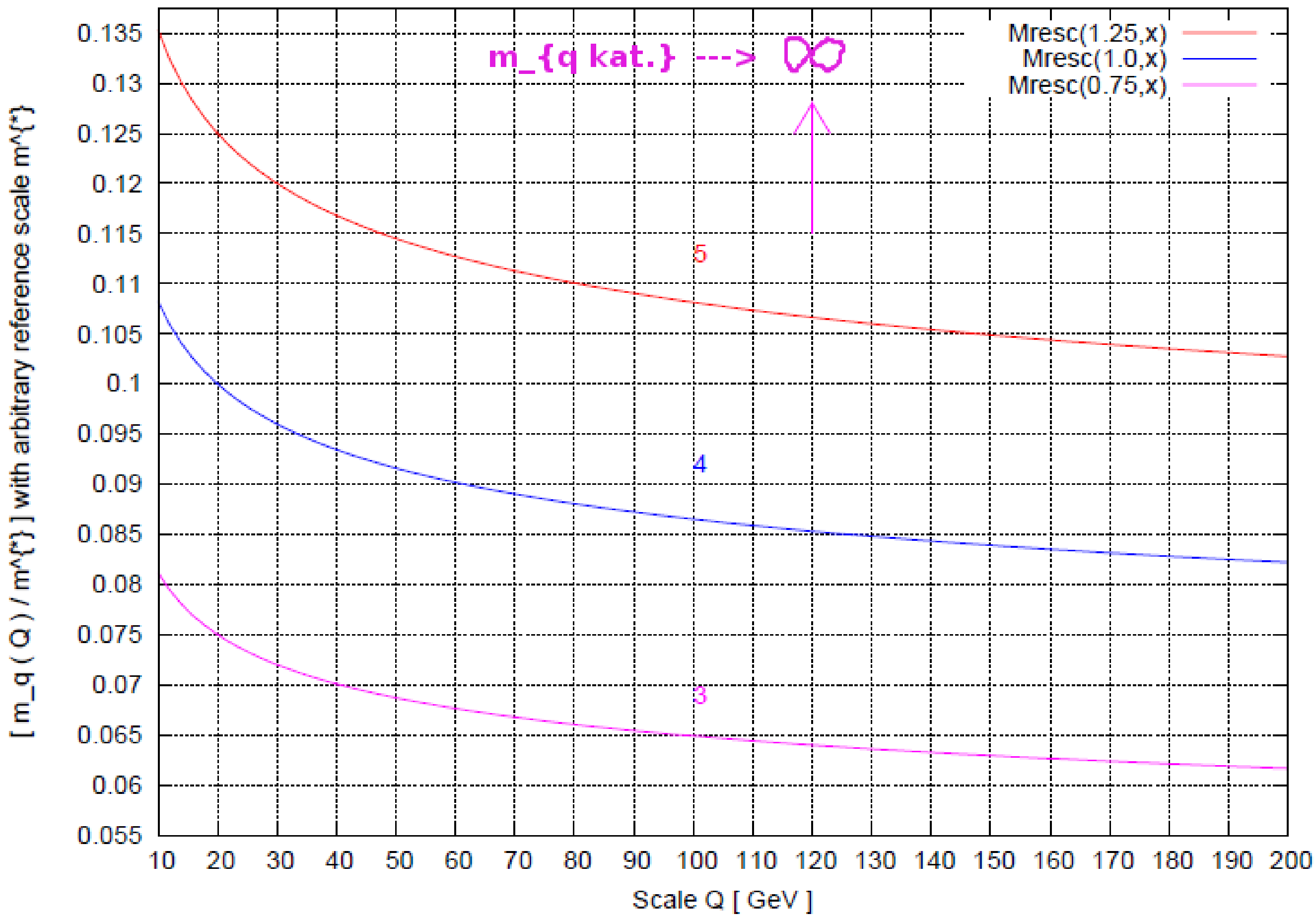

Fig. 4 : $m_q(Q)/m^*$ **with fixed ratio of rescaled quark masses**

$$m_u : \frac{1}{2}(m_d + m_u) : m_d = 3 : 4 : 5$$

→

3 – The filigran fabrics of gauge boson field complexes

(the word complex is used here in the association with 'complex chemistry')

3-1 – Shifting focus to the edge between perturbative and nonperturbative regions

3-1-1 – the nature of condensates and attempts to determine $\langle \Omega \mid \frac{1}{4} \left[B^{\,r}_{\mu\nu} B^{\,\mu\nu\, r} \right]_{\infty} \mid \Omega \rangle$ from QCD sum rules [7-1979]

Lets use as entry point the derivation of the scalar $\overline{u}\, u + \overline{d}\, d$ condensate (or vev) by Gell-Mann, Oakes and Renner [3-1968] and the axial current matrix element

$$\langle \Omega \mid j^{\,\varrho}_{5} (x) \left| \pi^{\,0} , q \right) \rangle = f_{\pi}\, i\, q^{\,\varrho}\, e^{-\,i\,x\,q} \;\; ; \; f_{\pi} \left(m^{\,2}_{\pi^{\,0}} \right) \sim 92.4 \text{ MeV}$$

(26) $$j^{\,\varrho}_{5} = \frac{1}{2} \left(\overline{u}\, \gamma^{\,\varrho} \gamma_{5}\, u - \overline{d}\, \gamma^{\,\varrho} \gamma_{5}\, d \right)$$

$$D_{5} (x) = \partial_{\varrho}\, j^{\,\varrho}_{5} (x) \;\longrightarrow\; \langle \Omega \mid D_{5} (x) \left| \pi^{\,0} , q \right) \rangle = f_{\pi}\, m^{\,2}_{\pi^{\,0}}\, e^{-i\,x\,q}$$

Associating a hermitian interacting field $\varphi (y)$ with the state $\left| \pi^{\,0} , q \right\rangle$ with the normalization and phase conditions leads to a special choice for the interpolating field φ

(27) $$\langle \Omega \mid \varphi (x) \left| \pi^{\,0} , q \right\rangle = e^{-i\,x\,q} \;\longrightarrow\; \varphi (y) = \frac{1}{f_{\pi}\, m^{\,2}_{\pi^{\,0}}}\, D_{5} (y)$$

Freezing the first argument – x – of D_{5} to $x = 0$ and using the Lehmann-Symanzik-Zimmermann reduction for the on shell pion we obtain

$\longrightarrow$

(28) $$f_\pi m^2_{\pi^0} = i \int d^4 y\, e^{-iyq} \langle \Omega | \, T \, \{ D_5(0) D_5(y) \} \, | \Omega \} \, \frac{m^2_{\pi^0} - q^2}{f_\pi m^2_{\pi^0}}$$

T denotes time ordered product, symmetric for Bose fields. results from the LSZ-reduction involving a neutral field φ refers to one half of the difference between an incoming state with momentum q , as in eq. 26 and the associated outgoing one

(29) $$\langle \pi^0 , p \, | \, j_5^\varrho (x) \, | \Omega \rangle = f_\pi \, i p^\varrho e^{+ixp} \; ; \; p = -q$$

Eq. 28 can be written

(30) $$\begin{aligned} & f_\pi^2 m^2_{\pi^0} = i \int d^4 y\, e^{-iyq} \langle \Omega | \, T \, \{ D_5(y) D_5(0) \} \, | \Omega \rangle \left(1 - q^2 / m^2_{\pi^0} \right) \\ & D_5(y) = \partial_\varrho j_5^\varrho (y) = i \left(m_u \overline{u} \gamma_5 u - m_d \overline{d} \gamma_5 d \right) (y) \end{aligned}$$

Upon partial integration we obtain

(31) $$f_\pi^2 m^2_{\pi^0} = \left(1 - q^2 / m^2_{\pi^0} \right) \int d^4 y\, e^{-iyq} \begin{bmatrix} - q_\varrho \langle \Omega | \, T \left\{ j_5^\varrho (y) D_5(0) \right\} | \Omega \rangle \\ - i \, \delta \left(y^0 \right) \langle \Omega | \, \left[j_5^0 (y) , D_5(0) \right] | \Omega \rangle \end{bmatrix}$$

Next we extrapolate the right hand side of eq. 31 off shell to the value $q = 0$, which induces relative errors $o \left(m^2_{\pi^0} / \widetilde{\Lambda}^2 \right)$, where $\widetilde{\Lambda}$ denotes a hadronic scale not vanishing in the chiral limit . →

Chiral perturbative expansions within effective theories is a dedicated discipline, pioneered by Steven Weinberg, Heinrich Leutwyler and Jürg Gasser, [4-1966, 5-1984, 6-1985] .

Continuing the derivation laid out here, eq. 31 reduces to the form

$$f_{\pi}^{2} m_{\pi^0}^{2} \sim \int d^4 y \, \delta(y^0) \, \langle \Omega | \left[j_5^0(y), \frac{1}{i} D_5(0) \right] | \Omega \rangle \tag{32}$$

The equal time commutator on the right hand side of eq. 32 leads to the contact term involving scalar quark-antiquark bilinears

$$j_5^{\varrho}(y) = \tfrac{1}{2} \left(\overline{u} \gamma^{\varrho} \gamma_5 u - \overline{d} \gamma^{\varrho} \gamma_5 d \right)(y)$$

$$\begin{aligned} &\delta(y^0) \left[j_5^0(y), \left(m_u \overline{u} \gamma_5 u - m_d \overline{d} \gamma_5 d \right)(0) \right] = \\ &\delta(y^0) \tfrac{1}{2} \left[\left(u^{\dagger} \gamma_5 u - d^{\dagger} \gamma_5 d \right)(y), \left(m_u u^{\dagger} \gamma_0 \gamma_5 u - m_d d^{\dagger} \gamma_0 \gamma_5 d \right)(0) \right] \\ &= \delta(y^0) \tfrac{1}{2} \sum_{q=u,d} m_q \left[q^{\dagger} \gamma_5 q(y), q^{\dagger} \gamma_0 \gamma_5 q(0) \right] \end{aligned} \tag{33}$$

Finally we use the *canonical structure* **pertaining to quark flavors yielding for the equal time commutators**

$$\delta(y^0) \left[q^{\dagger} \gamma_5 q(y), q^{\dagger} \gamma_0 \gamma_5 q(0) \right] = \delta^4(y) \, q^{\dagger} \left\{ \begin{array}{r} \gamma_5 \gamma_0 \gamma_5 \\ - \gamma_0 \gamma_5 \gamma_5 \end{array} \right\} q(0) \tag{34}$$

→

For clarity we rewrite eq. 34 in condensed form

(35) $$\delta \left(y^{0} \right) \left[q^{\dagger} \gamma_{5} q(y) , q^{\dagger} \gamma_{0} \gamma_{5} q(0) \right] = \underbrace{-} 2 \delta^{4}(y) \left\{ \overline{q} q(0) \right\}$$

Eqs. 34 and 35 are operator identities compatible precisely with the Ward identities of $\overline{q} \gamma^{\varrho} \gamma_{5} q$ nonanomalous combinations of axial vector local quark currents and associated spseudoscalar and scalar densities, including elimination of the single $q (\overline{q})$ wave function renormalization factors in QCD. The - sign is not a convention , as made explicit in the derivation of eqs. 26 - 33 .

Integrating out the local contact term eq. 32 becomes

(36) $$\begin{aligned} f_{\pi}^{2} m_{\pi^{0}}^{2} \sim \; & - \langle \Omega | \left\{ \begin{array}{l} m_{u}(\mu) \overline{u} u(\mu) + \\ m_{d}(\mu) \overline{d} d(\mu) \end{array} \right\} | \Omega \rangle \\ & - \langle \Omega | \left\{ \begin{array}{l} (m_{u} + m_{d})(\mu) \frac{1}{2} \left(\overline{u} u + \overline{d} d \right)(\mu) + \\ (m_{d} - m_{u})(\mu) \frac{1}{2} \left(\overline{d} d - \overline{u} u \right)(\mu) \end{array} \right\} | \Omega \rangle \end{aligned}$$

In eq. 36 we dropped the space time dependence of the local operators and reintroduced the explicit contragredient sliding scale dependence of u , d quark masses and scalar densities $\overline{u} u$, $\overline{d} d$ in the $\overline{\text{MS}}$ sliding scale renormalization scheme.

The last term on the right hand side of eq. 36 is of $o \left(m_{d} - m_{u} \right)$ (of second order in a power expansion in $m_{d} - m_{u}$) , i.e. of the same order as the approximation involving →

the off shell extrapolation, and thus can be omitted without affecting the $\sim$ approximate relation. Thus using as abbreviation i $(\overline{q}\,q)_{u\leftrightarrow d} = \frac{1}{2}\left(\overline{u}\,u + \overline{d}\,d\right)$ eq. 36 becomes

$$(\overline{q}\,q)_{u\leftrightarrow d} = \frac{1}{2}\left(\overline{u}\,u + \overline{d}\,d\right)$$

(37) $$f_{\pi}^{2}\, m_{\pi^0}^{2} \sim -\,(m_u + m_d)\,(\mu)\,\langle\,\Omega\,|\,(\overline{q}\,q)_{u\leftrightarrow d}\,(\mu)\,|\,\Omega\,\rangle$$

$$\langle\,\Omega\,|\,(\overline{q}\,q)_{u\leftrightarrow d}\,(\mu)\,|\,\Omega\,\rangle \sim \langle\,\Omega\,|\,(\overline{u}\,u)\,(\mu)\,|\,\Omega\,\rangle \sim \langle\,\Omega\,|\,(\overline{d}\,d)\,(\mu)\,|\,\Omega\,\rangle$$

We attempt to reach the chiral limit relative to quark flavors u and d $\lim\,(m_u\,,\,m_d)\;(\mu$ fixed $) \rightarrow 0$ in an analogous but opposite way to the infinit mass limit limit of a catalytic quark flavor , discussed in subsection 2-1-2a .

To this end let me use recent results approaching the above limit by lattice simulations extending to pion mass(es) below the physical one(s), along exact $SU2_{u,d}$ symmetry, by Stephan Dürr et al. [7-2012] .

We set following ref. [7-2012] but with liberal errors

(38)
$$f\,(0) = f_{\pi}\,(m_{\pi}^{2} = 0) = 87.2\ \text{MeV}\,(1 \pm 0.01) \rightarrow$$
$$f^{2}\,(0) = 0.006724\,(1 \pm 0.02)\ \text{GeV}^{2}$$

in order to test the asymptotic ($m_{u,d} \rightarrow 0)$) relation $\rightarrow$

$$f^2(0)\left[\frac{\partial m_\pi^2}{\partial(m_u+m_d)(\mu\ \textbf{fixed})}\right]_{m_u=m_d=0} = -\langle\Omega|\,(\overline{q}\,q)_{u\leftrightarrow d}(\mu)\,|\Omega\rangle$$

(39)

We do this estimate applying a linear interpolation to determine the quantity in eq. 39

(40)
$$\left[\frac{\partial m_\pi^2}{\partial(m_u+m_d)(\mu\ \textbf{fixed})}\right]_{m_u=m_d=0} \sim \frac{m_\pi^2}{(m_u+m_d)(\mu\ \textbf{fixed})}$$

with the following specifications . We choose a reference mean u,d quark mass of 6 ± 1 **MeV**

(41)
$$\widehat{m}(\mu_6^*) = 6\left(1 \pm \tfrac{1}{6}\right)\ \textbf{MeV} \ \leftarrow\ \widehat{m} = \tfrac{1}{2}(m_u+m_d)(\mu)$$

This can be compared with the $\overline{\widehat{m}}$ **at the scale** $\mu = 2$ **GeV in ref. [8-2011]**
$($ **Budapest-Marseille-Wuppertal Collaboration** $)$

(42)
$$\overline{\widehat{m}}(2\ \textbf{GeV}) = 3.503(48)(49)\ \textbf{MeV}$$

Substituting eqs. 41 and 38 in eq. 37 we obtain $\longrightarrow$

$$f^{2}(0)\left[\frac{m^{2}_{\pi^{0}}}{2\,\widehat{\overline{m}}(\mu_6^*)}\right] \sim -\langle\Omega|\,(\overline{q}\,q)_{u\leftrightarrow d}(\mu_6^*)\,|\Omega\rangle\Big|_{chir.lim.} \longrightarrow$$

(43)
$$f^{2}(0)\left[\frac{m^{2}_{\pi^{0}}}{2\,\widehat{\overline{m}}(\mu_6^*)}\right] = 1.154\,(1\pm 0.02)\left(1\pm\tfrac{1}{6}\right)10^{-2}\,\text{GeV}^{3}$$
$$= 1.154\,(1\pm 0.19)\,10^{-2}\,\text{GeV}^{3}$$

$$f^{2}(0)\,m^{2}_{\pi^{0}} = 1.385\,(1\pm 0.02)\,10^{-4}\,\text{GeV}^{4}$$

For the sake of comparison we can rescale the condensate as defined in eq. 43 from μ_6^* to $\mu = 2$ GeV by the ratio of mean up and down quark masses

$$\langle\Omega|\,(\overline{q}\,q)_{u\leftrightarrow d}(\mu = 2\,\text{GeV})\,|\Omega\rangle\,|_{chir.lim.} =$$

(44)
$$= \frac{\widehat{\overline{m}}(\mu_6^*)}{\widehat{\overline{m}}(\mu = 2\,\text{GeV})}\,\langle\Omega|\,(\overline{q}\,q)_{u\leftrightarrow d}(\mu_6^*)\,|\Omega\rangle\Big|_{chir.lim.}$$

→

This yields

(45) $\langle\,\Omega\,|\;(\,\overline{q}\,q\,)_{\,u\leftrightarrow d}\,(\,\mu\;=\;2\,\textbf{GeV}\,)\;|\,\Omega\,\rangle\,|_{\,chir.lim.}\;=\;-\,1.979\;(\,1\;\pm\;0.19\,)\;10^{\,-2}\;\textbf{GeV}^{\,3}$

The quantity of the $\overline{q}\,q\;;\;SU2_{\,fl}$ chiral limit with reversed sign , relative to eq. 45

$$-\;\langle\,\Omega\,|\;(\,\overline{q}\,q\,)_{\,u\leftrightarrow d}\,(\,\mu\;=\;2\,\textbf{GeV}\,)\;|\,\Omega\,\rangle\,|_{\,chir.lim.}\;=\;\Sigma\,(\,2\,\textbf{GeV}\,)$$

is denoted $\Sigma\,(\,2\,\textbf{GeV}\,)$ in Stephan Dürr et al., op.cit. [7-2012] . The value determined is compared with the result derived here in eq. 46 below

$$\Sigma\,(\,2\,\textbf{GeV}\,)\;=\;2.020\quad\pm 0.027\,\pm 0.031\quad .\,10^{\,-2}\;\textbf{GeV}^{\,3}$$

(46)
$$-\;\langle\,\Omega\,|\left\{\begin{array}{c}(\,\overline{q}\,q\,)_{\,u\leftrightarrow d}\\(\,\mu\;=\;2\,\textbf{GeV}\,)\\\textbf{chir. lim.}\end{array}\right\}|\,\Omega\,\rangle\;=\;1.979\quad\times\,(\,1\;\pm\;0.19\,)\quad .\,10^{\,-2}\;\textbf{GeV}^{\,3}$$

This concludes main derivations in this subsection, subject to the following remarks →

1) The numerical agreement of the best values

for the negative of the $\overline{q}\, q\, (\mu)$ condensate $(\equiv \Sigma\, (\mu))$ as displayed in eq. 46 is not due to a hidden 'mimicry' . The errors added in quadrature for ref. [7-2012] are 2 % against almost 20 % for my deivation.

2) The aim of the latter

is (was) to give an indication that the $\overline{q}\, q$ vacuum expected value remains different from zero also in the chiral limit, whence it represents not only a spontaneous parameter, but an intrinsic spontaneous symmetry breaking.

3) In conjunction with the katalytic relation in sbsection 2-1-2a

(47)
$$\lim_{m_{\,q\;kat.}(\mu)\to\infty} m_{\,q\;kat.}\,(\mu)\; \langle\, \Omega \,|\; \overline{q}_{\,kat.}\, q_{\,kat.}\,(\mu)\; |\, \Omega\, \rangle = -\, \frac{1}{12 \pi^{\,2}}\; \langle\, \Omega \,|\; \mathcal{B}^{\,+}_{\,\infty}\; |\, \Omega\, \rangle$$

the proof of a unique sign of the $\overline{q}\, q$ vacuum expected value for all quark mass values brings the gauge boson pair ccondensate into unique focus, yet far from proving its nonvanishing value.

$\longrightarrow$

3-1-Fig. 2

Fig. 2 : Filigran (latin filum = thread , granum = grain) fabric of gauge field complexes here supporting the wings of a dragon-fly

→

3-1-1a – attempts to determine $\langle \Omega | \frac{1}{4} [B^{\,r}_{\mu\nu} B^{\mu\nu\, r}]_{\infty} | \Omega \rangle$ from QCD sum rules [7-1979]

Let me start citing here a recent paper and result(s) by Stephan Narison [11-2011, 45-2011] in partcular with respect to the renormalization group invariant setting – in principle – of composite local field normalization

$$\alpha_s G^2 = \pi^{-1} \mathcal{X} \; ; \; \mathcal{X} \equiv \frac{1}{4} [B^{\,r}_{\mu\nu} B^{\mu\nu\, r}]_{\infty}$$

$$\langle \Omega | \alpha_s G^2 | \Omega \rangle = (7.0 \pm 1.3)\, 10^{-2} \text{ GeV}^4$$

(48) $$= \pi^{-1} (0.22 \pm 0.04) \text{ GeV}^4 \qquad [\mathbf{11\text{-}2011},\ \mathbf{45\text{-}2011}]$$

$$\langle \Omega | \frac{1}{4} [B^{\,r}_{\mu\nu} B^{\mu\nu\, r}]_{\infty} | \Omega \rangle = (0.22 \pm 0.04) \text{ GeV}^4 = \left(\left(0.685 {}^{+\,0.029}_{-\,0.036} \right) \text{GeV} \right)^4$$

The errors in the numerical values reported in ref. [11-2011, 45-2011] do correspond to a central hadronic scale for the gauge boson vacuum expected value driving the central anomaly.

Let us compare the above determination with the recent analysis of Cesareo Dominguez et al. in ref. [8-2011]

(49) $$\langle \Omega | \frac{1}{4} [B^{\,r}_{\mu\nu} B^{\mu\nu\, r}]_{\infty} | \Omega \rangle = (0.48 \pm 1.14) \text{ GeV}^4 \qquad [\mathbf{8\text{-}2011}]$$

$\longrightarrow$

To clarify conventions I repeat eq. 9 from op.cit. [8-2011] in eq. 50 below

(50) $$\langle\, \Omega \,|\, \frac{1}{4}\, \left[\, B^{\,r}_{\,\mu\nu}\, B^{\,\mu\nu\; r}\, \right]_{\,\infty}\, |\, \Omega\, \rangle \;=\; 3\, C_{\,4}\, (\, O_{\,4}\,)\; -\; \left(\, 12\, \pi^{2}\, \right)\, \langle\, \Omega\, |\, \sum_{\,q=u,d,s}\, \overline{q}\, q\, |\, \Omega\, \rangle$$

In evaluating the numerical values given in eq. 49 given the errors I have neglected the contribution of the u,d,s $\overline{q}\, q$ contributions .

$\longrightarrow$

4 - Concluding remarks and outlook

1) Perturbative accessibility of renormalization in asymptotically free theories

While the entire renormalization procedure thus comes within *perturbative accessibility* **– as explained in textbooks [21] , [22-1982] – the associated renormalization group equation serves to restore renormalization group invariant properties, in particular such definitions of operators , as well as subtleties of definition and** $\overline{MS}$ **sliding scale dependence of quark masses and** $\overline{q}\ q$ **flavor specific bilinears, including extrapolations to chiral limits.**

2) Infrared instability

is associated with all physical scales *not* **accessible to perturbative approximations .**
The embedding of chiral symmetry depends in a nontrivial way on the strength of the *gauge field strength pair-* **Bose condensate as does the excitation of binary and higher gauge boson compounds ('glueballs')** *and* **the phase structure of QCD . While the above topics have not been directly addressed here , the main objective continues to be to acces on a deeper level the consequences of the trace anomaly associated with the gauge boson field strength bilinear**

$$\frac{1}{4}\ \left[\ B_{\ \mu\nu}^{\ r}\ B^{\ \mu\nu\ r}\ \right]_{\ \infty}\ \equiv\ \mathcal{B}_{\ \infty}^{\ +} \qquad (51)$$

and its nonvanishing vacuum expected value.

$\longrightarrow$

4) Canonical structure and valid estimates for $\langle\, \Omega \,|\; \mathcal{B}^{\,+}_{\,\infty} \;|\, \Omega \,\rangle$

These topics as well as many others – and dicussed by others – have not yet received analytically and/or experimentally satisfactory answers . "Es bleibt noch viel zu tun" and this is fine .

5) Outlook

The completion of QCD as a gauge field theory in uncurved space time remains a far goal .

Along the way let us keep in mind that (for all we know) the physical reality transcends much further, in particular to curved space time of unknown dimensionality .

What I could do to achieve some of the goals set , I tried to show you here.

— Thank you —

Appendix 1 - expansion coefficients of the rescaling functions $\widehat{\beta}$, γ_m to four loops

$$-\beta / g = Y B(Y) \;;\; B(Y) = b_0 A(Y)$$

$$B(Y) \sim \sum_{n=0}^{\infty} b_n Y^n \;,\; A(Y) \sim \sum_{n=0}^{\infty} a_n Y^n$$

$\kappa = g^2 / (16 \pi^2)$ **generic** Y

$$b_0 = \frac{1}{3} \left(33 - 2 N_{fl} \right) \;,\; a_0 = 1 \;,\; a_n = b_n / b_0$$

$$b_1 = \frac{2}{3} \left(153 - 19 N_{fl} \right)$$

$$b_2 = \frac{1}{54} \left(77139 - 15099 N_{fl} + 325 N_{fl}^2 \right)$$

$$b_3 \sim 29243 - 6946.3 N_{fl} + 405.089 N_{fl}^2 + 1.49931 N_{fl}^3$$

$$-\gamma_m(Y) = (-\gamma_0) Y \Gamma(Y) = -\sum_{n=0}^{\infty} \gamma_n Y^{n+1} \;,\; \Gamma(Y) \sim \sum_{n=0}^{\infty} \Gamma_n Y^n$$

$$-\gamma_0 = 4 \;;\; \Gamma_n = \gamma_n / \gamma_0$$

(52)

$\longrightarrow$

$$-\gamma_m^0 = 4\,, \quad -\gamma_m^1 = \frac{202}{3} - n_{fl}\frac{20}{9} \qquad\qquad \Big| \; -\gamma_m^l \equiv \chi_m^l$$

$$-\gamma_m^2 = 1249 - \left[\frac{2216}{27} + \frac{160}{3}\zeta(3)\right] N_{fl} - \frac{140}{81} N_{fl}^2$$

$$-\gamma_m^3 = \left\{ \begin{array}{l} \frac{4603055}{162} + \frac{135680}{27}\zeta(3) - 8800\,\zeta(5) + \\ + \left[-\frac{91723}{27} - \frac{34192}{9}\zeta(3) + 880\,\zeta(4) + \frac{18400}{9}\zeta(5) \right] N_{fl} + \\ + \left[\frac{5242}{243} + \frac{800}{9}\zeta(3) - \frac{160}{3}\zeta(4) \right] N_{fl}^2 + \\ + \left[-\frac{332}{243} + \frac{64}{27}\zeta(3) \right] N_{fl}^3 \end{array} \right\} \tag{53}$$

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