Notes and remarks on the limiting velocity c

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Abstract

A1-a – Two points held in fixed space-distance as by the (extended) air crust , rotating around its axis – seen in an inertial frame , wherein ’marginally perturbed’ elementary ’particles’ evolve on straight space-time paths

It takes a few steps in concentration to find a simple and yet fully adequate coordinate system, which is defined as an inertial frame under complete neglect of gravitational effects, dominated in praxis by sun, earth and moon.

So we envisage the following inertial frame coordinates, but only upon searching and quoting literature on how to use geodesic information on reference points within CERN [A1-1-2002] and within the Gran Sasso Laboratory (LNGS) [A1-2] . I wish to stress, that refs. [A1-1-2002, A1-2] do not constitute a present day ’state of the art’ information, rather shall illustrate the prerequisite material mainly based on the GPS system which needs to be exposed, before details of the recent time of flight measurement of the OPERA collaboration [A1-3-2011] can be assessed.

As an example of ’a just angular specification’ of a particular point on the earth surface near entrance B of CERN I note : $46.234167\,^{\circ}$ N (Lat), $6.052778\,^{\circ}$ E (Long) .

To obtain the analogous geodetic angular coordinates of a definite GPS basepoint near the Gran Sasso Laboratories , I refer to the report by A. Amoruso and L. Crescentini from 2009 [A1-4-2009] : point B in op.cit. : $42.458598\,^{\circ}$ N (Lat), $13.577586\,^{\circ}$ E (Long) . The 6 digits after the decimal point are just ’freely’ interpolated and simply serve to illustrate the ensuing discussion and present day precision. $\longrightarrow$

Ap1-2

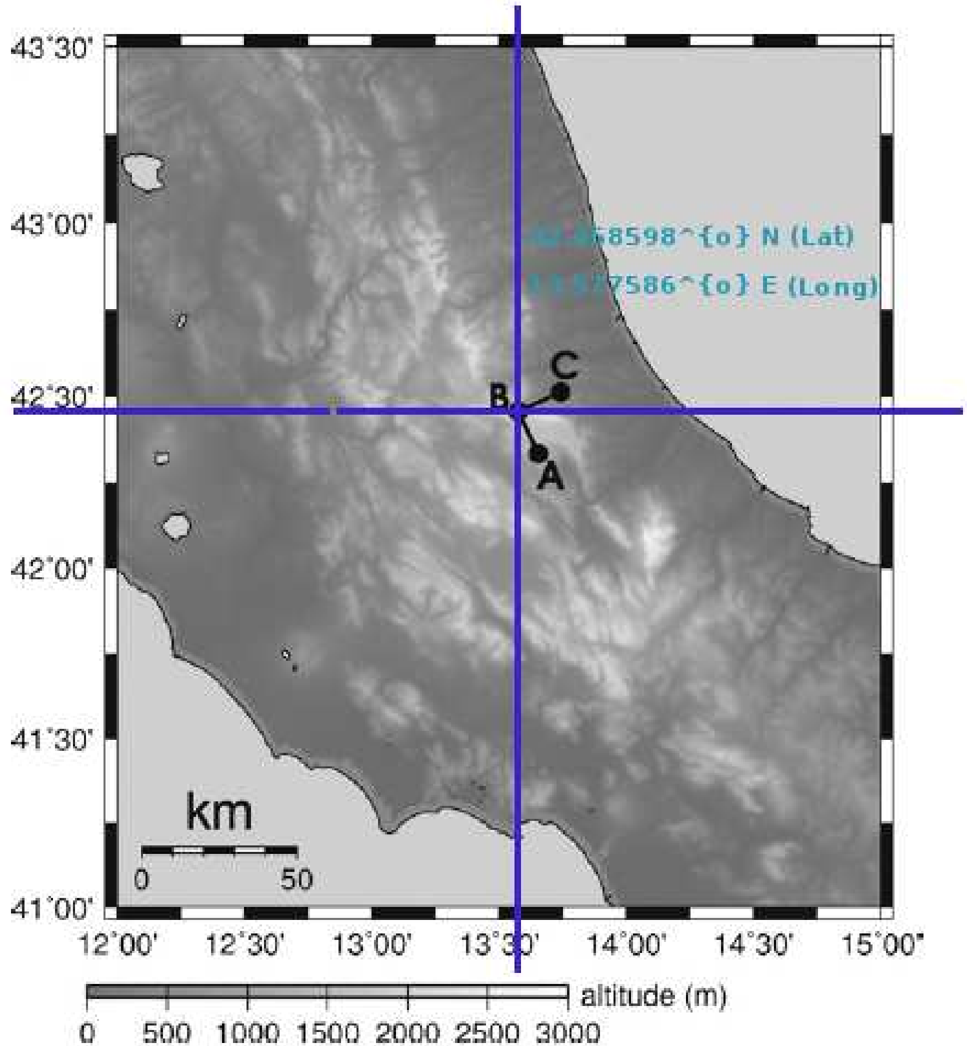

Fig. 1. Location and directions of the laser strainmeters operating at Gran Sasso.

Fig Ap1-1 : 'Geodetic' angular coordinates of a GPS basepoint near the Gran Sasso Laboratories.

Difficulties start from the physical requirement that the center of the earth is to be the actual center of mass of the earth, probably including the atmosphere , and all energy momentum densities inside the surface , either land-, ice- and water-surfaces, all this separating away other energy momentum densities assigned to the space making up the sun, earth and moon, then the entire solar system. In addition the so defined center of mass (point) of the earth is constantly accelerated by moon and sun, by the other planets (and of course many more causes) , and so an inertial frame to be defined by *neglecting gravity* **, is a logical impossibility , for exact form of just accounting for these accelerations in an 'exact' way theoretically, and using any form of distance definition as e.g. given by the GPS system and refering to $\{$relatively constant in time $\}$ $\{$ relative space-distances $\}$.**

Difficulty 1:

We go here in small historic steps (backwards in time from today) , adapting geodetic angles (φ , λ) for geographic latitude (φ) and geographic longitude (λ) . These angles are to be measured from 'ideal' geometric rays , as they intersect an ideal sphere, all comoving with the main earth crust , which is assumed ideally rigid.

$\longrightarrow$

Difficulty 1 (continued)

A first entry point to older determinations of geographic angles I take from the definition of the mean earth surface , formed by a reference ocean surface and assumed of the form of a rotational ellipsoid , with major and minor (half-)axes $(\,a\,,\,b\,)$:

(1)

Ellipsoid reference	Semi-major axis a	Semi-minor axis b	Inverse flattening
GRS 80	6 378 137.0 m	6 356 752.314 140 m	298.257 222 101
WGS 84	6 378 137.0 m	6 356 752.314 245 m	298.257 223 563

In the first column of the table in eq. 1 the numerals 80 and 84 refer to the year in which the given ellipsoids have been introduced, in the sense of a first approximation.

For full information and references to the ’World Geodetic System’ I refer to ref. [A1-5-1996] .

Let me state here, that inertial system coordinates are introduced for good reasons in satellite based geodesy. This is fine but leads to an alternative difficulty

$\longrightarrow$

Difficulty 1'

Geocentric coordinate systems used in geodesy can be divided naturally into two classes:

I) Inertial reference systems, where the coordinate axes retain their orientation relative to the fixed stars, or equivalently, to the rotation axes of ideal gyroscopes; the X axis points to the vernal equinox .

II) Co-rotating, also ECEF ("Earth Centred, Earth Fixed"), where the axes are attached to the solid body of the Earth. The X axis lies within the Greenwich observatory's meridian plane.

The quote from ref. [A1-5-1996] is (without any guarantee of it beeing correctly stated) :

"The coordinate transformation between these two systems is described to good approximation by (apparent) sidereal time, which takes into account variations in the Earth's axial rotation (length-of-day variations). A more accurate description also takes polar motion into account, a phenomenon closely monitored by geodesists."

The difficulty of successive chains of approximations for practical uses, not fully accounting for the difference between logically practical and logically theoretically consistent errors, persists and can be inferred from the 'exact wording' used in the sentence prior to this one.

In my understanding the inertial frame as described above should be transportable to the center of mass point, but this is also not an inertial point, yet important in reducing general angular ephemeride data to a consistent set of angles on the celestial sphere.

⟶

A1-b – Time dilatation in inertial frames with relative velocity v

preliminary setting

Because of the first 2 difficulties, presented in subsection A1-a, we postpone the task outlined there and turn towards the recent report by the OPERA collaboration [A1-3-2011].

The main result of the analysis in ref. [A1-3-2011] is interpreted by the authors as implying that on a baseline of length

(2)

$$L = L_1 + L_2 + L_\nu = 731278.0 \pm 0.2 \text{ m}$$

$$T_L \equiv L / c = (2'439'280.844 \pm 0.667) \text{ ns}$$

neutrino's – moving only on a flight path $L_\nu < L$ with $L_{1,2} \ll L_\nu , L$ – reached a velocity exceeding the limiting velocity in vacuo, denoted c throughout, realized by light at least in the simplified situation of inertial frames, wherein gravity can be neglected.

It must be mentioned at this point that only the total baseline can be precisely meausured using the GPS system. The three subdivisions of this baseline refer to the following distances along a straight line, to a sufficient approximation, between the 4 points, as described in ref. [A1-3-2011] :

(0) Beam Current Transformer - (1) Target - (2) Pion Decay Point - (3) Detector Base Point . $\longrightarrow$

Transforming from the base length L as given in meters to the the 'lighttime' equivalent T_L already makes use of the conceptual existence of a limiting velocity c, which thus becomes a c o n v e n t i o n, precluding in principle any valid measurement of an absolute velocity, but only of the ratio v/c .
All times which are thus derived are denoted by capital letters $T_L , T_N , \cdots$ for a general label N , and really simply serves to transform to a system of units for which the conventional value of $c \equiv 0.299792458$ m/ns becomes $c_{rational\ units} \equiv 1$. Thus keeping (e.g.) as fundammental unit of time – 1 ns – implies that the unit of length is (exactly) $[$ Length $] \equiv 0.299792458$ m .
The above state of affairs remains true including all effects of classical gravity in 4 space-time dimenions, upon also appropriate choice of local coordinates.
Finally let me specify that the actual production point of a given pion in the target, whose housing structure is approximately 2.7 m long, as well as the actual length of decay of this same pion into (mainly) a μ^+ and a neutrino ν_μ cannot be precisely determined , on an event by event basis nor on average. Yet it is clearly shown that both proton and pion as selected are highly relativistic and over all possible values of L_1 : length of proton path ending in the target , and the length of the pion path ending in the emission of muon flavored neutrinos : L_2 , of the order of several 100 m need not be measured at all , with respect to the overall time of flight of the $p \rightarrow \pi^+ \rightarrow \nu_\mu \rightarrow \mu^-$ detected in the far Opera detector at the Gran Sasso National Laborotory time sequenced signal over the total length $L = L_1 + L_2 + L_3$ as defined in eq. 2 , with similar properties also to the detection of neutral current induced events at Gran Sasso . $\longrightarrow$

Any coordinate system chosen, whether moving as inertial frame, and comoving with lets say point (0) about 100 m undergaround in the CERN-Gran Sasso (CNGS) proton beam tunnel at a given time $t_{(0)}$, established there, should then be to a sufficient approximation identifyable with the earthbound non-inertial system coincident at point (0) with the so chosen inertial frame,
over the entire time of flight **of the signal as travelling to the Gran Sasso laboratory and the Opera detector situated there over the time**

(3)
$$\Delta t \sim T_L \equiv L / c = \left[\begin{array}{l} (\, 2'439'280.844 \pm 0.667 \,) \text{ ns} \\ = \left(\begin{array}{r} 2.439280844 \\ \pm 0.000000667 \end{array} \right) . \, 10^{-3} \text{ s} \end{array} \right.$$

Given the two so described coordinate systems, the practical use of the earth bound or laboratory bound system, laboratory bound for both laboratories : CERN and Gran Sasso, can safely be chosen, which was also done by the OPERA collaboration . The time of flight as described was determined and reported as an average over 16'111 charged- and neutral current events recorded and time-transported to point (3) in the OPERA detector at Gran Sasso laboratory, as described accurately in ref. [A1-3-2011]

(4)
$$\Delta t = T_L - \delta t$$
$$\delta t = (\, 60.7 \pm 6.9 \,(\textbf{stat.}) \pm 7.4 \,(\textbf{syst.}) \,) \text{ ns}$$

→

Here we do not at all attempt to analyze the breaking apart of the systematic and statistical errors , but just combine the two errors given in eq. 4 in quadrature to a simplified overall error. Eq. becomes

(5)
$$\Delta\, t \;=\; T_{\,L} \;-\; \delta\, t$$
$$\delta\, t \;=\; (\, 60.7 \;\pm\; 10.12 \text{ overall} \,) \text{ ns} \;=\; 60.7\,(\, 1 \;\pm\; \tfrac{1}{6} \,)\;;\; 5.999324308 \;\rightarrow\; 6$$

The *relative* quantity in the above space time framework is the ratio, denoted $r_{CNGS} \;=\; \delta\, t \,/\, T_{\,L}$

(6)
$$r_{CNGS} \;\rightarrow\; r \;=\; 24.88438364(13)\,.\,10^{\,-6}\,(\, 1 \;\pm\; \tfrac{1}{6} \,)$$

The quantity we are interested in is derived from r as defined in eq. 6

(7)
$$r_{\frac{1}{2}} \;=\; \sqrt{r\,/\,2} \;=\; 3.527349121\,(\, 1 \pm\; \sim 0.087 \,)\,.\,10^{\,-3} \quad\longrightarrow$$
$$w\,/\,c \;=\; 7.054654354\,.\,10^{\,-3}\,(\, 1 \pm\; \sim 0.087 \,)$$

The velocity w in eq. 7 is not referring to inertial frames but rather parametrizes the quantity $r_{\,CNGS}$ **defined in eq. 6 as if entirely due to time dilatation because of relative motion of earthbound system and an appropriate inertial system, as if both were inertial systems with relative velocity w**

(8)
$$r_{\,CNGS} \;=\; \sqrt{1 \;-\; (\, w\,/\,c\,)^{\,2}}$$

$\longrightarrow$

The 'earthbound' system adapted by GPS is *not* an inertial system, and so the velocity w , derived from eqs. 7, 8

(9)
$$w = \left[\begin{array}{l} 7.054654354 \,.10^{\,-3}\, c \,(\,1 \pm\ \sim 0.087\,) \\ = \ 2114.93216912606\ \,(\,1 \pm\ \sim 0.087\,)\ \textbf{km/s} \end{array} \right.$$

$$c = 2.99792458\ .\ 10^{\,5}\ \textbf{km/s}$$

is not a physical velocity.

It may be useful here to remark that any fixed time rotating coordinate system assigns apparent superluminal velocities to all corotating relative to not corotating objects like stars, situated beyond the distance $c\,T_{period}\,/\,(\,2\,\pi\,)$ from the fixed center of the rotating plane.

As a consequence the velocity w, as derived in eqs. 7 - 9, while it appears fairly large relative to other velocities within the solar system, has to await a better interpretation.

A1-c – The complete set of inertial frames in the limit of exactly vanishing gravitational constant

The earthbound coordinates, which do not form an inertial frame, still can be transformed from any inertial one and thus we shall define a choice of associated coordinates, using capital greek letters The earth's rotation axis, approximated here as rigid and thus fixed in earthbound time, e.g. UTC.

(10) $$\Xi^{\mu} = \left(\Xi^{0} , \vec{\Xi}^{m} \right) \; ; \; \mu = 0,1,2,3 \; ; \; m = \mu \text{ for } \mu = 1,2,3$$

The spatial axes 1,2 (denoted X-,Y- axes) are in the equatorial plane , whereas the 3 (or Z-) axis is formed by the earth rotation axis. The 1 axis is convened to point from the center of the ellipsoid , independent of how the earth geoid is actually defined, thus allowing all of earth-bound space to be projected also beyond or below any assumed equilibrium surface formed by an idealized 'motionless' – with respect to the earthbound time $\Theta = \Xi^{0}$ – ocean surface, equilibrating centrifugal and gravitational forces.

The geographic survey material was improved in the CERN local earthbound system to the level denoted by ETRF2000 (the latest realization of the European reference system) . I base subsequent data on a note called 'OPERA public note 132, 22. September 2011', by the joint CERN and Universitá di Roma la Sapienza survey teams : ref. [A1-5-2011] . The directions of the X , Y Z axes follow the ETFR or ITRF conventions as shown in Fig. Ap1-2 from ref. [A1-6-2009] . $\longrightarrow$

2. Coordinate systems

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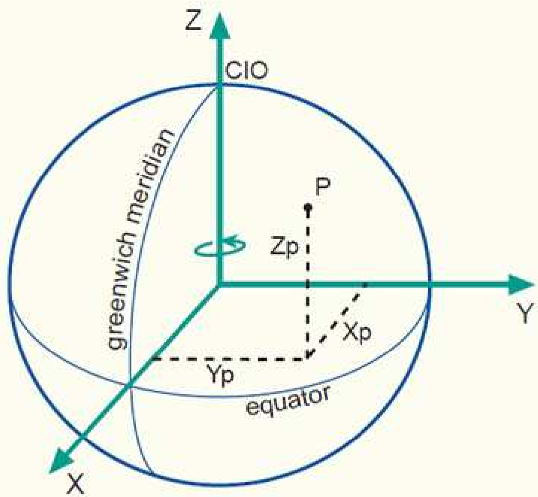
Presented by: R. Knippers

Greenwich, and the *Z*-axis coincides with the Earth's axis of rotation. The three axes are mutually orthogonal and form a right-handed system. Geocentric coordinates can be used to define a position on the surface of the Earth (point P in figure below).



An illustration of the geocentric coordinate system

It should be noted that the rotational axis of the Earth changes its position over time (referred to as ***polar motion***). To compensate for this, the mean position of the pole in the year 1903 (based on observations between 1900 and 1905) has been used to define the so-called 'Conventional International Origin' (CIO).

Fig Ap1-2 : 'Geodetic' Cartesian coordinates from ref. [A1-6-2009] . ⟷

We acknowledge the existence of a very large amount of data with respect to many reference points on the earth surface , together with measurements of the time variation, with respect to $\Theta = \Xi^0$, usually of the order of centimeters or fractions thereof per *year* .

It remains to derive appropriate coordinates in the neigbourhood of the earthbound points as determined in refs. [A1-3-2011] and [A1-5-2011] , beginning with the two points denoted T-40-S-CERN [in the CNGS target area at CERN] and A1-9999 [in Hall C at the far end, relative to CERN, of the OPERA detector]

(11)

	X (m)	Y (m)	Z (m)
$(3) =$ **A1-9999**	4582167.465	1106521.805	4283602.714
$(1) =$ **T-40-S-CERN**	4394369.327	467747.795	4584236.112
$\Delta = (3) - (1)$	187798.138	638774.010	−300633.396

This yields the Euclidean distance of $\vec{\Delta}$

(12)

$$D = \sqrt{(\Delta^1)^2 + (\Delta^2)^2 + (\Delta^3)^2} = 730534.609226859 \text{ m}$$

$$\rightarrow 730534.610 \pm 0.20 \text{ m}$$

in accordance with ref. [A1-5-2011] . The points $(0), (1), (2), (3)$ are as defined in subsection A1-b. →

At this point I wish to continue the enumeration of difficulties in subsection A1-a.

Difficulty 2 :

The main interest of GPS earthbound surface static length determinations brings in the center of mass of the entire earth as an important geodetic reference point. It is very hard to estimate the precision of the so determined distances from this earth center of mass.

We first determine in the earthbound system the static straight line passing through points (1) and (3) , keeping point (1) fixed

(13)
$$\vec{\Xi}(\Lambda) = \vec{\Xi}_{(1)} + \Lambda \vec{E} \; ; \; \vec{E} = \vec{\Delta} / D$$
$$\vec{\Xi}(\Lambda = 0) = \vec{\Xi}_{(1)} \; ; \; \vec{\Xi}(\Lambda = 1) = \vec{\Xi}_{(3)}$$

The cartesian coordinates of $\vec{\Xi}_{(1)}$ and $\vec{\Xi}_{(3)}$ are given as points (1) and (3) in eq. 11 . This is in order to determine the minimum distance between the $\Xi^3 = Z$ - axis and the straight line defined in eq. 13 . The corresponding coordinates are given by two parameters $\Lambda \, , \, Z$ with

(14)
$$\vec{\Xi}(Z) = Z \vec{E}_{(Z)} \; ; \; \vec{E}_{(Z)} = (0, 0, 1) \quad \text{with}$$
$$D^2(\Lambda, Z) = \left| \vec{\Xi}(\Lambda) - \vec{E}_{(Z)} \right|^2 = \left| \vec{\Xi}_{(1)} + \Lambda \vec{E} - Z \vec{E}_{(Z)} \right|^2$$

→

This exercise in Euclidean geometry is meant to illustrate eventual propagation of errors

The square distance in eq. 14 becomes

(15)
$$D^2(\Lambda, Z) = \\ = \vec{\Xi}^{\,2}_{(1)} + 2\,\Lambda\,\vec{\Xi}_{(1)}\,\vec{E} - 2\,Z\,\vec{\Xi}_{(1)}\,\vec{E}_{(Z)} + \Lambda^2 - 2\,\Lambda\,Z\,\vec{E}\,\vec{E}_{(Z)} + Z^2$$

The minimum distance implies the solution of two inhomogeneous linear equations

$$\left\{ \begin{array}{lcl} \Lambda - Z\,\vec{E}\,\vec{E}_{(Z)} & = & -\vec{\Xi}_{(1)}\,\vec{E} \\ Z - \Lambda\,\vec{E}\,\vec{E}_{(Z)} & = & \vec{\Xi}_{(1)}\,\vec{E}_{(Z)} \end{array} \right\} \rightarrow$$

(16)
$$\left(1 - \left(\vec{E}\,\vec{E}_{(Z)} \right)^2 \right) \Lambda = -\vec{\Xi}_{(1)}\,\vec{E} + \left(\vec{\Xi}_{(1)}\,\vec{E}_{(Z)} \right) \left(\vec{E}\,\vec{E}_{(Z)} \right)$$
$$\left(1 - \left(\vec{E}\,\vec{E}_{(Z)} \right)^2 \right) Z = \vec{\Xi}_{(1)}\,\vec{E}_{(Z)} - \left(\vec{\Xi}_{(1)}\,\vec{E} \right) \left(\vec{E}\,\vec{E}_{(Z)} \right)$$

$$\vec{E} = \vec{\Delta} \,/\, D$$

In between let me compute the transverse distances, denoted B, of points (1) and (3) from the coordinates given in eq. 11 $\longrightarrow$

(17) $$B(\vec{\Xi}) = \sqrt{(\Xi^1)^2 + (\Xi^2)^2}$$

(18)

	B (**m**)	Z (**m**)
$(3) =$ **A1-9999**	4713878.359	4283602.714
$(1) =$ **T-40-S-CERN**	4419193.341	4584236.112

The ratio Z / B $= \tan \beta$ **determines the latidtude** β **, positive for North and negative for South .**
We determine β **for points (1) and (3) and compare with the surface benchmarks**

(19)

	β (**degrees**)
$(3) =$ **A1-9999**	42.2620991514113
point B GS	42.458598
$(1) =$ **T-40-S-CERN**	46.0501770808146
CERN $\sim$ **B**	46.234167

→

The latitudes follow from the ratios Y $/$ X $= \tan\ \lambda$, for points (1) , (3) in eq. 11 and are compared to the rspective surface benchmarks

(20)

	λ (**degrees**)
$(3)\ =$ **A1-9999**	13.5761220955767
point B GS	13.577586
$(1)\ =$ **T-40-S-CERN**	6.07583031822059
CERN $\sim$ **B**	6.052778

In this 'orientational' determination of variables we also compute the total distance of points (1) and (2) from the center of mass point assigned to the earth from the relation

(21) $R\,(\,\vec{\Xi}\,) \;=\; \sqrt{X^{\,2} + Y^{\,2} + Z^{\,2}} \;=\; \sqrt{B^{\,2} + Z^{\,2}}$ **with** $\vec{\Xi} \;=\; (\,X\,,\,Y\,,\,Z\,)$

$\longrightarrow$

Ap2-18

(22)

	R (**m**)
$(3) \; = $ **A1-9999**	6369450.63515160
$(1) \; = $ **T-40-S-CERN**	6367455.57600292
$R_{(3)} \; - \; R_{(1)}$	1995.05914868228

The elevation of Gran Sasso peak is given as [A1-7-2011] : 'Gran Sasso d'Italia. Gran Sasso mountain, the highest peak in the Apennines, elevation, 2912 m .'

A1-d – An imperfect approximate realization of general coordinate invariance

In this sbsection we illustrate with an inaccurate, at best locally valid approximation of a neutrino projectile being sent from point (1) (T-40-S-CERN, a fixed reference point along the target in the CNGS tunnel) to point (3) (A1-9999, a definite point at the downstream end of the OPERA detector) as given in eq. 11 .

It represents the path of a pointlike football (neutrino) on an upward-downward tilted football field underground choosing Cartesian coordinates containing point (1) and respecting the constancy of the limiting velocity together with the parabolic approximation of the flight path in the plane subtended by points (1) and (2) and the center of mass of the earth

z = hight above 'sea level'
approximated to be flat near points (1) and (2) and passing through point (1)   (23)

x = orthogonal direction to z with origin at point (1)

$\longrightarrow$

The coordinates in eq. 23 are chosen such, that the center of the earth is projected to infinity. The Newtonian acceleration underground is approximated by a mean value

(24)
$$\vec{a}_{\,underground} \rightarrow \vec{a} = (0 , - K g_{\,s})$$
$$\text{with} \quad g_{\,s} \rightarrow 10 \, \text{m} / \text{sec}^{\,2} \; ; \; K = O (1) > 0 \; ; \; \text{G} = K g_{\,s}$$

In eq. 24 $g_{\,s}$ approximates the gravitational acceleration at the earth surface and the positive constant K depends on the density profile of the energy density inside the earth , assumed rotationally invariant, but it depends in reality also on the coordinates chosen, and is 2 for constant energy density modulo higher general relativistic corrections, in Newtonian approximation . We use G for the Newtonian acceleration inside the earths surface to distingish it from the metric induced densities discussed below The Newtonian-like second order eqations then take the form

(25)
$$\ddot{z} = - \text{G} \; ; \; x (t) = v^{\,1 \,(1)} \, t$$

Switching to planar vector notation we have

(26)
$$\vec{x} (t) = (x (t) , z (t)) \; ; \; \vec{v} (t) = \left(v^{\,1} (t) , v^{\,2} (t) \right)$$

We here assume that the cartesian planar coordinates introdced in eqs. 23 - 26 also respect the constancy of the limiting velocity, also with velocities of the order of (yet $\leq$) c

(27)
$$\vec{v}^{\,2} \leq c^2 \quad \text{and for massless neutrino's} \quad \vec{v}^{\,2} \equiv c^{\,2}$$

→

In eq. 25 the superfix $^{(1)}$ refers to initial velocity components at point (1), i.e. $\vec{x} = 0 \, ; \, t = 0$.
It is interesting that it is at this stage precisely, that neglecting beyond the malso neutrino masses is logically perfectly allowed , but the effects of gravity both inside the earth crust as well as outside cannot be dodged in retaining consistency, even for very small accelerations .
We need to this end either the Christoffel- or the spin-connection for the associated general second order equation, valid – within general relativity – for massless quanta of any kind

(28) $$\left(\frac{d}{d\,\tau} \right)^{2} x^{\,\mu} + \left(\frac{d\,x^{\,\varrho}}{d\,\tau} \right) \Gamma_{\,\varrho}{}^{\mu}{}_{\sigma} \, (\,x\,) \left(\frac{d\,x^{\,\sigma}}{d\,\tau} \right) = 0$$

$$\Gamma_{\,\varrho}{}^{\mu}{}_{\sigma} = g^{\,\mu\nu} \, \Gamma_{\,\nu\,;\,\varrho \quad \sigma} \; ; \; \Gamma_{\,\nu\,;\,\varrho \quad \sigma} = \tfrac{1}{2} \left(\partial_{\,\varrho} \, g_{\,\nu\sigma} + \partial_{\,\sigma} \, g_{\,\nu\varrho} - \partial_{\,\nu} \, g_{\,\varrho\sigma} \right)$$

In eq. 28 the parameter τ can be thought of as independent of coordinate transformations , it can not simply be taken as the invariant path length along the path

(29) $$\left. \begin{array}{l} x^{\,\mu} = x^{\,\mu} \, (\,\tau\,) \; ; \; \longrightarrow \; \bullet = d \, / \, d\,\tau \\ s^{\,B \leftarrow A} = \displaystyle\int_{A}^{B} \sqrt{ g_{\,\mu\nu} \, \dot{x}^{\,\mu} \, \dot{x}^{\,\nu} } \, d\,\tau \end{array} \right\} = 0$$ **for trajectories of massless quanta**

$\longrightarrow$

Since in previous work, e.g. ref. [A1-8-2007] , I always used for the Riemann curvature tensor the 'negative' definition for the curl , and would like to adapt to the normal 2-form convention , here the Riemann curvature tensor, as defined below, shall be denoted by $\Re \equiv - R$ **.**

We begin with the (Christoffel-) connection *matrix valued* **1-form**

(30)
$$\Gamma_{\varrho}{}^{\mu}{}_{\nu}(x) = (\Gamma_{\varrho})^{\mu}{}_{\nu} \quad \rightarrow \quad \left(\Gamma^{(1)}\right)^{\mu}{}_{\nu} = dx^{\varrho} \, (\Gamma_{\varrho})^{\mu}{}_{\nu}$$
$$\Gamma_{\varrho}{}^{\mu}{}_{\nu} = \Gamma_{\nu}{}^{\mu}{}_{\varrho} \qquad \downarrow$$
$$\Gamma^{(1)}$$

In the last line of eq. 30 the matrix indices are omitted for compactness of notation.

The curvature relative to the (Christoffel-) connection determines the Riemann curvatre (4-) tensor generated from the 2-form

(31)
$$\Re^{(2)} = \mathbf{d}\,\Gamma^{(1)} + \left(\Gamma^{(1)}\right)^{2} \quad \text{in matrix components} \quad \rightarrow$$
$$\left(\Re^{(2)}\right)^{\mu}{}_{\nu} = dx^{\varrho} \wedge \partial_{\varrho} \left(\Gamma^{(1)}\right)^{\mu}{}_{\nu} + \left(\Gamma^{(1)}\right)^{\mu}{}_{\alpha} \wedge \left(\Gamma^{(1)}\right)^{\alpha}{}_{\nu}$$

in (full) 4-tensor components $\rightarrow$

$$\tfrac{1}{2}\, dx^{\varrho} \wedge dx^{\sigma} \, (\Re_{\varrho\sigma})^{\mu}{}_{\nu} \equiv \tfrac{1}{2}\, dx^{\varrho} \wedge dx^{\sigma} \;\; \Re^{\mu}{}_{\nu\,;\,\varrho\sigma}$$

$\rightarrow$

In eq. 31 the antisymmetric wedge product of two differentials $d\,x^{\,\varrho}$ and $d\,x^{\,\sigma}$ is introduced, as well as the derivative d as acting on the connection 1-form, and p-forms in general. For a coherent exposition I refer to (e.g.) the textbook(s) by Kobayashi and Nomizu [A1-9-1963] .

The rearrangement of the two pairs of tensor indices in the last line of eq. 31

(32) $$(\Re_{\,\varrho\,\sigma})^{\,\mu}{}_{\nu} \;\equiv\; \Re^{\,\mu}{}_{\nu\,;\,\varrho\,\sigma} \qquad \equiv\; -\,R^{\,\mu}{}_{\nu\,;\,\varrho\,\sigma}$$

is a convention, as well as the second identity in eq. 32 : $\Re \;\equiv\; -R$.

We write out in detail the term quadratic in $\Gamma^{\,(1)}$ in the second line of eq. 31

(33) $$\left(\Gamma^{\,(1)} \right)^{\,\mu}{}_{\alpha} \wedge \left(\Gamma^{\,(1)} \right)^{\,\alpha}{}_{\nu} \;=\; \tfrac{1}{2}\, d\,x^{\,\varrho} \wedge d\,x^{\,\sigma} \left[\begin{array}{c} \Gamma_{\,\varrho}{}^{\mu}{}_{\alpha}\, \Gamma_{\,\sigma}{}^{\alpha}{}_{\nu} \\ -\,\Gamma_{\,\sigma}{}^{\mu}{}_{\alpha}\, \Gamma_{\,\varrho}{}^{\alpha}{}_{\nu} \end{array} \right]$$

The first term in the expression for $\Re^{\,(2)}$ in eq. 31 , which contains the second derivatives of the metric, is of the form

(34) $$d\,x^{\,\varrho} \wedge \partial_{\,\varrho} \left(\Gamma^{\,(1)} \right)^{\,\mu}{}_{\nu} \;=\; \tfrac{1}{2}\, d\,x^{\,\varrho} \wedge d\,x^{\,\sigma} \left[\begin{array}{c} \partial_{\,\varrho}\, \Gamma_{\,\sigma}{}^{\mu}{}_{\nu} \\ -\,\partial_{\,\sigma}\, \Gamma_{\,\varrho}{}^{\mu}{}_{\nu} \end{array} \right]$$

$\longrightarrow$

The full representation of the Riemann curvature tensor shall be derived here using eqs. 28 , 33 and 34

$$\mathfrak{R}^{\,\mu}{}_{\nu\,;\,\varrho\,\sigma} = g^{\,\mu\kappa}\,\mathfrak{R}_{\,\kappa\,\nu\,;\,\varrho\,\sigma} =$$

(35)
$$= g^{\,\mu\kappa}\left[\begin{array}{c} g_{\,\kappa\mu'}\,\partial_{\,\varrho}\left(g^{\,\mu'\,\eta}\,\Gamma_{\,\eta\,;\,\sigma\,\nu}\right) - \partial_{\,\sigma}\left(g^{\,\mu'\eta}\,\Gamma_{\,\eta\,;\,\varrho\,\nu}\right) \\ +\,g_{\,\kappa\mu'}\,g^{\,\mu'\eta}\,g^{\,\alpha\zeta}\,(\Gamma_{\,\eta\,;\,\varrho\,\alpha}\,\Gamma_{\,\zeta\,;\,\sigma\,\nu} - \Gamma_{\,\eta\,;\,\sigma\,\alpha}\,\Gamma_{\,\zeta\,;\,\varrho\,\nu}) \end{array}\right]$$

$$= g^{\,\mu\kappa}\left[\begin{array}{c} g_{\,\kappa\mu'}\,\partial_{\,\varrho}\left(g^{\,\mu'\,\eta}\,\Gamma_{\,\eta\,;\,\sigma\,\nu}\right) - \partial_{\,\sigma}\left(g^{\,\mu'\eta}\,\Gamma_{\,\eta\,;\,\varrho\,\nu}\right) \\ +\,g^{\,\alpha\zeta}\,(\Gamma_{\,\kappa\,;\,\varrho\,\alpha}\,\Gamma_{\,\zeta\,;\,\sigma\,\nu} - \Gamma_{\,\kappa\,;\,\sigma\,\alpha}\,\Gamma_{\,\zeta\,;\,\varrho\,\nu}) \end{array}\right]$$

The structure of the Riemann curvature tensor is a subtle one and the steps followed here are meant to reduce all derivatives to act exclusively on the metric and its defining , i.e. lower two, components. To this end we can use the identity

(36)
$$g_{\,\kappa\mu'}\,\partial_{\,\varrho\,(or\ \sigma)}\,g^{\,\mu'\,\eta} = -\,g^{\,\mu'\eta}\,\partial_{\,\varrho\,(or\ \sigma)}\,g_{\,\kappa\mu'}$$

and obtain

$$\mathfrak{R}^{\,\mu}{}_{\nu\,;\,\varrho\,\sigma} = g^{\,\mu\kappa}\,\mathfrak{R}_{\,\kappa\,\nu\,;\,\varrho\,\sigma} =$$

(37)
$$= g^{\,\mu\kappa}\left[\begin{array}{c} \partial_{\,\varrho}\,\Gamma_{\,\kappa\,;\,\sigma\,\nu} - \partial_{\,\sigma}\,\Gamma_{\,\kappa\,;\,\varrho\,\nu} \\ -\,g^{\,\mu'\eta}\,\left[(\partial_{\,\varrho}\,g_{\,\kappa\mu'})\,\Gamma_{\,\eta\,;\,\sigma\,\nu} - (\partial_{\,\sigma}\,g_{\,\kappa\mu'})\,\Gamma_{\,\eta\,;\,\varrho\,\nu}\right] \\ +\,g^{\,\alpha\zeta}\,\left[\Gamma_{\,\kappa\,;\,\varrho\,\alpha}\,\Gamma_{\,\zeta\,;\,\sigma\,\nu} - \Gamma_{\,\kappa\,;\,\sigma\,\alpha}\,\Gamma_{\,\zeta\,;\,\varrho\,\nu}\right] \end{array}\right]$$

→

We turn to the cyclic structure of the $\Gamma_{\nu\,;\,\varrho\,\sigma}$ symbols (eq. 28)

(38)

$$\Gamma_{\nu\,;\,\varrho\,\sigma} \;=\; \tfrac{1}{2}\,(\,\partial_{\varrho}\,g_{\nu\sigma} \,+\, \partial_{\sigma}\,g_{\nu\varrho} \,-\, \partial_{\nu}\,g_{\varrho\sigma}\,)$$

$$\Gamma_{\sigma\,;\,\nu\,\varrho} \;=\; \tfrac{1}{2}\,(\,\partial_{\nu}\,g_{\varrho\sigma} \,+\, \partial_{\varrho}\,g_{\nu\sigma} \,-\, \partial_{\sigma}\,g_{\nu\varrho}\,) \quad \longrightarrow$$

$$\Gamma_{\nu\,;\,\varrho\,\sigma} \,+\, \Gamma_{\sigma\,;\,\nu\,\varrho} \;=\; \partial_{\varrho}\,g_{\sigma\nu}$$

$$\leftrightarrow \quad \partial_{\varrho}\,g_{\nu\sigma} \,-\, \Gamma_{\varrho}{}^{\alpha}{}_{\nu}\,g_{\alpha\sigma} \,-\, \Gamma_{\varrho}{}^{\alpha}{}_{\sigma}\,g_{\nu\alpha} \;=\; D_{\varrho}\,g_{\nu\sigma} \;=\; 0$$

The preserving of scalar products through parallel tranport , giving rise to the last relation in eq. 38 is at the very origin of all cyclic relations therein.

Substituting

(39)

$$\partial_{\varrho}\,g_{\kappa\mu'} \;=\; \Gamma_{\kappa\,;\,\varrho\,\mu'} \,+\, \Gamma_{\mu'\,;\,\varrho\,\kappa} \quad \textbf{and} \quad \varrho \,\rightarrow\, \sigma$$

in eq. 37 it follows $\longrightarrow$

(40)

$$\mathfrak{R}_{\kappa\,\nu\,;\,\varrho\,\sigma} = $$
$$= \left[\begin{array}{c} \partial_{\varrho}\Gamma_{\kappa\,;\,\sigma\,\nu} - \partial_{\sigma}\Gamma_{\kappa\,;\,\varrho\,\nu} \\ -\,g^{\alpha\zeta} \left[\begin{array}{c} (\Gamma_{\kappa\,;\,\varrho\,\alpha} + \Gamma_{\alpha\,;\,\varrho\,\kappa})\,\Gamma_{\zeta\,;\,\sigma\,\nu} \\ -\,(\Gamma_{\kappa\,;\,\sigma\,\alpha} + \Gamma_{\alpha\,;\,\sigma\,\kappa})\,\Gamma_{\zeta\,;\,\varrho\,\nu} \end{array} \right] \\ +\,g^{\alpha\zeta}\left[\Gamma_{\kappa\,;\,\varrho\,\alpha}\Gamma_{\zeta\,;\,\sigma\,\nu} - \Gamma_{\kappa\,;\,\sigma\,\alpha}\Gamma_{\zeta\,;\,\varrho\,\nu}\right] \end{array} \right]$$

Going step by step lets multiply out the inner bracket in eq. 40

(41)

$$\mathfrak{R}_{\kappa\,\nu\,;\,\varrho\,\sigma} = $$
$$= \left[\begin{array}{c} \partial_{\varrho}\Gamma_{\kappa\,;\,\sigma\,\nu} - \partial_{\sigma}\Gamma_{\kappa\,;\,\varrho\,\nu} \\ -\,g^{\alpha\zeta} \left[\begin{array}{c} \left(\Gamma_{\kappa\,;\,\varrho\,\alpha}\Gamma_{\zeta\,;\,\sigma\,\nu} + \Gamma_{\alpha\,;\,\varrho\,\kappa}\Gamma_{\zeta\,;\,\sigma\,\nu}\right) \\ -\left(\Gamma_{\kappa\,;\,\sigma\,\alpha}\Gamma_{\zeta\,;\,\varrho\,\nu} + \Gamma_{\alpha\,;\,\sigma\,\kappa}\Gamma_{\zeta\,;\,\varrho\,\nu}\right) \end{array} \right] \\ +\,g^{\alpha\zeta}\left[\Gamma_{\kappa\,;\,\varrho\,\alpha}\Gamma_{\zeta\,;\,\sigma\,\nu} - \Gamma_{\kappa\,;\,\sigma\,\alpha}\Gamma_{\zeta\,;\,\varrho\,\nu}\right] \end{array} \right]$$

In eq. 41 the terms on the last line cancel with the first terms in the two lines within the inner bracket to yield $\longrightarrow$

$$\mathfrak{R}_{\mu\,\nu\,;\,\varrho\,\sigma} =$$

(42)
$$= \left[\begin{array}{c} \partial_{\varrho} \Gamma_{\mu\,;\,\sigma\,\nu} - \partial_{\sigma} \Gamma_{\mu\,;\,\varrho\,\nu} \\ -g^{\alpha\beta} \left[\Gamma_{\alpha\,;\,\varrho\,\mu} \Gamma_{\beta\,;\,\sigma\,\nu} - \Gamma_{\alpha\,;\,\sigma\,\mu} \Gamma_{\beta\,;\,\varrho\,\nu} \right] \end{array} \right]$$

Next we substitute the Christoffel symbols in the first line of eq. 42

(43)
$$\partial_{\varrho} \Gamma_{\mu\,;\,\sigma\,\nu} = \frac{1}{2} \partial_{\varrho} \begin{pmatrix} \partial_{\sigma} g_{\mu\nu} \\ +\partial_{\nu} g_{\mu\sigma} \\ -\partial_{\mu} g_{\sigma\nu} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \partial_{\varrho} \partial_{\sigma} g_{\mu\nu} \\ +\partial_{\varrho} \partial_{\nu} g_{\mu\sigma} \\ -\partial_{\varrho} \partial_{\mu} g_{\sigma\nu} \end{pmatrix}$$

$$\partial_{\sigma} \Gamma_{\mu\,;\,\varrho\,\nu} = \frac{1}{2} \begin{pmatrix} \partial_{\varrho} \partial_{\sigma} g_{\mu\nu} \\ +\partial_{\sigma} \partial_{\nu} g_{\mu\varrho} \\ -\partial_{\sigma} \partial_{\mu} g_{\varrho\nu} \end{pmatrix}$$

$\longrightarrow$

Thus the 4-temsor $\Re_{\,\mu\,\nu\,;\,\varrho\,\sigma}$ becomes

$$\Re_{\,\mu\,\nu\,;\,\varrho\,\sigma} =$$

(44)
$$= \left[\begin{array}{c} \frac{1}{2}\,[\,\partial_{\,\varrho}\,\partial_{\,\nu}\,g_{\,\mu\sigma} - \partial_{\,\sigma}\,\partial_{\,\nu} g_{\,\mu\varrho} - \partial_{\,\varrho}\,\partial_{\,\mu}\,g_{\,\nu\sigma} + \partial_{\,\sigma}\,\partial_{\,\mu}\,g_{\,\nu\varrho}\,] \\ -\,g^{\,\alpha\beta} \left[\; \Gamma_{\,\alpha\,;\,\varrho\,\mu}\,\Gamma_{\,\beta\,;\,\sigma\,\nu} - \Gamma_{\,\alpha\,;\,\sigma\,\mu}\,\Gamma_{\,\beta\,;\,\varrho\,\nu} \; \right] \end{array} \right]$$

Several linear identities follow from the structure of the Riemann curvature tensor as displayed in eq. 44

(45)
$$\Re_{\,\mu\,\nu\,;\,\varrho\,\sigma} = \Re_{\,\varrho\,\sigma\,;\,\mu\,\nu} = -\Re_{\,\nu\,\mu\,;\,\varrho\,\sigma} = -\Re_{\,\mu\,\nu\,;\,\sigma\,\varrho}$$

Furthermore **we inspect if from the 4-tensor $\Re_{\,\mu\,\nu\,;\,\varrho\,\sigma}$ we cold form a 3-form**

(46)
$$\Re_{\,\mu}^{\,(3)} = d\,x^{\,\nu} \wedge d\,x^{\,\varrho} \wedge d\,x^{\,\sigma}\,\Re_{\,\mu\,\nu\,;\,\varrho\,\sigma}$$

To this end we form the cyclic 3-index sum

(47)
$$\left(A_{\,\mu}^{\,(3)} \right)_{\,\nu\,\varrho\,\sigma} = \left(\begin{array}{c} \Re_{\,\mu\,\nu\,;\,\varrho\,\sigma} \\ \Re_{\,\mu\,\sigma\,;\,\nu\,\varrho} \\ \Re_{\,\mu\,\varrho;\,\sigma\,\nu} \end{array} \right) \longrightarrow 0$$ **as shown below in eq. 48 and text**

$\longrightarrow$

We list the three cyclic permutations of the indices $\nu \, \varrho \, \sigma$

(48)

	$+$	$-$	$-$	$+$
$\begin{pmatrix} \nu & \varrho & \sigma \\ \nu & \varrho & \sigma \end{pmatrix}$	$\partial_\varrho \, \partial_\nu \, g_{\,\mu\sigma} (1)$	$\partial_\sigma \, \partial_\nu \, g_{\,\mu\varrho} (2)$	$\partial_\varrho \, \partial_\mu \, g_{\,\nu\sigma} (3)$	$\partial_\sigma \, \partial_\mu \, g_{\,\nu\varrho} (4)$
$\begin{pmatrix} \nu & \varrho & \sigma \\ \sigma & \nu & \varrho \end{pmatrix}$	$\partial_\nu \, \partial_\sigma \, g_{\,\mu\varrho} (5)$	$\partial_\varrho \, \partial_\sigma \, g_{\,\mu\nu} (6)$	$\partial_\nu \, \partial_\mu \, g_{\,\sigma\varrho} (7)$	$\partial_\varrho \, \partial_\mu \, g_{\,\sigma\nu} (8)$
$\begin{pmatrix} \nu & \varrho & \sigma \\ \varrho & \sigma & \nu \end{pmatrix}$	$\partial_\sigma \, \partial_\varrho \, g_{\,\mu\nu} (9)$	$\partial_\nu \, \partial_\varrho \, g_{\,\mu\sigma} (10)$	$\partial_\sigma \, \partial_\mu \, g_{\,\varrho\nu} (11)$	$\partial_\nu \, \partial_\mu \, g_{\,\varrho\sigma} (12)$

(1) cancels with (10) , (5) cancels with (2) , (8) cancels with (3) , (4) cancels with (11) , (9) cancels with (6) , (12) cancels with (7) ,
yielding complete cancellation of the sum of all cyclic $\nu \, \varrho \, \sigma$ contribtions , selecting only the terms with second derivatives in $\mathfrak{R}_{\,\mu \, \nu \, ; \, \varrho \, \sigma}$. This suffices since equal type index symmetries among $\longrightarrow$

tensor components are preserved by general coordinate transformations and in a given point of interest all first derivatives of the metric can be so transformed to 0 . Of course the terms proportinal to $(\Gamma (^2$ in the expression for $\mathfrak{R}_{\mu \nu ; \varrho \sigma}$ in eq. 44 also satisfy the cyclic addition to 0 .

We display all linear tensor component symmetries of the Riemann curvatre 4-tensor $\mathfrak{R}_{\mu \nu ; \varrho \sigma}$, subsumming eqs. 45 - 47 in eq. 49 below

(49)
$$\begin{array}{l} \mathfrak{R}_{\mu \nu ; \varrho \sigma} = \mathfrak{R}_{\varrho \sigma ; \mu \nu} = - \mathfrak{R}_{\nu \mu ; \varrho \sigma} = - \mathfrak{R}_{\mu \nu ; \sigma \varrho} \\ \mathfrak{R}_{\mu \nu ; \varrho \sigma} = + \mathfrak{R}_{\mu \sigma ; \nu \varrho} + \mathfrak{R}_{\mu \varrho ; \sigma \nu} = 0 \end{array}$$

A1-e – The discssion of general relativity basics becomes unavoidable

As a shorter approximate treatment as annonced in the subsection A1-d showed, it necessitated the discussion of the Riemann curvature tensor $\mathfrak{R}_{\mu \nu ; \varrho \sigma}$ (eq. 31 - 42) leading to its linear tensor symmetry relations summarized in eq. 49 .

Beyond the linear relations displayed in eq. 49 there exists a differential one →

Bianchi identity

We return to the Riemann curvature 2-form in eq. 31 repeated below

$$\mathfrak{R}^{(2)} = \mathbf{d}\,\Gamma^{(1)} + \left(\Gamma^{(1)}\right)^2 \quad \textbf{in matrix components} \quad \longrightarrow \left(\mathfrak{R}^{(2)}\right)^{\mu}{}_{\nu}$$

(50)

$$\left(\mathfrak{R}^{(2)}\right)^{\mu}{}_{\nu} = d\,x^{\varrho} \wedge \partial_{\varrho}\left(\Gamma^{(1)}\right)^{\mu}{}_{\nu} + \left(\Gamma^{(1)}\right)^{\mu}{}_{\alpha} \wedge \left(\Gamma^{(1)}\right)^{\alpha}{}_{\nu}$$

in (full) 4-tensor components $\longrightarrow$

$$\tfrac{1}{2}\,d\,x^{\varrho} \wedge d\,x^{\sigma}\,\left(\mathfrak{R}_{\varrho\,\sigma}\right)^{\mu}{}_{\nu} \equiv \tfrac{1}{2}\,d\,x^{\varrho} \wedge d\,x^{\sigma}\;\mathfrak{R}^{\mu}{}_{\nu\,;\,\varrho\,\sigma}$$

The Bianchi identity involves the 3-form derived from matrix valued curvature 2-form $\left(\mathfrak{R}^{(2)}\right)^{\mu}{}_{\nu}$ **.**

Dropping the matrix indices it follows in matrix notation

$$\mathfrak{B}^{(3)} = \mathbf{d}\,\mathfrak{R}^{(2)} + \left[\Gamma^{(1)}, \mathfrak{R}^{(2)}\right] = 0 \;\longleftarrow\; \mathbf{d}\,\mathfrak{R}^{(2)} = \mathbf{d}\,\mathbf{d}\,\Gamma^{(1)} + \mathbf{d}\left(\Gamma^{(1)}\right)^2$$

(51)

$$\begin{aligned} \mathbf{d}\,\mathfrak{R}^{(2)} &= \left(\mathbf{d}\,\Gamma^{(1)}\right)\Gamma^{(1)} - \Gamma^{(1)}\,\mathbf{d}\,\Gamma^{(1)} \\ \left[\Gamma^{(1)}, \mathfrak{R}^{(2)}\right] &= \left\{ \begin{array}{l} \Gamma^{(1)}\,\mathbf{d}\,\Gamma^{(1)} - \left(\mathbf{d}\,\Gamma^{(1)}\right)\Gamma^{(1)} \\ +\,\Gamma^{(1)}\left(\Gamma^{(1)}\right)^2 - \left(\Gamma^{(1)}\right)^2\Gamma^{(1)} \end{array} \right\} \\ &= \Gamma^{(1)}\,\mathbf{d}\,\Gamma^{(1)} - \left(\mathbf{d}\,\Gamma^{(1)}\right)\Gamma^{(1)} \qquad \surd \end{aligned}$$

$\longrightarrow$

While the algebraic derivation of $\left(\mathfrak{B}^{\,(3)} \right)^{\mu}_{\ \ \nu} = 0$ is straightforward as shown in eq. 51 , it involves a mixed 5-tensor

(52) $$\left(\mathfrak{B}^{\,(3)} \right)^{\mu}_{\ \ \nu} = \frac{1}{6}\, d\,x^{\,\tau} \wedge d\,x^{\,\varrho} \wedge d\,x^{\,\sigma}\, \mathfrak{B}^{\,\mu}_{\ \ \nu\,;\,\tau\varrho\sigma} = 0$$

which thus should be expressible through covariant derivatives of $\mathfrak{R}^{\,\mu}_{\ \ \nu\,;\,\varrho\sigma}$ upon antisymmetrization of the indices $\tau\varrho\sigma$

(53) $$\boxed{\begin{array}{l} \mathfrak{B}^{\,\mu}_{\ \ \nu\,;\,\tau\varrho\sigma} = \frac{1}{2}\left(D_{\tau}\,\mathfrak{R}^{\,\mu}_{\ \ \nu\,;\,\varrho\sigma} + D_{\sigma}\,\mathfrak{R}^{\,\mu}_{\ \ \nu\,;\,\tau\varrho} + D_{\varrho}\,\mathfrak{R}^{\,\mu}_{\ \ \nu\,;\,\sigma\tau} \right) \quad \rightarrow \\ D_{\tau}\,\mathfrak{R}^{\,\mu}_{\ \ \nu\,;\,\varrho\sigma} + D_{\sigma}\,\mathfrak{R}^{\,\mu}_{\ \ \nu\,;\,\tau\varrho} + D_{\varrho}\,\mathfrak{R}^{\,\mu}_{\ \ \nu\,;\,\sigma\tau} = 0 \end{array}}$$

Eqs. 49 and 53 contain all nontrivial relations pertaining to the Riemann tensor, curvature relative to a metric preserving connection .

The covariant derivative follows the chain rule of composite and arbitrarily mixed (contra- and co-variant) tensors, here we restrict the definition to $D_{\tau}\,\mathfrak{R}^{\,\mu}_{\ \ \nu\,;\,\varrho\sigma}$ as displayed in eq. 55 below.

First let me cite here an excellent textbook "Gravitation" , by C. W. Misner, K. S. Thorne and J. A. Wheeler, ref. [A1-10-1970] . $\longrightarrow$

We use the matrix valued connection as defined in eq. 30 $\Gamma_{\varrho}{}^{\mu}{}_{\nu}(x) = (\Gamma_{\varrho})^{\mu}{}_{\nu}$ and the associated covariant derivatives of contra- and co-variant vectors

(54)
$$
\begin{aligned}
(D_{\tau}v)^{\mu} &= \partial_{\tau}v^{\mu} + \Gamma_{\tau}{}^{\mu}{}_{\mu'}v^{\mu'}\\
(D_{\tau}w)_{\nu} &= \partial_{\tau}w_{\nu} - \Gamma_{\tau}{}^{\nu'}{}_{\nu}w_{\nu'} \quad \rightarrow\\
D_{\tau}(w_{\varrho}v^{\varrho}) &= \partial_{\tau}(w_{\varrho}v^{\varrho}) - v_{\varrho'}\Gamma_{\tau}{}^{\varrho'}{}_{\varrho}w^{\varrho} + v_{\varrho'}\Gamma_{\tau}{}^{\varrho'}{}_{\varrho}w^{\varrho}\\
&= \partial_{\tau}(w_{\varrho}v^{\varrho})
\end{aligned}
$$

From the chain rule we thus obtain

(55)
$$
\begin{aligned}
&D_{\tau}\mathfrak{R}^{\mu}{}_{\nu\,;\,\varrho\sigma} =\\
&= \partial_{\tau}\mathfrak{R}^{\mu}{}_{\nu\,;\,\varrho\sigma} + \left[\begin{array}{l}\Gamma_{\tau}{}^{\mu}{}_{\mu'}\mathfrak{R}^{\mu'}{}_{\nu\,;\,\varrho\sigma} - \Gamma_{\tau}{}^{\nu'}{}_{\nu}\mathfrak{R}^{\mu}{}_{\nu'\,;\,\varrho\sigma} -\\ -\Gamma_{\tau}{}^{\varrho'}{}_{\varrho}\mathfrak{R}^{\mu}{}_{\nu\,;\,\varrho'\sigma} - \Gamma_{\tau}{}^{\sigma'}{}_{\sigma}\mathfrak{R}^{\mu}{}_{\nu\,;\,\varrho\sigma'}\end{array}\right]
\end{aligned}
$$

A few remarks are in place at this stage $\longrightarrow$

1) Quantum mechanical field variables

Since the derivations presented here are based on notations also inherent to *classical* tensorial structures, it may appear that the quantities discussed , including local coordinates as e.g. arguments of the connection $\Gamma_{\tau}{}^{\mu}{}_{\nu}(x)$ (eqs. 30, 54 , 55) are conceived as classical quantities. This is *not* the case .

2) Clocks are (or should be) based on quantum mechanical systems

i.e. periodic states exhibiting specific frequency eigenvalues under specified external conditions, as is the case for Cesium clocks . This is notwithstanding the existence of very precise motions of binary star systems, which are usually described 'classically' , in particular the binary pulsar PSR 1913+1916 discovered in 1974 by Hulse and Taylor 1974 [A1-11-1974] , giving rise to the observation of acceleration of the revolution period with time in accordance with the energy loss due to emission of gravitational radiation [A1-12-1981] and related work on general relativistic motion, e.g. in ref. [A1-13-1994] .

3) From the nature of clocks space becomes quantum mechanical

This is represented in the most precise form by wavelengths and also used practically in the GPS form of geodetic and other length measurements. as described e.g. in ref. [A1-14-2003] .

4) The present discssionen is limited to classical configrations ; nevertheless .

$\longrightarrow$

Volume integrals and -densities

We begin with a 2d ($d \rightarrow 1+3$ general dimension of time-space with signature $1+d-1$)

(56) $$T_{\alpha_1 \alpha_2 \cdots \alpha_d ; \beta_1 \beta_2 \cdots \beta_d}(x) = \Pi_j \, g_{\alpha_j \beta_j}(x)$$

Coordinate transformation induce the associated co- and covariant vector transformations

(57)
$$x'^{\mu'} = x'^{\mu'}(x) \quad \leftrightarrow \quad x^{\nu} = x^{\nu}(x')$$
$$d\,x'^{\mu'} = M^{\mu'}{}_{\nu}\, d\,x^{\nu} \quad \leftrightarrow \quad d\,x^{\nu} = N^{\nu}{}_{\mu'}\, d\,x'^{\mu'}$$
$$M^{\mu'}{}_{\nu} = \partial_{x^{\nu}}\left(x'^{\mu'}\right)(x) \quad \leftrightarrow \quad N^{\nu}{}_{\mu'} = \partial_{x'^{\mu'}}\left(x^{\nu}\right)(x')$$
$$\left(N = M^{-1}\right)^{\nu}{}_{\mu'}$$

From eq. 57 the transformation of co- and contra-variant vector fields follow

$$w'^{\mu'}(x') = M^{\mu'}{}_{\nu}\, w^{\nu}(x) \quad \leftarrow \quad w^{\varrho}(x)$$
$$v'_{\mu'}(x') = N^{\nu}{}_{\mu'}\, v_{\nu}(x) \quad \leftarrow \quad v_{\varrho}(x)$$
$$x' = x'(x) \quad \leftarrow \quad x$$
$$f'(x') = v'_{\mu'}\, w'^{\mu'} = N^{\nu}{}_{\mu'}\, v_{\nu}\, M^{\mu'}{}_{\sigma}\, w^{\sigma} = v_{\nu}\, w^{\nu} = f(x) \quad \leftarrow \quad f(x)$$
(58)

→

We return to the 2d tensor in eq. 56 , which is totally symmetric under the interchanges

(59) $$\alpha_j \beta_j \leftrightarrow \alpha_k \beta_k \quad \forall \textbf{ pairs } j \leftrightarrow k$$

Next we take the antisymmetric sum over the permutations of the indices $1\,2 \cdots d$ **in** $\alpha_1 \alpha_2 \cdots \alpha_d$

(60) $$T^{(a)}_{\alpha_1 \alpha_2 \cdots \alpha_d \,;\, \beta_1 \beta_2 \cdots \beta_d} = \\ = \sum_{\Pi \begin{pmatrix} 1 & 2 & \cdots & d \\ \kappa_1 & \kappa_2 & \cdots & \kappa_d \end{pmatrix}} T_{\alpha_{\kappa_1} \alpha_{\kappa_2} \cdots \alpha_{\kappa_d} \,;\, \beta_1 \beta_2 \cdots \beta_d} \textbf{ sign } (\Pi)$$

It follows from the symmetries in eq. 59

(61) $$T^{(a)}_{\alpha_1 \alpha_2 \cdots \alpha_d \,;\, \beta_1 \beta_2 \cdots \beta_d} = \\ = \left[\begin{array}{l} \varepsilon_{\alpha_1 \alpha_2 \cdots \alpha_d} \varepsilon_{\beta_1 \beta_2 \cdots \beta_d} \times \\ \times \sum_{\Pi \begin{pmatrix} 0 & 1 & \cdots & d-1 \\ \kappa_1 & \kappa_2 & \cdots & \kappa_d \end{pmatrix}} T_{\kappa_1 \kappa_2 \cdots \kappa_d \,;\, 0\, 1 \cdots d-1} \textbf{ sign } (\Pi) \end{array} \right] \\ = \varepsilon_{\alpha_1 \alpha_2 \cdots \alpha_d} \varepsilon_{\beta_1 \beta_2 \cdots \beta_d} Det (g_{\mu\nu})$$

→

In eq. 61 the qantities $\varepsilon_{\alpha_1 \alpha_2 \cdots \alpha_d}$, $\varepsilon_{\beta_1 \beta_2 \cdots \beta_d}$ are totally antisymmetric in the d respective indices and normalized to 1 for the natural ordering thereof ; they are not tensors

$$\varepsilon_{\alpha_1 \alpha_2 \cdots \alpha_d} = \begin{cases} +1 & \\ -1 & \text{for } \Pi \begin{pmatrix} 0 & 1 & \cdots & d-1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_d \end{pmatrix} \\ 0 & \end{cases} \quad \begin{matrix} \text{even permutation} \\ \text{odd permutation} \\ \text{otherwise} \end{matrix}$$

(62)

Now we transform the 2d tensor $T_{\cdot}^{\;a}(x) \to T'^{\;a}_{\cdot}(x')$ according to eq. 58

(63) $$T'^{\;a}_{\alpha'_1 \alpha'_2 \cdots \alpha'_d ; \beta'_1 \beta'_2 \cdots \beta'_d}(x') = \begin{bmatrix} N^{\alpha_1}_{\;\alpha'_1} \cdots N^{\alpha_d}_{\;\alpha'_d} \varepsilon_{\alpha_1 \alpha_2 \cdots \alpha_d} \times \\ \times N^{\beta_1}_{\;\beta'_1} \cdots N^{\beta_d}_{\;\beta'_d} \varepsilon_{\beta_1 \beta_2 \cdots \beta_d} \times \\ \times Det\left(g_{\mu\nu}(x)\right) \end{bmatrix}$$

$$N^{\nu}_{\;\mu'} = \partial_{x'\,\mu'}\left(x^{\nu}(x')\right)$$

Next we use the d-fold antisymmetric logic for the terms containing the functional derivatives

(64) $$N^{\alpha_1}_{\;\alpha'_1} \cdots N^{\alpha_d}_{\;\alpha'_d} \varepsilon_{\alpha_1 \alpha_2 \cdots \alpha_d} = \varepsilon_{\alpha'_1 \alpha'_2 \cdots \alpha'_d} \; Det\left(N^{\nu}_{\;\mu'}\right)$$

and $\underline{\alpha}_{\,\underline{\alpha'}} \to \underline{\beta}_{\,\underline{\beta'}}$

→

In eq. 64 we have introdced the notation for an array of n equal type vector indices (either contra- or co-variant ones)

(65) $$\underline{\alpha} = (\alpha_1 \alpha_2 \cdots \alpha_n) \quad (\updownarrow)$$

Substituting eq. 64 in eq. 63 $T'^{\,a}_{\cdot}(x')$ becomes

(66)
$$\begin{aligned} & T'^{\,a}_{\alpha'_1 \alpha'_2 \cdots \alpha'_d \,;\, \beta'_1 \beta'_2 \cdots \beta'_d}(x') = \\ & = \varepsilon_{\alpha'_1 \alpha'_2 \cdots \alpha'_d} \varepsilon_{\beta'_1 \beta'_2 \cdots \beta'_d} \left(Det \left(N^{\sigma}{}_{\tau'} \right) \right)^2 Det \left(g_{\mu\nu}(x) \right) \\ & = \varepsilon_{\alpha'_1 \alpha'_2 \cdots \alpha'_d} \varepsilon_{\beta'_1 \beta'_2 \cdots \beta'_d} Det \left(N^{\sigma}{}_{\mu'} \, g_{\sigma\tau}(x) \, N^{\tau}{}_{\nu'} \right) \\ & = \varepsilon_{\alpha'_1 \alpha'_2 \cdots \alpha'_d} \varepsilon_{\beta'_1 \beta'_2 \cdots \beta'_d} Det \left(g'_{\mu'\nu'}(x') \right) \end{aligned}$$

$$g'_{\mu'\nu'}(x') = N^{\sigma}{}_{\mu'} \, g_{\sigma\tau}(x) \, N^{\tau}{}_{\nu'} \quad ; \quad N^{\nu}{}_{\mu'} = \partial_{x'\,\mu'} \left(x^{\nu}(x') \right)$$

The content of eq. 66 shows the 2d tensor structure of the quantity $T^{\,a}_{\cdot}(x)$ introdced in eqs. 56 and 60 to be equivalent to quantities representing powers of density .

→

The first such quantity a square density is the determinant of the metric $Det\ (\ g_{\mu\nu}\ (\ x\ )\ )\ \equiv\ -$ g , with the density-powers associated transformation laws

(67)
$$\mathbf{g'}\ (\ x'\ )\ =\ \left(\ Det\ \left(\ N^{\ \sigma}{}_{\tau'}\ \right)\ \right)^{\ p\ (\ \mathbf{g}\ )\ =\ 2}\ \mathbf{g}\ (\ x\ )\quad \leftarrow\quad \mathbf{g}\ (\ x\ )$$
$$\mathbf{g}\ (\ x\ )\ =\ -\ Det\ (\ g_{\mu\nu}\ (\ x\ )\ )\quad ;\quad p\ (\ \mathbf{g}\ )\ =\ 2$$

As becomes clear from the discussion between eqs. 56 - 66 g is modulo multiples the only quantity of given density power ($p\ (\ \mathbf{g}\ )\ =\ 2$) which can be formed from a sum of polynomials of matrix elements pertaining to the metric tensor $g_{\mu\nu}\ (\ x\ )$.
From this it can be inferred that the requirements of signature $(\ +\ ,\ -\ ,\ -\ ,\ -\ (d-1)$ times $)$ on the metric implies – we assume d even in the following –

(68)
$$\mathbf{g}\ =\ -\ Det\ (\ g_{\mu\nu}\ (\ x\ )\ )\ >\ 0$$

The signature restriction enforces to introduce the vielbein (d-bein for d dimensions) variables as basic field variables of the gravitational dynamics, together with the notion of tangent space tensors and spinors. We will take up tangent space in the next subsection.
Here we define – as is common – the base density of density-power $p\ =\ 1$, which follows from eq. 68

(69)
$$\mathbf{g}_{\ 1/2}\ =\ {}^{+}\sqrt{\mathbf{g}}\quad ;\quad p\ (\ \mathbf{g}_{\ 1/2}\ )\ =\ 1\quad \rightarrow$$
$$\mathbf{g'}_{\ 1/2}\ (\ x'\ )\ =\ Det\ \left(\ N^{\ \sigma}{}_{\tau'}\ \right)\ \mathbf{g}_{\ 1/2}\ (\ x\ )\quad \leftarrow\quad \mathbf{g}_{\ 1/2}\ (\ x\ )$$

→

We conclude this subsection with the coordinate-transformation covariant structure of d-volumes and associated integrals obtained from the base density $\mathbf{g}_{1/2}(x) > 0$ (eq. 69) pertaining to a metric , and subjected to orientation preserving coordinate transformations

(70) $$V = V^{d} \; : \; V(\{g_{\mu\nu}\}) = \int_{V} d^{d}x\, \mathbf{g}_{1/2}(x) = \int_{V'} d^{d}x'\, \mathbf{g'}_{1/2}(x')$$

Let the volume V be delimited by the vanishing of a suitable function $f(x)$, whence the x-coordinates are used, whereas the same volume shall be given in x'-coordinates by

(71) $$\begin{aligned} & x = x(x') \leftrightarrow x' = x'(x) \quad \longrightarrow \\ & V : f(x) = 0 \quad \leftrightarrow \quad V' : f(x(x')) = f'(x') = 0 \end{aligned}$$

Then it follows using the transformation law a the density $\mathbf{g}_{1/2}(x)$ in eq. 69

(72) $$\begin{aligned} \int_{V'} d^{d}x'\, \mathbf{g'}_{1/2}(x') &= \int_{V'} d^{d}x' \, \frac{\partial(x^{1}, \cdots, x^{d})}{\partial(x'^{1}, \cdots, x'^{d})} \, \mathbf{g}_{1/2}(x) \\ &= \int_{V} d^{d}x \;\; \mathbf{g}_{1/2}(x) \end{aligned}$$

$$Det(N^{\sigma}{}_{\tau'}) \equiv \frac{\partial(x^{1}, \cdots, x^{d})}{\partial(x'^{1}, \cdots, x'^{d})}$$

Vielbein's : from coordinate space tensors to tangent space ones and spinor representations

We take up the discussion of the signature requirement $(+\,,\,-\,,\,-\,,\,-\,(d-1)$ **times** $)$ **, with d even. The 4 (d) signs indicate the signs of the eigenvalues of the metric matrix** $g_{\mu\nu}$ **– and repeat the text before eq. 69 below.**

5) The signature restriction enforces to introduce the vielbein (d-bein for d dimensions) variables as basic field variables of the gravitational dynamics, together with the notion of tangent space tensors and spinors.

Thus we compose metric $g_{\mu\nu}$ **qadratically from from the vielbein** $e_\mu{}^a$

$$e_\mu{}^a(x) \rightarrow g_{\mu\nu}(x) = e_\mu{}^a \eta_{ab} e_\nu{}^b \;\; ; \;\; \mu\,,\,\nu\,,\,{}^a\,,\,{}^b = 0, 1, \cdots, d-1$$

(73)

$$\eta_{ab} = \delta_{ab}\, sgn_a \;\; ; \;\; sgn_a = \begin{cases} +1 \;\; (c^2) & \textbf{for} \quad a = 0 \\ -1 & \textbf{for} \quad a = 1, \cdots, d-l \end{cases}$$

In eq. 73 η_{ab} **denotes the tangent space normalized metric, which fixes – by convention relative to overall reversed signs – the signature of the metric through the relations therein. In so doing a Lorentz-invariant scalar product is established in tangent space (at x) , denoted** $\mathcal{T}(x)$ **through** η_{ab}

(74)

$$v, w\,(x) \; \textbf{contravariant vectors} \in \mathcal{T}(x) \;\; : \;\; v^a\,,\, w^a \;\; : \;\; {}^a = 0, 1, \cdots, d-1$$

$$(v\,,\,w) = v^a \eta_{ab} w^b = v^0 w^0 - \sum_{k=1}^{k=d-1} v^k w^k$$

→

The evental entry $\eta_{00} = c^2$ in eq. 73 a mere *convention* within the logic of general relativity and its embodied special relativistic limit, which can safely be transformed to rational units where

(75) $$c \equiv c_{limiting} = 1$$

6) Assuming the above logic to be falsified , as tentatively concluded by the OPERA collaboration [A1-3-2011] , simply implies that – even if only by the "weak" intermediary action of neutrino's and antinetrino's – the theoretical structure of gravity and its limiting Poincaré invariant setting for non gravitational interactions have to be abandoned .

The above remark notwithstanding we consider together with a given coordinate space contravariant vector field $w^{\mu}(x)$, transforming as defined in eq. 58, the tangent space components $w^{a}(x)$, indced by the $d \times d$ matrix valued vielbein , acting as an x dependent substitution of basis from coordinate space to tangent space , generating the metric structure associated with gravity, as displayed in eq. 73 – by assumption .

(76)
$$w^{a}(x) = e_{\mu}{}^{a} w^{\mu}(x)$$
$$\text{with} \quad \mathbf{e} = Det\left(e_{\mu}{}^{a}\right) > 0 \, \forall x \quad \text{possible for an orientable manifold}$$
$$\mathbf{e}(x) = \mathbf{g}_{1/2}(x)$$

Thus reducing to vielbein as basic variables the density e with density power $p(\mathbf{e}) = 1$ $\rightarrow$

becomes a polynomial (local) function of the vielbein (matrix-) elements $e_{\mu}{}^{a}$.
Some technical or "kinematical" properties of tangent space vector fields in parallel with coordinate space such shall be added here .

Lowering and raising indices : contra- and co-variant vectors in tangent- and coordinate-space bases

Let coordinate space be denoted $\mathcal{C}(x)$ and tangent space $\mathcal{T}(x)$. Then contra- and co-variant vector fields in $\mathcal{T}$ and $\mathcal{C}$ are equipped with the indices $w^{a}\,,\ w_{b}\,;\ w^{\mu}\,,\ w_{\nu}$ respectively

(77)

qantity	$\mathcal{T}(x)$	$\mathcal{C}(x)$
metric	η_{ab}	$g_{\mu\nu} = e_{\mu}{}^{a}\,\eta_{ab}\,e_{\nu}{}^{b}$
contra-	$w^{a} = e_{\mu}{}^{a}\,w^{\mu}$	$w^{\mu} = e_{a}{}^{\mu}\,w^{a}$
co-	$w_{b} = \eta_{ba}\,w^{a}$	$w_{\nu} = g_{\nu\mu}\,w^{\mu}$
contra-	$w^{a} = \eta^{ab}\,w_{b}$	$w^{\mu} = g^{\mu\nu}\,w_{\nu}$
base	$e_{\mu}{}^{a}\,,\ \eta_{ab}$	$g_{\mu\nu}$
inverses	$e_{a}{}^{\mu}\,;\ \eta^{ab}$	$g^{\mu\nu}$
with	$e_{a}{}^{\mu}\,e_{\mu}{}^{b} = \delta_{a}{}^{b}\,;\ \eta^{ad}\,\eta_{db} = \delta^{a}{}_{b}$	$g^{\mu\sigma}\,g_{\sigma\nu} = \delta^{\mu}{}_{\nu}$

$\longrightarrow$

The table in eq. 77 shows a sufficient set of relations among tangent space and coordinate space local base variables , that can easily be extended if necessary.

What appears a drawback , i.e. the increase in local degrees of freedom from $d\,(d+1)\,/\,2\;(\,10$ **for** $d = 4\,)$ **pertaining to the metric** $g_{\,\mu\nu}\,(\,x\,)$ **to** $d^{\,2}\,(\,16$ **for** $d = 4\,)$ **pertaining to the vielbein** $e_{\,\mu}{}^{\,a}\,(\,x\,)$ **, turns out to be essential : not only to ensure the signature structure of the metric , but also to introduce the Lorentz group – with intrinsically** ***universal*** **limiting velocity** $c_{\,limiting}$ **– as a locally gauged Lie group [A1-15-1938] .**

I use the notion 'gauging orientation', orientation referring to the axes of local tangent spaces $\mathcal{T}\,(\,x\,)$ **.**

The spin connection $(\,\omega_{\,\mu}\,)^{\,a}{}_{\,b}\,(\,x\,)$

The spin connection shall define parallel transport of a contravariant tangent vector in the (coordinate space) $\mu -$ **direction**

(78) $$\delta_{\,\|}\,w^{\,a}\,(\,x\,) \;=\; -\,d\,x^{\,\mu}\;(\,\omega_{\,\mu}\,)^{\,a}{}_{\,b}\;(\,x\,)\,w^{\,b}\,(\,x\,)\;\;;\;\; w^{\,a} \;=\; e_{\,\mu}{}^{\,a}\,w^{\,\mu}$$

The parallel transport in eq. 78 shall preserve the scalar product defined in eq. 74

(79) $$\begin{aligned} &\left(\,v \,+\, \delta_{\,\|}\,v\,,\, w \,+\, \delta_{\,\|}\,w\,\right) \;=\; (\,v\,,\,w\,)\;\;\forall\, v\,,\,w\,,\,d\,x^{\,\mu} \;\;\rightarrow \\ &v^{\,a}\,\left[\,\omega_{\,\mu}{}^{\,b}{}_{\,a}\,\eta_{\,br} \,+\, \eta_{\,ab}\,\omega_{\,\mu}{}^{\,b}{}_{\,r}\,\right]\,w^{\,r} \;=\; 0 \\ &(\,\omega_{\,\mu}\,)_{\,ar} \;=\eta_{\,ab}\,\omega_{\,\mu}{}^{\,b}{}_{\,r} \;\;\rightarrow\;\; (\,\omega_{\,\mu}\,)_{\,ar} \;=\; -\,(\,\omega_{\,\mu}\,)_{\,ra} \end{aligned}$$

$\rightarrow$

Eq. 79 shows the way towards infinitesimal (integer spin) Lorentz-transformations associated with the spin connection, but not yet the inclusion of halfinteger spins, which shall be attacked below .
To this end we first transscribe the antisymmetry relation (eq. 79)

(80) $\omega_{\mu}{}^{a}{}_{b} = \eta^{ar} \left(\omega_{\mu} \right)_{rb} \; ; \; \left(\omega_{\mu} \right)_{ar} = \eta_{ab} \, \omega_{\mu}{}^{b}{}_{r} \;\leftarrow\; \left(\omega_{\mu} \right)_{ar} = - \left(\omega_{\mu} \right)_{ra}$

The relation in eq. 80 identifies all $\mu -$ components to belong to the d- dimensional representation of $Lie \left\{ \Lambda \right\}$, denoted by matrices $\left\{ \Omega^{a}{}_{b} \right\} = \mathcal{D}_{v}$ forming the (tangent space -) vector representation of the (restricted i.e. connected part of the) Lorentz group in d dimensions $L_{1,d-1}$

$$\Omega^{a}{}_{b} \; : \; \text{with a basis decomposition} \;\leftrightarrow\; \mathcal{D}_{v}$$

$$\Omega^{a}{}_{b} = \tfrac{1}{2} \psi^{[a'b']} \left(\Omega_{[a'b']} \right)^{a}{}_{b} \; : \; \Omega_{[a'b']} = - \Omega_{[b'a']}$$

(81) $$\psi^{[a'b']} = - \psi^{[b'a']} \; \text{angles} \; ; \; {}^{a} , {}_{b} , {}_{a'} , {}_{b'} = 0, \cdots, d-1$$

$$\left(\Omega_{[a'b']} \right)^{a}{}_{b} = \delta_{a'}^{a} \, \eta_{b'b} - \delta_{b'}^{a} \, \eta_{a'b} \;\longrightarrow$$

$$\Omega^{a}{}_{b} = \eta_{bd} \, \psi^{[ad]} \equiv \psi^{a}{}_{b}$$

$\longrightarrow$

From the basis of the Lie algebra representation $\mathcal{D}_{v}$ we read off the associated structure constants of $Lie\,(L_{1,d-1})$ of the commutation relations

$$[\,\Omega_{[a'b']}\,,\,\Omega_{[d'e']}\,]^{a}{}_{b} =$$
$$= (\,\delta_{a'}{}^{a}\,\eta_{b'd} - a' \leftrightarrow b'\,)\,(\,\delta_{d'}{}^{d}\,\eta_{e'b} - d' \leftrightarrow e'\,) - (a'\,b' \leftrightarrow d'\,e'\,) =$$
$$= \begin{pmatrix} \delta_{a'}{}^{a}\,\eta_{b'd'}\,\eta_{e'b} - \delta_{b'}{}^{a}\,\eta_{a'd'}\,\eta_{e'b} - \delta_{a'}{}^{a}\,\eta_{b'e'}\,\eta_{d'b} + \delta_{b'}{}^{a}\,\eta_{a'e'}\,\eta_{d'b} \\ -\delta_{d'}{}^{a}\,\eta_{e'a'}\,\eta_{b'b} + \delta_{e'}{}^{a}\,\eta_{a'd'}\,\eta_{b'b} + \delta_{d'}{}^{a}\,\eta_{b'e'}\,\eta_{a'b} - \delta_{e'}{}^{a}\,\eta_{d'b'}\,\eta_{a'b} \end{pmatrix} =$$
$$= \begin{pmatrix} \eta_{a'd'}\,(\,\delta_{e'}{}^{a}\,\eta_{b'b} - \delta_{b'}{}^{a}\,\eta_{e'b}\,) - \eta_{a'e'}\,(\,\delta_{d'}{}^{a}\,\eta_{b'b} - \delta_{b'}{}^{a}\,\eta_{d'b}\,) \\ -\eta_{b'd'}\,(\,\delta_{e'}{}^{a}\,\eta_{a'b} - \delta_{a'}{}^{a}\,\eta_{e'b}\,) + \eta_{b'e'}\,(\,\delta_{d'}{}^{a}\,\eta_{a'b} - \delta_{a'}{}^{a}\,\eta_{d'b}\,) \end{pmatrix} =$$
$$= (\,\eta_{a'e'}\,\Omega_{[b'd']} - \eta_{b'e'}\,\Omega_{[a'd']} - \eta_{a'd'}\,\Omega_{[b'e']} + \eta_{b'd'}\,\Omega_{[a'e']}\,)^{a}{}_{b}$$
(82)

We repeat eq. 82 stripped of intermediary steps and the matrix indices $^{a}\,,\,_{b}$ below $\longrightarrow$

(83)
$$\left[\,\Omega_{\,[\,a'b'\,]}\,,\,\Omega_{\,[\,d'e'\,]}\,\right] =$$
$$= \eta_{\,a'e'}\,\Omega_{\,[\,b'd'\,]} - \eta_{\,b'e'}\,\Omega_{\,[\,a'd'\,]} - \eta_{\,a'd'}\,\Omega_{\,[\,b'e'\,]} + \eta_{\,b'd'}\,\Omega_{\,[\,a'e'\,]}$$

We wish to check the commutation rules for the rotation subgroup $R_{\,d-1}$ of $L_{\,1,d-1}$:

$a'\,,\,b'\,,\,d'\,,\,e' \;=\; 1,\cdots,d-1$

(84)
$$a' = 2\,,\,b' = 3\,;\,d' = 3\,,\,e' = 1$$
$$\left[\,\Omega_{\,[\,23\,]}\,,\,\Omega_{\,[\,31\,]}\,\right] = \eta_{\,33}\,\Omega_{\,[\,21\,]} = \Omega_{\,[\,12\,]}$$
$$\Omega_{\,[\,23\,]} = \widetilde{\Omega}_{\,1} \;\;,\;\; \Omega_{\,[\,31\,]} = \widetilde{\Omega}_{\,2} \;\;,\;\; \Omega_{\,[\,12\,]} = \widetilde{\Omega}_{\,3}$$
$$\left[\,\widetilde{\Omega}_{\,1}\,,\,\widetilde{\Omega}_{\,2}\,\right] = \widetilde{\Omega}_{\,3} \;\; \text{and cyclic} \;\;,\;\cdots\;(\surd)$$

A remark should be made at this stage

7) Since assuming a higher velocity for e.g. neutrino or antinetrino modes from the velocity of photon modes is not compatible with the universal structure of Lorentz invarariance , not even in appropriate local adapted coordinates valid for local tangent space as described in this and the previos subsection, it is mandatory to adapt in a radical way the underlying dynamical laws, in way unknown within the general relativistic logic, no direct consequence partially keeping the established rules is applicable. As a conseqence we contine to develop the general relativistic logic and come back to the qestion of how the new dynamics shold be conceived later. $\longrightarrow$

We thus return to the spin connection as left in eq. 80 and define its antisymmetric upper tangent space components

(85) $$\left(\omega_{\mu}{}^{[ab]} = \eta^{bb'} \omega_{\mu}{}^{a}{}_{b'} = - \omega_{\mu}{}^{[ba]} \right) (x) \ \text{ at } \ \mathcal{T}(x)$$

The representation $\mathcal{D}_v$ of $Lie\left(L_{1,d-1} \right)$ in eqs. 81 - 83 , partially repeated below, extends straightforwardly to all (irreducible) finite dimensional such representations $\mathcal{D}_{general}\left(Lie(L_{1,d-1}) \right)$ over the real numbers

$$\left[\Omega_{[a'b']} , \Omega_{[d'e']} \right] =$$

$$= \eta_{a'e'} \Omega_{[b'd']} - \eta_{b'e'} \Omega_{[a'd']} - \eta_{a'd'} \Omega_{[b'e']} + \eta_{b'd'} \Omega_{[a'e']}$$

$$\left(\Omega_{[a'b']} \right)^{a}{}_{b} = \delta_{a'}^{a} \eta_{b'b} - \delta_{b'}^{a} \eta_{a'b} \in \mathcal{D}_v \left(Lie(L_{1,d-1}) \right) \longrightarrow$$

---

(86) $$\left(\Omega_{[a'b']} \right)^{a}{}_{b} \quad \rightarrow \quad \left(s_{[a'b']} \right)^{\mathcal{A}}{}_{\mathcal{B}} \in \mathcal{D}_s \left(Lie(L_{1,d-1}) \right)$$

$$\left[s_{[a'b']} , s_{[d'e']} \right] =$$

$$= \eta_{a'e'} s_{[b'd']} - \eta_{b'e'} s_{[a'd']} - \eta_{a'd'} s_{[b'e']} + \eta_{b'd'} s_{[a'e']}$$

$$\mathcal{A} , \mathcal{B} = 1 , \cdots , dim \left(Lie(L_{1,d-1}) \right)$$

→

The spinor representations $(s_{[\,ab\,]})^{\mathcal{A}}{}_{\mathcal{B}}$ denoted $\mathcal{D}_{s}^{\pm} \left(Lie\,(L_{1,d-1}) \right)$ – d even = 2 n

The spinor representations derive from the $d = 2\,n$ representations of the associated (associative) Clifford algebra

(87) $$(\gamma_{a})^{\mathcal{A}}{}_{\mathcal{B}} \; , \; a = 1, \cdots, d \, ; \; \mathcal{A} \, , \, \mathcal{B} = 1, \cdots, 2^{\,n}$$ **with the anticommutation relations**

$$\{ \gamma_{a} \, , \, \gamma_{b} \} = 2 \, \eta_{ab} \, ¶_{(\, 2^{n} \times 2^{n} \,)}$$

Within the irreducible Clifford algebras for $d = 2\,n$ of dimension $2^{\,n}$ there is a twofold degenerate set of irreducible and nonequivalent representations of the Lie group $spin\,(1 \, , \, d - 1)$ denoted by $D_{s}^{\pm}$ of dimension $2^{\,n-1} = 2^{\,\frac{d}{2} \, - \, 1}$ each . We refer to refs. [A1-17-1982] , [A1-18-2008] , for reality conditions, which are the same for $d \, mod \, 8$ and exist only for $d = 2 \, , \, 4 \, mod \, 8$.

We construct the reducible spinor representations $D_{s}^{+} \oplus D_{s}^{+}$ first, given by a multiple of the commutator of any pair of distinct gamma matrices

$$s_{[\,ab\,]} = \frac{1}{4} \, [\gamma_{a} \, , \, \gamma_{b}] \quad \rightarrow$$

(88) $$\left[s_{[\,ab\,]} \, , \, s_{[\,de\,]} \right] =$$

$$= \eta_{ae} \, s_{[\,bd\,]} - \eta_{be} \, s_{[\,ad\,]} - \eta_{ad} \, s_{[\,be\,]} + \eta_{bd} \, s_{[\,ae\,]}$$

The correctness of the representation algebra of $Lie(\, D_{s} \,)$ (eq. 88) follows directly $\rightarrow$

from the defining Clifford algebra relations in eq. 87 . Still it shall be sketched how this comes about considering the chain of relations

$$a \neq b \, ; \, d \neq e \quad : \quad \gamma_a \gamma_b \gamma_d \gamma_e =$$

(89)
$$= \left[\begin{array}{ll} & -2\,\eta_{be}\,\gamma_a\gamma_d + \gamma_a\gamma_e\gamma_b\gamma_d \\ = & -2\,\eta_{be}\,\gamma_a\gamma_d + 2\,\eta_{ae}\,\gamma_b\gamma_d - \gamma_e\gamma_a\gamma_b\gamma_d \\ = & 2\,(\eta_{ae}\,\gamma_b\gamma_d - \eta_{be}\,\gamma_a\gamma_d - \eta_{bd}\,\gamma_e\gamma_a) \\ & +\gamma_e\gamma_a\gamma_d\gamma_b \\ = & 2\,(\eta_{ae}\,\gamma_b\gamma_d - \eta_{be}\,\gamma_a\gamma_d - \eta_{bd}\,\gamma_e\gamma_a + \eta_{ad}\,\gamma_e\gamma_b) \\ & +\gamma_d\gamma_e\gamma_a\gamma_b \end{array} \right] \rightarrow$$

$$\frac{1}{4}\,[\,[\gamma_a\,,\gamma_b]\,,[\gamma_d\,,\gamma_e]\,] =$$

$$= \eta_{ae}\,[\gamma_b\,,\gamma_d] - \eta_{be}\,[\gamma_a\,,\gamma_d] - \eta_{ad}\,[\gamma_b\,,\gamma_e] + \eta_{bd}\,[\gamma_a\,,\gamma_e] \quad (\surd)$$

The projection on the irredcible representations $\mathcal{D}^{\pm}_{s}\left(Lie\,(L_{1,d-1})\right)$ **(for d even) is made** $\rightarrow$

by the prodct of the γ matrices, rendered hermition by an imaginary phase factor for $d = 4\, mod\, 4$

$$\gamma_{d+1} = e^{-i\varphi} \Pi_a \gamma_a \;\; ; \;\; (\gamma_{d+1})^2 = ¶_{(2^n \times 2^n)} \;\; ; \;\; \gamma_{d+1} = (\gamma_{d+1})^\dagger$$

(90)
$$e^{-i\varphi} = \begin{cases} 1/i & \text{for } d = 4\, mod\, 4 \\ 1 & \text{else} \end{cases}$$

The representations $\mathcal{D}_s^{\pm}\,(Lie\,(L_{1,d-1}))$ are then formed through the hermitian projection matrices

$$P_\pm = \frac{1}{2}\left(¶_{(2^n \times 2^n)} \pm \gamma_{d+1}\right) \; ; \; (P_\pm)^2 = P_\pm \; ; \; P_+ P_- = 0$$

(91)
$$\mathcal{D}_s^{\pm} \simeq P_\pm\, s_{[ab]}\, P_\pm \equiv s^{\pm}_{[ab]} \;\; : \;\; 2^{n-1} \times 2^{n-1} \text{ matrices respectively}$$

$$\left(s_{[ab]} = \frac{1}{4}\,[\gamma_a , \gamma_b]\right)^{\mathcal{A}}_{\mathcal{B}} \;\; ; \;\; [P_\pm , s_{[ab]}] = 0$$

In all cases (always for d even) the two representations $\mathcal{D}_s^{\pm}$ are inequivalent, while general direct products of arbitrary multiples of both of them generate all irredcible representations of $Lie\,(L_{1,d-1})$ and from there by the exponential mapping all irreducible finite dimensional (nonunitary) representations of $spin\,(1, d-1)$, completing essentially the integer spin tensorial part .
Many further details relative to $spin\,(1, d-1)$ are here omitted and we return to the spin connection in the next subsection .

Back to the spin connection $(\omega_{\mu})^{a}{}_{b} (x)$ with respect to a general (irreducible) representations $(\mathbf{d}_{[ab]})^{\mathcal{A}}{}_{\mathcal{B}}$ denoted $\mathcal{D} \left(Lie (L_{1,d-1}) \right)$ – d even = 2 n

We repeat below the definition and doubly contravariant form of the spin connection given in eqs. 80 , 85

(92) $\omega_{\mu}{}^{a}{}_{b} = \eta^{ar} (\omega_{\mu})_{rb} \ ; \ (\omega_{\mu})_{ar} = \eta_{ab} \omega_{\mu}{}^{b}{}_{r} \ \leftarrow \ (\omega_{\mu})_{ar} = - (\omega_{\mu})_{ra}$

(93) $$\left(\omega_{\mu}{}^{[ab]} = \eta^{bb'} \omega_{\mu}{}^{a}{}_{b'} = - \omega_{\mu}{}^{[ba]} \right) (x) \ \text{at} \ \mathcal{T} (x)$$

As made sxplicit in the title of this subsection we wish to extend the action of the spin connection to arbitrary irreducible (or nearly so, eventually in the case of the spinor representations $D^{\pm}_{s}$, if taken together) , representation(s) called generically

(94)
$$\mathcal{D} \leftarrow \mathcal{D} \left(Lie (L_{1,d-1}) \right) \ni \leftrightarrow (d_{[ab]})^{\mathcal{A}}{}_{\mathcal{B}}$$
$$a \, , \, b = 0 , \cdots , d-1 \ ; \ \mathcal{A} , \mathcal{B} = 1 , \cdots , dim (\mathcal{D})$$

$$\left[\mathbf{d}_{[ab]} , \mathbf{d}_{[de]} \right] =$$
$$= \eta_{ae} \mathbf{d}_{[bd]} - \eta_{be} \mathbf{d}_{[ad]} - \eta_{ad} \mathbf{d}_{[be]} + \eta_{bd} \mathbf{d}_{[ae]}$$

In eq. 94 **d** , d , $_{d}$ have very distinct meaning, which I hope are well kept apart . →

The form of the covariant derivative of a field $\varphi^{\mathcal{A}}(x)$ transforming under the representation of the integral $spin(1, d-1)$ group exponentiated from $\mathcal{D}\left(Lie(L_{1,d-1})\right)$ follows from the derivations in the last two subsections, summarized in eqs. 92 - 94

(95)
$$(D_{\mu}\varphi)^{\mathcal{A}}(x) = \partial_{\mu}\varphi^{\mathcal{A}}(x) + \frac{1}{2}\omega_{\mu}^{[ba]}(x)\left(\mathbf{d}_{[ab]}\right)^{\mathcal{A}}{}_{\mathcal{B}}\varphi^{\mathcal{B}}(x)$$
$$\mathbf{d}_{[fg]} \in \mathcal{D}\left(Lie(L_{1,d-1})\right)$$

The spin connection $(\omega_{\mu})^{a}{}_{b}(x)$ as a function of the vielbein and its first time-space derivatives

The covariant derivatives of the vielbein vanish as a consequence of leaving the metric structure invariant. This leads to the relations

$$(D_{\tau}e)_{\mu}{}^{a} = \partial_{\tau}e_{\mu}{}^{a} - \Gamma_{\tau}{}^{\nu}{}_{\mu}e_{\nu}{}^{a} + \omega_{\tau}{}^{a}{}_{d}e_{\mu}{}^{d} = 0 \;\mid\; e_{b}{}^{\mu} = \left(e^{-1}\right)_{b}{}^{\mu}$$

(96)
$$\begin{aligned}\omega_{\tau}{}^{a}{}_{b} &= e_{b}{}^{\mu}\Gamma_{\tau}{}^{\nu}{}_{\mu}e_{\nu}{}^{a} - e_{b}{}^{\mu}\partial_{\tau}e_{\mu}{}^{a} \\ &= e_{b}{}^{\mu}\Gamma_{\varrho;\tau\mu}g^{\varrho\nu}e_{\nu}{}^{a} - e_{b}{}^{\mu}\partial_{\tau}e_{\mu}{}^{a} \\ &= e_{b}{}^{\mu}\Gamma_{\varrho;\tau\mu}e^{a\,\varrho} - e_{b}{}^{\mu}\partial_{\tau}e_{\mu}{}^{a}\end{aligned}$$

It is an inherent difficulty of the 'gauging of orientation', that a clear dependence of the connection variables, be it spin- or Christoffel-connection, is not easy to achieve and mainly to 'oversee' . $\longrightarrow$

In trying to fully exhibit the situation I do show here the derivation of the spin connection dependence on the first derivatives of the vielbein , step by step . To this end we decompose the lower 3-index Christoffel symbols , defined in eqs. 28 , 38 and 39 with respect to the vielbein fields, defined in eq. 73

$$e_{\mu}{}^{a}(x) \rightarrow g_{\mu\nu}(x) = e_{\mu}{}^{a}\,\eta_{ab}\,e_{\nu}{}^{b} \;\; ; \;\; \mu , \nu , {}^{a} , {}^{b} = 0, 1, \cdots, d-1$$

$$\eta_{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 0 & 0 & \cdots \\ 0 & 0 & -1 & 0 & \cdots \\ \cdots & & & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & -1 \end{pmatrix}_{d \times d} : \left\{ \begin{array}{l} \textbf{x- independent metric, aligning all} \\ \mathcal{T}(x) - \textbf{bases} \end{array} \right\}$$

$$\Gamma_{\varrho\,;\,\tau\,\mu} = \tfrac{1}{2}\left(\partial_{\tau}\,g_{\varrho\mu} + \partial_{\mu}\,g_{\varrho\tau} - \partial_{\varrho}\,g_{\tau\mu}\right)$$

(97)

In the next step we substitute the vielbein in the Christoffel symbol

$$\Gamma_{\varrho\,;\,\tau\,\mu} \rightarrow \Gamma_{\varrho\,;\,\tau\,\mu}\left(e_{\nu}{}^{b},\,\partial_{\chi}\,e_{\nu}{}^{b}\right)$$

$\rightarrow$

$$2\,\Gamma_{\varrho\,;\,\tau\;\mu} = \partial_\tau\, g_{\varrho\mu} + \partial_\mu\, g_{\varrho\tau} - \partial_\varrho\, g_{\tau\mu}$$

(98)

$$\begin{aligned} &= + \left(\partial_\tau\, e_\varrho{}^g\right) e_{\mu\,g} + \left(\partial_\tau\, e_\mu{}^g\right) e_{\varrho\,g} && |1 + 2 \\ &\quad + \left(\partial_\mu\, e_\varrho{}^g\right) e_{\tau\,g} + \left(\partial_\mu\, e_\tau{}^g\right) e_{\varrho\,g} && |3 + 4 \\ &\quad - \left(\partial_\varrho\, e_\tau{}^g\right) e_{\mu\,g} - \left(\partial_\varrho\, e_\mu{}^g\right) e_{\tau\,g} && |-5-6 \\ &= + \left(\partial_\tau\, e_\varrho{}^g - \partial_\varrho\, e_\tau{}^g\right) e_{\mu\,g} && |1 - 5 \\ &\quad + \left(\partial_\mu\, e_\varrho{}^g - \partial_\varrho\, e_\mu{}^g\right) e_{\tau\,g} && |3 - 6 \\ &\quad + \left(\partial_\tau\, e_\mu{}^g + \partial_\mu\, e_\tau{}^g\right) e_{\varrho\,g} && |2 + 4 \end{aligned}$$

Next we compute $\Gamma_{\varrho\,;\,\tau\mu}\, e^{a\,\varrho}$ as it appears in eq. 96

(99)

$$\begin{aligned} 2\,\Gamma_{\varrho\,;\,\tau\;\mu}\, e^{a\,\varrho} &= + \left(\partial_\tau\, e_\varrho{}^g - \partial_\varrho\, e_\tau{}^g\right) e_{\mu\,g}\, e^{a\,\varrho} && |1 - 5 \\ &\quad + \left(\partial_\mu\, e_\varrho{}^g - \partial_\varrho\, e_\mu{}^g\right) e_{\tau\,g}\, e^{a\,\varrho} && |3 - 6 \\ &\quad + \left(\partial_\tau\, e_\mu{}^g + \partial_\mu\, e_\tau{}^g\right) e_{\varrho\,g}\, e^{a\,\varrho} && |2 + 4 \end{aligned}$$

The last term (2 + 4) in eq. 99 allows a simplification since $e_{\varrho\,g}\, e^{a\,\varrho} = \delta_g{}^a$ leading to the form $\longrightarrow$

(100)
$$\begin{aligned} 2\,\Gamma_{\varrho\,;\,\tau\;\mu}\, e^{a\,\varrho} \;&=\; +\left(\partial_\tau\, e_\varrho{}^g - \partial_\varrho\, e_\tau{}^g\right) e_{\mu\,g}\; e^{a\,\varrho} && |1 - 5 \\ &\quad +\left(\partial_\mu\, e_\varrho{}^g - \partial_\varrho\, e_\mu{}^g\right) e_{\tau\,g}\, e^{a\,\varrho} && |3 - 6 \\ &\quad +\left(\partial_\tau\, e_\mu{}^a + \partial_\mu\, e_\tau{}^a\right) && |2 + 4 \end{aligned}$$

Now we cast eq. 96 into the form

(101)
$$\begin{aligned} 2\,\omega_\tau{}^a{}_b \;&=\; e_b{}^\mu\,\left(2\,\Gamma_{\varrho\,;\,\tau\;\mu}\, e^{a\,\varrho} - 2\,\partial_\tau\, e_\mu{}^a\right) \\ &=\; \left[\begin{array}{l} +\left(\partial_\tau\, e_\varrho{}^g - \partial_\varrho\, e_\tau{}^g\right) e_{\mu\,g}\, e_b{}^\mu\; e^{a\,\varrho} \\ +\left(\partial_\mu\, e_\varrho{}^g - \partial_\varrho\, e_\mu{}^g\right) e_{\tau\,g}\, e^{a\,\varrho}\, e_b{}^\mu \\ -\left(\partial_\tau\, e_\mu{}^a - \partial_\mu\, e_\tau{}^a\right) e_b{}^\mu \end{array}\right] \end{aligned}$$

In order to display the antisymmetric tangent space component structre we raise the index b

(102)
$$\omega_\tau{}^{[\,a\;b\,]} = \eta^{bd}\,\omega_\tau{}^a{}_d$$

and rewrite eq. 101 for $\omega_\tau{}^{[\,a\;b\,]}$ →

(103)
$$2\,\omega_{\tau}{}^{[\,a\,b\,]} \;=\; \left[\begin{array}{l} +\left(\partial_{\tau}\, e_{\varrho}{}^{g} - \partial_{\varrho}\, e_{\tau}{}^{g}\right)\, e^{a\,\varrho}\, e_{\mu\,g}\, e^{b\,\mu} \\ -\left(\partial_{\tau}\, e_{\mu}{}^{a} - \partial_{\mu}\, e_{\tau}{}^{a}\right)\, e^{b\,\mu} \\ +\left(\partial_{\mu}\, e_{\varrho}{}^{g} - \partial_{\varrho}\, e_{\mu}{}^{g}\right)\, e_{\tau\,g}\, e^{a\,\varrho}\, e^{b\,\mu} \end{array}\right]$$

In the first term in eq. 103 we again substitute $e_{\mu\,g}\, e^{b\,\mu} = \delta_{g}{}^{b}$ and obtain

(104)
$$\boxed{\omega_{\tau}{}^{[\,a\,b\,]} \;=\; \frac{1}{2}\left[\begin{array}{l} +\left(\partial_{\tau}\, e_{\varrho}{}^{b} - \partial_{\varrho}\, e_{\tau}{}^{b}\right)\, e^{a\,\varrho} \\ -\left(\partial_{\tau}\, e_{\varrho}{}^{a} - \partial_{\varrho}\, e_{\tau}{}^{a}\right)\, e^{b\,\varrho} \\ +\left(\partial_{\mu}\, e_{\varrho}{}^{g} - \partial_{\varrho}\, e_{\mu}{}^{g}\right)\, e_{\tau\,g}\, e^{a\,\varrho}\, e^{b\,\mu} \end{array}\right]}$$

The linear antisymmetry relation $\omega_{\tau}{}^{[\,a\,b\,]} = -\,\omega_{\tau}{}^{[\,b\,a\,]}$ is assured combining the first two terms of the expression in eq. 104 , whereas the third term is antisymmetric by itself . The basic matrix valued spin connection $\omega_{\tau}{}^{a}{}_{b}$ is recovered from the relation

(105)
$$\omega_{\tau}{}^{a}{}_{b} \;=\; \eta_{\,bd}\, \omega_{\tau}{}^{[\,a\,d\,]}$$

Of course it is to be noted that the spin connection only depends as far as the derivatives of the vielbein is concerned on the antisymmetric combinations $\partial_{\chi}\, e_{\psi}{}^{g} - \partial_{\psi}\, e_{\chi}{}^{g}$ for generic indices $\psi\,,\,\chi\,,\,{}^{g}$.

From spin connection $(\omega_{\mu})^{a}{}_{b}(x)$ to spin curvature $\Re^{a}{}_{b\,;\,\varrho\sigma}(x)$ and the relation relative to the Riemann curvature $\Re^{\mu}{}_{\nu\,;\,\varrho\sigma}(x)$

Before taking up the topic in the title of this subsection 2 remarks are in order here

8) The sign of the tangentspace metric and associated signatre convention

The signature convention $+\,,\,-\,,\,\cdots\,,\,-$ is generated by and logically identical to the diagonal elements of the tangentspace metric $\eta_{ab} = \eta^{ab}$ as defined here in eq. 97 . Many authors in older as e.g. in ref. [A1-19-1921-1963] , as well as contemporary literature use the opposite convention $\widetilde{\eta}_{ab} \equiv -\eta_{ab} \;\; \rightarrow$ signature $-\,,\,+\,,\,\cdots\,,\,+$.
There is no difference in the intrinsic conseqences and thus *only* a clear observation of the corresponding convention chosen is necessary .

9) On the role of torsion, neglected intentionally here

The Christoffel- as well as spin-connection as discssed here, bear the attribte 'minimal' , or with vanishing torsion . Despite interesting special case applications – deviating from the minimal connections – exist, e.g. in supergravity frameworks, the torsion necessarily brings into play spinor and tensor fields, matter fields in a sense to be specified from case to case. The justification for the introdction of such matter fields is however, usually left very unclear. I refer to a remark in the textbook by Wolfgang Pauli, ref. [A1-19-1921-1963] in this connection . $\rightarrow$

Returning to the topic in the title of this subsection we first introduce the spin connection and vielbein associated 1-forms

(106) $$\left(\omega^{(1)} \right)^{a}{}_{b} = dx^{\mu} \left(\omega_{\mu} \right)^{a}{}_{b} (x) \; ; \; \left(e^{(1)} \right)^{a} = dx^{\mu} e_{\mu}{}^{a} (x)$$

The metric preserving restriction in eq. 95 a fortiori induces the relation between $e^{(1)}$ **and** $\omega^{(1)}$ **which, using vector- and matrix-notation with respect to the – suppressed – tangent space indices, reads**

(107) $$\mathbf{d}\, e^{(1)} + \omega^{(1)} e^{(1)} = 0$$

Eq. 107 follows from eq. 96. It is called the first Cartan structure relation, as discussed in ref. [A1-9-1963], including a ***restricted*** **form of torsion (see ref. [A1-8-2007]) , represented by a 2-form** $\left(T^{(2)} \right)^{a}$

$$\mathbf{d}\, e^{(1)} + \omega^{(1)} e^{(1)} = 2\, T^{(2)} \quad \textbf{in components} \quad \longrightarrow$$

(108)
$$\partial_{\tau} e_{\mu}{}^{a} - \partial_{\mu} e_{\tau}{}^{a} + \left(\omega_{\tau} \right)^{a}{}_{b} e_{\mu}{}^{b} - \left(\omega_{\mu} \right)^{a}{}_{b} e_{\tau}{}^{b} = 2\, T_{[\,\tau}{}^{a}{}_{\mu\,]}$$
$$T_{[\,\tau}{}^{a}{}_{\mu\,]} = - T_{[\,\mu}{}^{a}{}_{\tau\,]} \quad \textbf{with} \quad e_{a}{}^{\nu}\, T_{[\,\tau}{}^{a}{}_{\mu\,]} = T_{[\,\tau}{}^{\nu}{}_{\mu\,]} \; ; \; e_{a}{}^{\nu} = \left(e^{-1} \right)_{a}{}^{\nu}$$
$$T_{[\,\tau}{}^{\nu}{}_{\mu\,]} = \tfrac{1}{2} \left(\Gamma_{\tau}{}^{\nu}{}_{\mu} - \Gamma_{\mu}{}^{\nu}{}_{\tau} \right)$$

In eq. 108 $T_{[\,\tau}{}^{\nu}{}_{\mu\,]}$ **represents a once contra- and twice co-variant coordinate space tensor of rank 3 , which is given by the antisymmetric part of the Christoffel connection.** $\longrightarrow$

We continue – for the time beeing – to assume all torsion contribtions to vanish.

The curvature pertaining to the spin connection is a tangent space matrix valued 2-form

$$\left(\mathfrak{R}^{(2)} \right)^{a}_{\;\;b} = \left(\mathbf{d}\,\omega^{(1)} + \left(\omega^{(1)} \right)^{2} \right)^{a}_{\;\;b} \quad \text{in components}$$

(109) $$\left(\mathfrak{R}^{(2)} \right)^{a}_{\;\;b} = \tfrac{1}{2}\, d x^{\sigma} \wedge d x^{\tau}\, \mathfrak{R}^{a}_{\;\;b\,;\,\sigma\tau} \quad \longrightarrow$$

$$e_{a}^{\;\;\mu}\, e_{\nu}^{\;\;b}\; \mathfrak{R}^{a}_{\;\;b\,;\,\sigma\tau} = \mathfrak{R}^{\mu}_{\;\;\nu\,;\,\sigma\tau} \quad : \quad \text{Riemann curvature , coordinate space 4-tensor}$$

From eq. 107 we find

(110) $$\left(\mathbf{d} + \omega^{(1)} \right) \left(\mathbf{d}\, e^{(1)} + \omega^{(1)} e^{(1)} \right) = 0 \quad \longrightarrow$$
$$\mathbf{d} \left(\omega^{(1)} e^{(1)} \right) + \omega^{(1)}\, \mathbf{d}\, e^{(1)} + \left(\omega^{(1)} \right)^{2} e^{(1)} = \mathfrak{R}^{(2)} e^{(1)} = 0$$

The linear relation in eq. 110 clearly has to be identical to the ciclic symmetry relation in eq. 49 , which follows from the component decomposition in eq. 110

(111) $$\mathfrak{R}^{a}_{\;\;b\,;\,\sigma\tau}\, e_{\nu}^{\;\;b} + \left(\nu\sigma\tau \rightarrow \tau\nu\sigma \right) + \left(\nu\sigma\tau \rightarrow \sigma\tau\nu \right) = 0 \quad (\surd)$$

Tracing the characteristics of second order partial derivatives of the vielbein inside curvature pertaining to the spin connection $\partial_{\sigma}\, \omega_{\tau}{}^{[\,a\, b\,]} \, - \, (\, \sigma \, \leftrightarrow \, \tau \,)$

For clarity we repeat the structure of the spin connection in eq. 104 below

(112)
$$\boxed{\omega_{\tau}{}^{[\,a\,b\,]} \;=\; \frac{1}{2}\left[\begin{array}{l} +\left(\partial_{\tau}\, e_{\varrho}{}^{b} - \partial_{\varrho}\, e_{\tau}{}^{b}\right) e^{\,a\,\varrho} \\ -\left(\partial_{\tau}\, e_{\varrho}{}^{a} - \partial_{\varrho}\, e_{\tau}{}^{a}\right) e^{\,b\,\varrho} \\ +\Big(\partial_{\psi}\, e_{\varrho}{}^{g} - \partial_{\varrho}\, e_{\psi}{}^{g}\Big)\, e_{\tau\, g}\, e^{\,a\,\varrho}\, e^{\,b\,\psi} \end{array}\right]}$$

We denote the second order in partial derivatives contribution to the curvature of the spin connection by $[\,\mathfrak{R}\,]^{[\,ab\,]}{}_{;\,[\,\sigma\tau\,]}$ and obtain

$$[\,\mathfrak{R}\,]^{[\,ab\,]}{}_{;\,[\,\sigma\tau\,]} \;=$$

(113)
$$=\;\frac{1}{2}\left[\begin{array}{r} e^{\,b\,\varrho}\,\left(\partial_{\varrho}\,(\partial_{\sigma}\, e_{\tau}{}^{a} - \partial_{\tau}\, e_{\sigma}{}^{a}\,)\right) - e^{\,a\,\varrho}\,\Big(\partial_{\varrho}\,(\partial_{\sigma}\, e_{\tau}{}^{b} - \partial_{\tau}\, e_{\sigma}{}^{b}\,)\Big) \\ +\, e_{\tau\,g}\, e^{\,a\,\varrho}\, e^{\,b\,\psi}\,\Big(\partial_{\sigma}\,(\,\partial_{\psi}\, e_{\varrho}{}^{g} - \partial_{\varrho}\, e_{\psi}{}^{g}\,)\Big) \\ -\, e_{\sigma\,g}\, e^{\,a\,\varrho}\, e^{\,b\,\psi}\,\Big(\partial_{\tau}\,(\,\partial_{\psi}\, e_{\varrho}{}^{g} - \partial_{\varrho}\, e_{\psi}{}^{g}\,)\Big) \end{array}\right]$$

$\longrightarrow$

We transform all indices of the Riemann-spin curvature to coordinate space ones, as in eq. 42 and 43

$$[\mathfrak{R}]_{\mu\,\nu\,;\,\sigma\,\tau} = e_{\mu a}\, e_{\nu b}\, [\mathfrak{R}]^{[ab]}{}_{;[\sigma\tau]} =$$

(114)
$$= \frac{1}{2} \left[\begin{array}{l} + e_{\mu g} \left(\partial_\nu \left(\partial_\sigma e_\tau{}^g - \partial_\tau e_\sigma{}^g \right) \right) - e_{\nu g} \left(\partial_\mu \left(\partial_\sigma e_\tau{}^g - \partial_\tau e_\sigma{}^g \right) \right) \\ - e_{\tau g} \left(\partial_\sigma \left(\partial_\mu e_\nu{}^g - \partial_\nu e_\mu{}^g \right) \right) + e_{\sigma g} \left(\partial_\tau \left(\partial_\mu e_\nu{}^g - \partial_\nu e_\mu{}^g \right) \right) \end{array} \right]$$

Indeed eq. 114 can be simplified to the form

(115)
$$[\mathfrak{R}]_{\mu\,\nu\,;\,\sigma\,\tau} == \frac{1}{2} \left[\begin{array}{l} + \left[e_{\mu g}\, \partial_\nu - e_{\nu g}\, \partial_\mu \right] \left(\partial_\sigma e_\tau{}^g - \partial_\tau e_\sigma{}^g \right) \\ + \left[e_{\sigma g}\, \partial_\tau - e_{\tau g}\, \partial_\sigma \right] \left(\partial_\mu e_\nu{}^g - \partial_\nu e_\mu{}^g \right) \end{array} \right]$$

from which the linear symmetry relations in eq. 49 follow .

Second order partial derivatives of the vielbein in Ricci tensor and curvature scalar

The quantities in the title of this subsection are obtained from the relation in eq. 115 by one and two contractions with the inverse metric respectively

(116)
$$[\mathfrak{R}]_{\nu\,\tau} = g^{\alpha\beta}\, [\mathfrak{R}]_{\alpha\,\nu\,;\,\beta\,\tau} \quad ; \quad [\mathfrak{R}] = g^{\gamma\delta}\, [\mathfrak{R}]_{\gamma\,\delta}$$

→

(117)

$$[\,\mathfrak{R}\,]_{\nu\,\tau} = \frac{1}{2}\left[\begin{array}{l} +e_g{}^{\alpha}\,\partial_{\alpha}\,\partial_{\nu}\,e_{\tau}{}^{g} - e_g{}^{\alpha}\,\partial_{\nu}\,\partial_{\tau}\,e_{\alpha}{}^{g} \\ +e_g{}^{\alpha}\,\partial_{\alpha}\,\partial_{\tau}\,e_{\nu}{}^{g} - e_g{}^{\alpha}\,\partial_{\tau}\,\partial_{\nu}\,e_{\alpha}{}^{g} \\ -e_{\nu g}\,g^{\alpha\beta}\,\partial_{\alpha}\,\partial_{\beta}\,e_{\tau}{}^{g} + e_{\nu g}\,g^{\alpha\beta}\,\partial_{\alpha}\,\partial_{\tau}\,e_{\beta}{}^{g} \\ -e_{\tau g}\,g^{\alpha\beta}\,\partial_{\alpha}\,\partial_{\beta}\,e_{\nu}{}^{g} + e_{\tau g}\,g^{\alpha\beta}\,\partial_{\alpha}\,\partial_{\nu}\,e_{\beta}{}^{g} \end{array}\right]$$

The form of eq. 117 again serves to allow full traceback of all expressions derived from eqs. 114, 115 , first relative to the Ricci tensor, and can be simplified

(118)

$$\boxed{[\,\mathfrak{R}\,]_{\nu\,\tau} = [\,\mathfrak{R}\,]_{\tau\,\nu} = \frac{1}{2}\left[\begin{array}{l} +e_g{}^{\alpha}\left(\partial_{\alpha}\,\partial_{\nu}\,e_{\tau}{}^{g} + \partial_{\alpha}\,\partial_{\tau}\,e_{\nu}{}^{g} - 2\,\partial_{\nu}\,\partial_{\tau}\,e_{\alpha}{}^{g}\right) \\ -\left(e_{\nu g}\,g^{\alpha\beta}\,\partial_{\alpha}\,\partial_{\beta}\,e_{\tau}{}^{g} + e_{\tau g}\,g^{\alpha\beta}\,\partial_{\alpha}\,\partial_{\beta}\,e_{\nu}{}^{g}\right) \\ +\left(e_{\nu g}\,g^{\alpha\beta}\,\partial_{\alpha}\,\partial_{\tau}\,e_{\beta}{}^{g} + e_{\tau g}\,g^{\alpha\beta}\,\partial_{\alpha}\,\partial_{\nu}\,e_{\beta}{}^{g}\right) \end{array}\right]}$$

We turn to the second contraction, yielding the second order derivative contribution →

to the curvature scalar

$$[\,\mathfrak{R}\,] =$$

(119)
$$= \frac{1}{2}\left[\begin{array}{r} +2\,e_{\alpha g}\,g^{\alpha\beta}\,g^{\gamma\delta}\left(\partial_\beta\,\partial_\gamma\,e_\delta{}^g - \partial_\delta\,\partial_\gamma\,e_\beta{}^g\right) \\ -2\,e_{\alpha g}\,g^{\alpha\beta}\,g^{\gamma\delta}\,\partial_\delta\,\partial_\gamma\,e_\beta{}^g \\ +2\,e_{\alpha g}\,g^{\alpha\beta}\,g^{\gamma\delta}\,\partial_\beta\,\partial_\gamma\,e_\delta{}^g \end{array}\right]$$

The two terms in the last two lines of the right hand side of eq. 119 add to the same expression as in the first line and we obtain

(120)
$$\boxed{[\,\mathfrak{R}\,] = 2\,e_{\alpha g}\,g^{\alpha\beta}\,g^{\gamma\delta}\left(\partial_\beta\,\partial_\gamma\,e_\delta{}^g - \partial_\delta\,\partial_\gamma\,e_\beta{}^g\right)}$$

On the lookout for a Lagrangean density $\mathcal{L}_{grav}$ in accord with a Hamiltonian structure $\longleftrightarrow$ lack of dependence on second order partial derivatives of metric or vielbein

This is a good place to cite the work and a book by Richard Tolman [A1-20-1934-1987] , even though the topic of this subsection has received original , seminal contributions by Albert Einstein, David Hilbert and (probably incognito) Emma Nöther (~ 1915) – as well as many others [A1-21-1915, A1-22-1915] .
The Ansatz for the thought Lagrangean density , pertinent to the gravitational field with a minimum of eventually *adjoined* ones as e.g. in what is called today dilaton-gravity

(121)

$$\mathcal{L}_{grav}(x) \sim sgn\, \mathbf{e}(x)\, f\left(\underline{\varphi}(x)\right) \Re\,(g(x))$$

$$\mathbf{e} = Det\left(e_{\mu}{}^{a}\right) > 0\,\forall x \quad \text{possible for an orientable manifold}$$

$$\mathbf{e}(x) = \mathbf{g}_{1/2}(x) \;\; ; \;\; \mathbf{g}(x) = -Det\,(g_{\mu\nu}(x))$$

$$\mathbf{g}_{1/2} = {}^{+}\sqrt{\mathbf{g}}$$

In eq. 121 $sgn = \mp 1$ denotes a sign, to be determined. The determinant of the vielbein e serves as base density according to eqs. 67 - 69 and 76 . $f > 0$ for the ground state configuration, denotes a scalar function of a finite number of local scalar fields $\underline{\varphi} = (\varphi_1, \cdots, \varphi_N)$, with $N \geq 0$, understood to be intrinsically adjoined to the gravitational interactions for $N > 0$, and $\Re$ is the scalar curvature : $\Re = g^{\mu\sigma} g^{\nu\tau} \Re_{\mu\,\nu\,;\,\sigma\,\tau}$. $\rightarrow$

It is important that $f\,(\,\underline{\varphi}\,)$ does not depend on derivatives of the scalar fields $\varphi_{\,n}\,(\,x\,)$ for $n = 1, \cdots, N$.

Because the form of $\mathcal{L}_{\,grav}$ in eq. 121 explicitely depends on second order derivatives of the vielbein (or metric) contained in the curvature scalar $\Re$, as given in eq. 120, it is not of Hamiltonian form. This (Hamiltonian) form can always be achieved modulo a divergence , which we discuss next.

To this end we repeat the 1- and 2-forms as defined in eqs. 106 below, first the spin connection and vielbein 1-forms

(122) $$\left(\omega^{\,(1)}\right)^{a}_{\;\;b} = d\,x^{\,\mu}\,(\,\omega_{\,\mu}\,)^{a}_{\;\;b}\,(\,x\,) \;\;;\;\; \left(e^{\,(1)}\right)^{a} = d\,x^{\,\mu}\,e_{\,\mu}^{\;\;a}\,(\,x\,)$$

For the following it is suitable to redefine the spin connection 1-form to exhibit antisymmetric tangent space indices, from eq. 105

(123) $$\omega_{\,\tau}^{\;\;[\,a\,b\,]} = \eta^{\,bd}\,\omega_{\,\tau\;\;d}^{\;\;a}$$

Similarly we prepare the tangent space curvature 2-form to the antisymmetric logic for tangent space tensors from eq. 109 repeated below

$$\left(\Re^{\,(2)}\right)^{a}_{\;\;\;b} = \left(\mathbf{d}\,\omega^{\,(1)} + \left(\omega^{\,(1)}\right)^{2}\right)^{a}_{\;\;\;b} \quad \textbf{in components}$$

(124) $$\left(\Re^{\,(2)}\right)^{a}_{\;\;\;b} = \tfrac{1}{2}\,d\,x^{\,\sigma}\wedge d\,x^{\,\tau}\,\Re^{\,a}_{\;\;\,b\,;\,\sigma\tau} \quad \longrightarrow$$

$$e_{\,a}^{\;\;\mu}\,e_{\,\nu}^{\;\;b}\;\Re^{\,a}_{\;\;\,b\,;\,\sigma\tau} = \Re^{\,\mu}_{\;\;\,\nu\,;\,\sigma\tau} \quad : \quad \textbf{Riemann curvature , coordinate space 4-tensor} \qquad \longrightarrow$$

We begin with the curvature 2-form in eq. 124

$$\left(\mathfrak{R}^{(2)}\right)^{a}{}_{b} = \left(\mathbf{d}\,\omega^{(1)} + \left(\omega^{(1)}\right)^{2}\right)^{a}{}_{b} =$$

$$= \frac{1}{2}\,d x^{\sigma} \wedge d x^{\tau} \left[\begin{array}{r} \partial_{\sigma}\,\omega_{\tau}{}^{a}{}_{b} - \partial_{\tau}\,\omega_{\sigma}{}^{a}{}_{b} \\ +\,\omega_{\sigma}{}^{a}{}_{g}\,\omega_{\tau}{}^{g}{}_{b} - \omega_{\tau}{}^{a}{}_{g}\,\omega_{\sigma}{}^{g}{}_{b} \end{array} \right] \longrightarrow$$

(125)

$$\left(\mathfrak{R}^{(2)}\right)^{[\,a\,b\,]} = \eta^{bg}\left(\mathfrak{R}^{(2)}\right)^{a}{}_{g} = -\left(\mathfrak{R}^{(2)}\right)^{[\,a\,b\,]} =$$

$$= \frac{1}{2}\,d x^{\sigma} \wedge d x^{\tau} \left[\begin{array}{r} \partial_{\sigma}\,\omega_{\tau}{}^{[\,a\,b\,]} - \partial_{\tau}\,\omega_{\sigma}{}^{[\,a\,b\,]} \\ +\,\omega_{\sigma}{}^{a}{}_{g}\,\omega_{\tau}{}^{[\,g\,b\,]} - \omega_{\tau}{}^{a}{}_{g}\,\omega_{\sigma}{}^{[\,g\,b\,]} \end{array} \right]$$

Next we consider the trial density times the d-dimensional infinitesimal volume element

(126)

$$d^{d} x\,\mathcal{L}_{tr} =$$

$$= sgn\, f\, \mathcal{N} \left(\varepsilon_{a_1 \cdots a_{d-2}\, g\, h} \right) e^{(1)\,a_1}\, e^{(1)\,a_2} \ldots e^{(1)\,a_{d-2}} \left(\mathfrak{R}^{(2)}\right)^{[\,g\,h\,]}$$

In eq. 126 $sgn = \pm$ denotes a sign , $\mathcal{N}$ a positive normalization constant and $\varepsilon_{\alpha_1 \alpha_2 \cdots \alpha_d}$ the totally antisymmetric symbol introduced in eq. 62. f denotes the scalar function introduced in eq. 121. $\longrightarrow$

The curvature two form $\left(\mathfrak{R}^{(2)}\right)^{[a\,b]}$ in eq. 125 is substituted in the following way

(127) $e_{\mu}{}^{a}\, e_{\nu}{}^{b} \left(\mathfrak{R}^{(2)}\right)^{[\mu\,\nu]} ; \left(\mathfrak{R}^{(2)}\right)^{[\mu\,\nu]} = \frac{1}{2}\, dx^{\sigma_{d-1}} \wedge dx^{\sigma_d}\, \mathfrak{R}^{\mu\nu}{}_{;\sigma_{d-1}\sigma_d}$

into eq. 126 to take the form

(128)
$$
\begin{aligned}
d^d x\, \mathcal{L}_{tr} &= \\
&= f \left\{ \begin{array}{l} (d!)^{-1}\, sgn\, \mathcal{N} \left(\varepsilon_{a_1 \cdots a_{d-2}\, a_{d-1}\, a_d}\right) \times \\ \times \left[\, e_{\sigma_1}{}^{a_1}\, e_{\sigma_2}{}^{a_2} \cdots e_{\sigma_{d-2}}{}^{a_{d-2}}\, e_{\mu}{}^{a_{d-1}}\, e_{\nu}{}^{a_d} \,\right] \times \\ \times\, dx^{\sigma_1} \wedge \cdots \wedge dx^{\sigma_{d-1}} \wedge dx^{\sigma_d} \left(\frac{1}{2}\, \mathfrak{R}^{\mu\nu}{}_{;\sigma_{d-1}\sigma_d}\right) \end{array} \right\} \\
&= d^d x\, \mathbf{e}\, f \left\{ \begin{array}{l} (d!)^{-1}\, sgn\, \mathcal{N} \times \\ \times \left(\varepsilon_{\sigma_1 \cdots \sigma_{d-2}\, \mu\, \nu}\right) \left(\widetilde{\varepsilon}^{\,\sigma_1 \cdots \sigma_{d-2}\, \sigma_{d-1}\, \sigma_d}\right) \times \\ \times \left(\frac{1}{2}\, \mathfrak{R}^{\mu\nu}{}_{;\sigma_{d-1}\sigma_d}\right) \end{array} \right\}
\end{aligned}
$$

The notation shall be clarified of the quantities $\varepsilon_{\sigma_1 \cdots \sigma_{d-2}\, \mu\, \nu}$ and $\widetilde{\varepsilon}^{\,\sigma_1 \cdots \sigma_{d-2}\, \sigma_{d-1}\, \sigma_d}$ in eq. 128 , which despite carrying lower and upper tensor indices are *not* tensors . →

Rather upon choosing identical indices they are identical and as defined in eq. 62

$$\varepsilon_{\alpha_1 \cdots \alpha_{d-1} \alpha_d} \equiv \widetilde{\varepsilon}^{\alpha_1 \cdots \alpha_{d-1} \alpha_d} =$$

(129)
$$= \begin{cases} +1 & \\ -1 & \textbf{for } \Pi \begin{pmatrix} 0 & 1 & \cdots & d-1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_d \end{pmatrix} \\ 0 & \end{cases} \quad \begin{matrix} \textbf{even permutation} \\ \textbf{odd permutation} \\ \textbf{otherwise} \end{matrix}$$

For the product in the last relation of eq. 128 it follows

(130)
$$\left(\varepsilon_{\sigma_1 \cdots \sigma_{d-2} \mu \nu} \right) \left(\widetilde{\varepsilon}^{\sigma_1 \cdots \sigma_{d-2} \sigma_{d-1} \sigma_d} \right) =$$
$$= k \left(\delta_{\mu}^{\ \sigma_{d-1}} \delta_{\nu}^{\ \sigma_d} - \delta_{\mu}^{\ \sigma_d} \delta_{\nu}^{\ \sigma_{d-1}} \right)$$

with $k = (d!)/2$

Substituting eq. 130 into eq. 128 and dropping the common volume element $d^d x$ **we obtain**

(131) $\mathcal{L}_{tr} = \mathbf{e} \frac{1}{2} sgn f \mathcal{N} \mathfrak{R} \leftrightarrow \mathcal{L}_{grav}(x) \sim sgn \mathbf{e}(x) f \left(\underline{\varphi}(x) \right) \mathfrak{R}(g(x))$

identical up to the numerical factor $\frac{1}{2} \mathcal{N}$ **with the form in eq. 121 .**

A3 - Inertial frames in neglect of gravity do make predictions, provided they are inertial

AEC miniworkshop on the Opera measurement, 7. December 2011, PM

A3 - 1 The earth center of mass bound frame not corotating in daily and erratic motion around the axis

The aim of the frame , as mentioned in the title of this (sub)section, is a *generous* **idealization, which is by definition only approximately realizable by measurement, i.e.** *experiment* **.**

It starts from two points in the two GPS reference systems

- **P1 at CERN, somewhere well specified along the beamline for the CERN-Gran Sasso OPERA experiment, as described by the collaboration [A1-3-2011] and surveyed by a dedicated GPS implementation group at CERN [A1-1-2002] .**

- **P2 at the Gran Sasso underground laboratories, also surveyed by recently one and over the years several GPS implementation groups [A1-5-2011] .**

The references (for P1 and P2) are very incomplete . These points are implemented by the GPS network with UTC time in a noninertial but very practical system of time zones on earth's crust , shown in the next figure below, following tradition of astronomical observations and conventions, for the better and for the worth .

In the second (following) figure a GPS base station near Gran Sasso is also shown . $\longrightarrow$

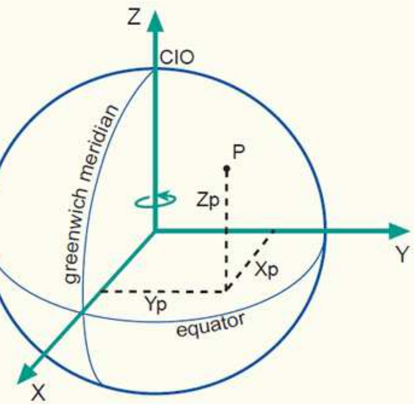
Ap3-71

2. Coordinate systems

August, 2009

Presented by: R. Knippers

Greenwich, and the *Z*-axis coincides with the Earth's axis of rotation. The three axes are mutually orthogonal and form a right-handed system. Geocentric coordinates can be used to define a position on the surface of the Earth (point P in figure below).



An illustration of the geocentric coordinate system

It should be noted that the rotational axis of the Earth changes its position over time (referred to as ***polar motion***). To compensate for this, the mean position of the pole in the year 1903 (based on observations between 1900 and 1905) has been used to define the so-called 'Conventional International Origin' (CIO).

Fig Ap1-2 : 'Geodetic' Cartesian coordinates from ref. [A1-6-2009] . ⟷

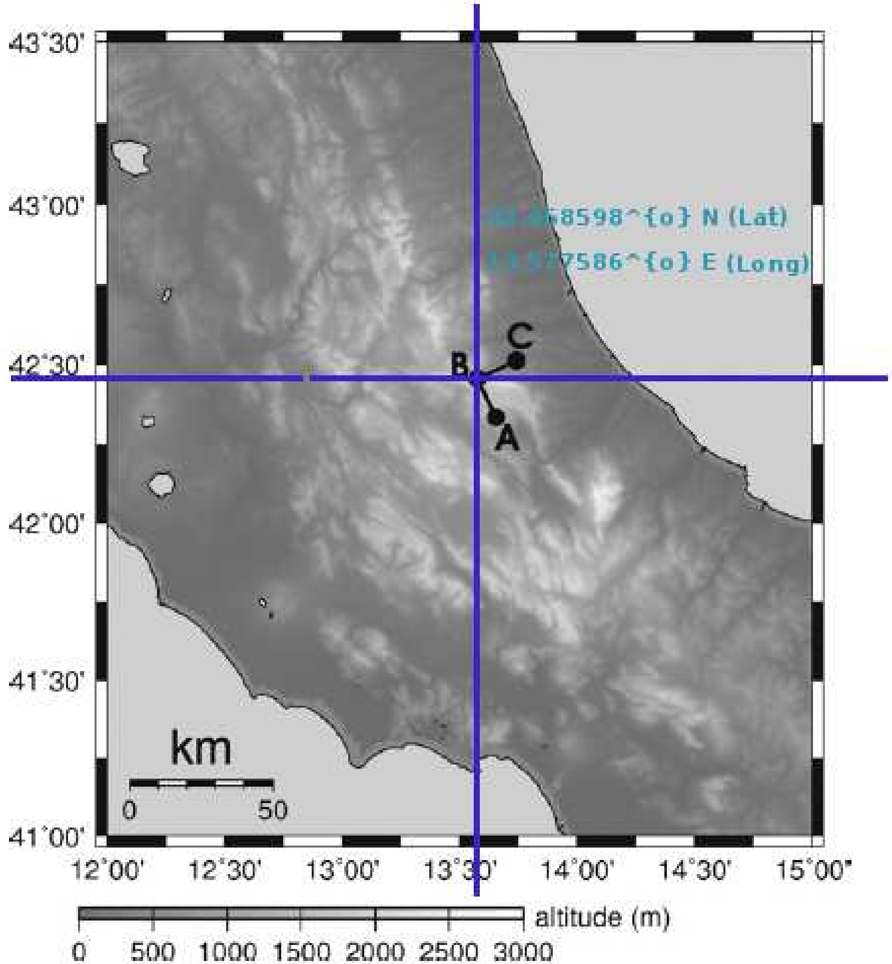


Fig. 1. Location and directions of the laser strainmeters operating at Gran Sasso.

Fig Ap1-1 : 'Geodetic' angular coordinates of a GPS basepoint near the Gran Sasso Laboratories.

→

Let me report the *earth-crust-bound* coordinates in the neigbourhood of the points determined in refs. [A1-3-2011] and [A1-5-2011] , beginning with the two points denoted T-40-S-CERN [in the CNGS target area at CERN] and A1-9999 [in Hall C at the far end, relative to CERN, of the OPERA detector]

(1)

	X (**m**)	Y (**m**)	Z (**m**)
$(3) \; = $ **A1-9999**	4582167.465	1106521.805	4283602.714
$(1) \; = $ **T-40-S-CERN**	4394369.327	467747.795	4584236.112
$\Delta \; = \; (3) \; - \; (1)$	187798.138	638774.010	-300633.396

This yields the Euclidean distance of $\vec{\Delta}$

(2)

$$D \; = \; \sqrt{(\,\Delta^{\,1}\,)^{\,2} \, + \, (\,\Delta^{\,2}\,)^{\,2} \, + \, (\,\Delta^{\,3}\,)^{\,2}} \quad = \quad 730534.609226859 \quad \textbf{m}$$

$$\rightarrow \quad 730534.610 \pm \; 0.20 \quad \textbf{m}$$

in accordance with ref. [A1-5-2011] . It remains to regress by $743.391 \; \pm \; 0.002$ **m , along** $(-) \, \vec{\Delta}$ **, to the point (0) , at the Beam Current Transformer (BCT) , as shown in the next figure** $\longrightarrow$

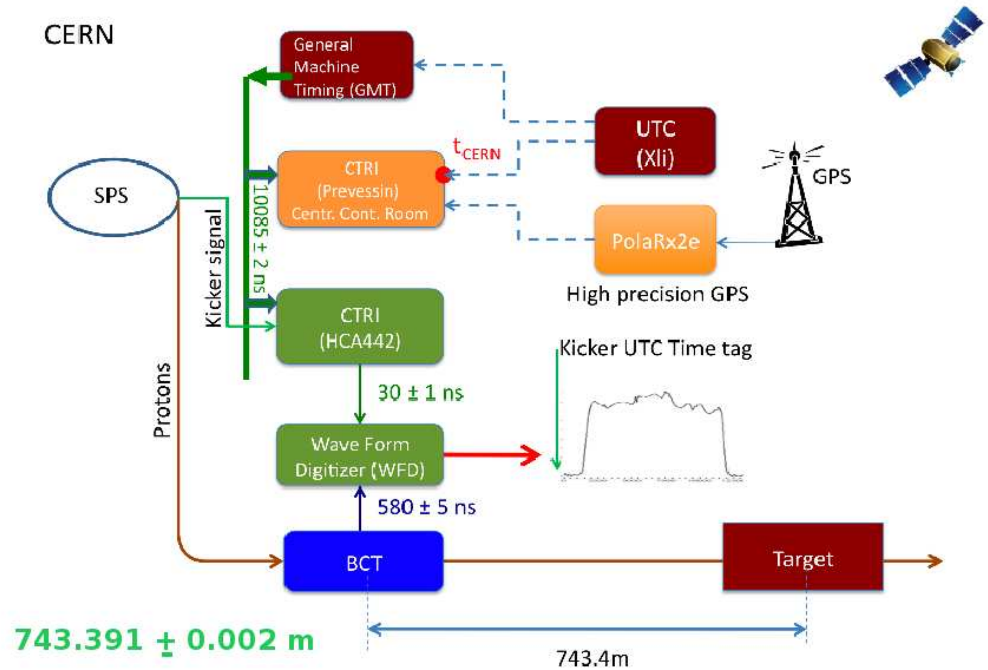


Fig. 3: Schematic of the CERN SPS/CNGS timing system. Green boxes indicate detector time-response. Orange boxes refer to elements of the CNGS-OPERA synchronisation system. Details on the various elements are given in section 6.

Fig Ap3-1 : Beam Current Transformer (BCT) and CNGS layout at CERN from ref. [A1-3-2011] .

→

Three points in earth crust bound noninertial frames , differing in the direction of the $X \rightarrow 1-$ axis :

$\oplus$ = cm of the earth , (0) = BCT [CERN] , (3) A1-9999 [Opera detector]

We give the two vectors from point $\oplus$ to points (0) and (3) in capital roman letters as $\vec{X}\,(0)$, $\vec{X}\,(3)$ respectively with coordinates denoted $\left(X^{\,1} , X^{\,2} , X^{\,3} \right) (P)$ with P = 0 , 3 respectively . We shall use two earth crust bound (noninertial) frames :

A) $X^{\,1}$ axis pointing in the equatorial plane towards the meridian through the Greenwich observatory

and

B) $X^{\,1}$ axis pointing equivalently towards the meridian through point (0) at CERN

$A)$	$X^{\,1}$ (m)	$X^{\,2}$ (m)	$X^{\,3}$ (m)
$(3) =$ **A1-9999**	4582167.465	1106521.805	4283602.714
$(0) =$ **BCT-CERN**	4394178.22387656	467097.779457861	4584542.03612673
$\Delta 30 = (3) - (0)$	187989.241123443	639424.025542139	-300939.322126728

(3)

$\longrightarrow$

The geographical latitudes and longitudes (projected on an ideal sphere of the two points (0) [BCT] and (3) [OPERA detector] are

(4)

	λ **long E deg**	β **lat N deg**
$(3)\ =$ **A1-9999**	13.5761	42.2621
$(0)\ =$ **BCT-CERN**	6.0677	46.0538

This gives rise to the Euclidean distance L between points (3) and (0)

(5) $$L\ =\ 731278.0\ \pm\ 0.2\ \textbf{m}\ \ ;\ \ T_L\ \equiv\ L\,/\,c\ =\ (\,2'439'280.844\ \pm 0.667\,)\ \textbf{ns}$$

according to and at the responsibility for the quoted errors of both GPS and the OPERA collaboration.

$B)$	$X^{\,1}$ (**m**)	$X^{\,2}$ (**m**)	$X^{\,3}$ (**m**)
$(3)\ =$ **A1-9999**	4673460.11017689	615970.600621801	4283602.714
$(0)\ =$ **BCT-CERN**	4418934.55470495	0	4584542.03612673
$\Delta 30\ =\ (3)\ -\ (0)$	254525.555471941	615970.600621801	-300939.322126728

(6)

$\longrightarrow$

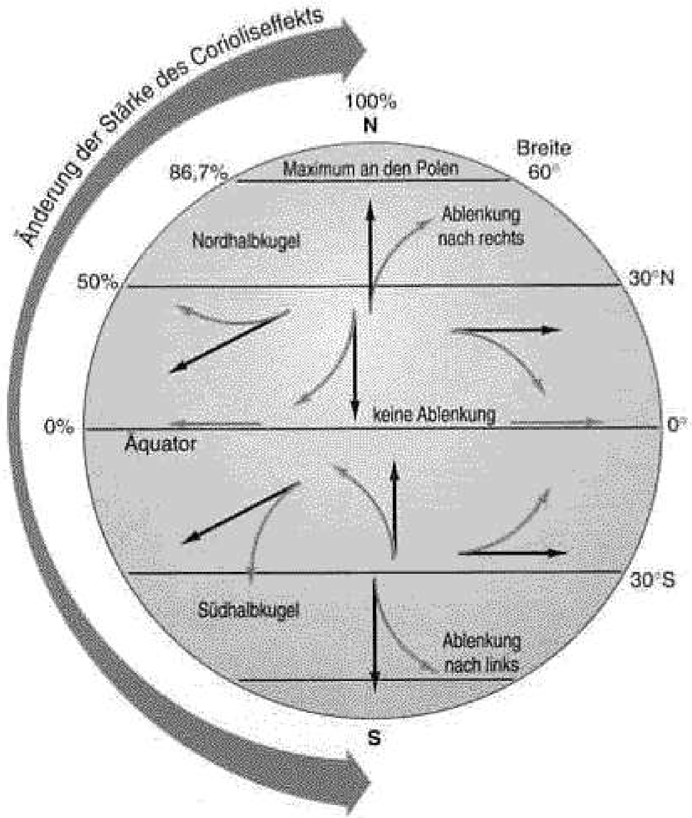
The radii of the latitude circles through points (0) BCT and (3) OPERA are (from eqs. 3 and 6) , denoted $B(0)$, $B(3)$ respectively, are

(7) $$B(0) \;=\; 4418934.55470495 \text{ m} \quad ; \quad B(3) \;=\; 4713878.35887234 \text{ m}$$

Correspondingly the daily motion velocities $v(0)$, $v(3)$ in m/s and in rational units $v(0)/c$, $v(3)/c$ become

(8) $$\begin{aligned} v(0) &= 321.353989207295 \text{ m/s} \quad \leftrightarrow \quad 1.07192152648248e - 06 \\ v(3) &= 342.802907920122 \text{ m/s} \quad \leftrightarrow \quad 1.14346741811671e - 06 \end{aligned}$$

We proceed to discuss the Coriolis effect in the earth crust bound frame relative, yet mainly in the center of mass of the earth bound one *alone* for conciseness and simplicity. Hereby we neglect the influence of all other inertial frames or platforms in the GPS satellite assembly as well as gravitational interactions in solar system and beyond. Such platforms may have much larger relative velocities whereby time dilatation/contraction can not be neglected. An overview of the Coriolis effect for nonrelativistic projectile and motion verlocities is given in the next figure [Fig Ap3-2] below.
To this end coordinates including simplified time in the earth crust bound frame shall be identified with frame B) in eq. 6 and denoted with symbols coninuing to use blue color .
In the earth center of mass inertial frame – neglecting all residual gravity interactions, persisting in this frame – we shall use for time- and space coordinates cyan colored symbols . $\longrightarrow$



Die Ablenkende Wirkung der Erddrehung.
Aus Strahler/Strahler (1999), S. 100.

Fig Ap3-2 : Nonrelativistic Coriolis effect illustration : $\vec{F} = -2\,m\;(\vec{\omega} \wedge \vec{v})$**.**

$\rightarrow$

Moving in the center of mass earthbound (almost) inertial frame

Generic coordinates , with spacial origin in the center of mass of the earth shall be

(9)
$$\Xi = \Xi^{\mu} = \left(c\,\tau , \Xi^{1} , \Xi^{2} , \Xi^{3} \right) \quad ; \quad c = c_{limiting} = 299'792'458 \text{ m/s}$$
$$\vec{\Xi} = \left(\Xi^{1} , \Xi^{2} , \Xi^{3} \right)$$

We set the clock time $\tau = 0$, when a given proton is recorded passing the BCT . How this is done and within what timing errors is assessed by the OPERA collaboration, with the result of beeing negligible. This notwithstanding the initial coordinates – $\Xi(0)$ – are [from eqs. 6 and 7]

(10)
$$\Xi(0) = \left(0 , B(0) \cos \varphi(0) , B(0) \sin \varphi(0) , \Xi^{3}(0) \right)$$
$$B(0) = 4418934.55470495 \text{ m} \quad ; \quad \Xi^{3}(0) = 4584542.03612673 \text{ m}$$

Next we assume the 3 stage process between BCT (0) - target (1) - pion decay point (2) - OPERA detector (3) to involve sufficiently relativistic and colinear particles p , π^{+} , ν_{μ} such that they can be thought of as one substrate traveling with the limiting velocity , specifically in the cyan inertial system .
Next we follow in time – τ – the trajectory of the substrate collection of p , π^{+} , ν_{μ} on a straight line and with velocity c $\longrightarrow$

denoting it $\Xi\,(\,\tau\,)$

(11)
$$\Xi\,(\,\tau\,)\;=\;\Xi\,(0)\;+\;\Delta\,(\,\tau\,)\quad;\quad\Delta\,(\,\tau\,)\;=\;(\,c\,\tau\,,\,c\,\tau\,\vec{e}\,)$$
$$\vec{e}^{\,2}\;=\;1\quad\leftrightarrow\quad\vec{e}\;=\;(\,\sin\,\vartheta\,\cos\,\varphi\,,\,\sin\,\vartheta\,\sin\,\varphi\,,\,\cos\,\vartheta\,)\quad;\quad\partial_{\,\tau}\,\vec{e}\;=\;0$$

Finally, with respect to the three points considered, we need the trajectory of point (3) [OPERA detector] in cyan – $\Xi(3)\,(\,\tau\,)$ **– which we transpose rigidly from blue coordinates at** $\tau\;=\;0$

(12)
$$\Xi\,(3)\,(\,\tau\,)\;=\;\big(\,c\,\tau\,,\,B\,(3)\,\cos\,\varphi\,(\,\tau\,)\,,\,B\,(3)\,\sin\,\varphi\,(\,\tau\,)\,,\,\Xi^{\,3}\,(3)\,\big)$$
$$\varphi\,(\,\tau\,)\;=\;\varphi\,(0)\;+\;\Delta\,\lambda\,(30)\;+\;\omega\,\tau$$
$$\Delta\,\lambda\,(30)\;=\;0.131046480477406\;\textbf{rad}\quad;\quad B(3)\;=\;4713878.35887234\;\textbf{m}$$
$$\Xi^{\,3}\,(3)\;=\;4283602.714$$
$$\omega\;=\;2\,\pi\,/\;\textbf{day}\;=\;7.27220521664304e-05\;,\;7.29211505392569e-05\;\textbf{rad/s}$$

For completeness of data we repeat eq. 10 below

(13)
$$\Xi\,(0)\;=\;\big(\,0\,,\,B\,(0)\,\cos\,\varphi\,(0)\,,\,B\,(0)\,\sin\,\varphi\,(0)\,,\,\Xi^{\,3}\,(0)\,\big)$$
$$B(0)\;=\;4418934.55470495\;\textbf{m}\quad;\quad\Xi^{\,3}\,(0)\;=\;4584542.03612673\;\textbf{m}$$

The arrival time $\tau\,(3)$ **is the solution of the equation , to which we turn next**

(14)
$$\Xi\,(\,\tau\,)\;=\;\Xi\,(0)\;+\;\Delta\,(\,\tau\,)\;=\;\Xi\,(3)\,(\,\tau\,)$$

→

Back to mainly blue symbols

Using the axial symmetry to set $\varphi(0) = 0$ **eq. 14 becomes**

$$\Xi(\tau) - \Xi(0) = \Delta(\tau) = \Xi(3)(\tau) - \Xi(0)$$

$$\Delta(\tau) = (c\tau, c\tau\,\vec{e}) \quad ; \quad \varphi(\tau) = \Delta\lambda(30) + \omega\tau$$

(15)
$$\Xi(3)(\tau) - \Xi(0) =$$
$$= \begin{pmatrix} c\tau & B(3)\cos\varphi(\tau) & B(3)\sin\varphi(\tau) & \Xi^3(3) \\ -0 & -B(0) & -0 & -\Xi^3(0) \end{pmatrix}$$

$$\Xi^3(3) - \Xi^3(0) = \Delta 30^3 = -300939.322126728 \text{ m}$$

The numerical solution to eq. 15 yields

(16)
$$\tau = 2'439'283.0497\,[3279] \text{ ns}$$
$$T_L \equiv L/c = 2'439'280.844 \pm 0.667 \text{ ns}$$
$$\tau - T_L = \sim +2.206 \pm 0.667 \text{ ns}$$
$$-\delta t = TOF - TOFc = -57.8 \pm 7.8\,^{+5.9}_{-8.3} \text{ ns}$$ **[OPERA coll.]**

The way to a linear interpolation with adequate precision is shown in the next figure below →

Ap3-82

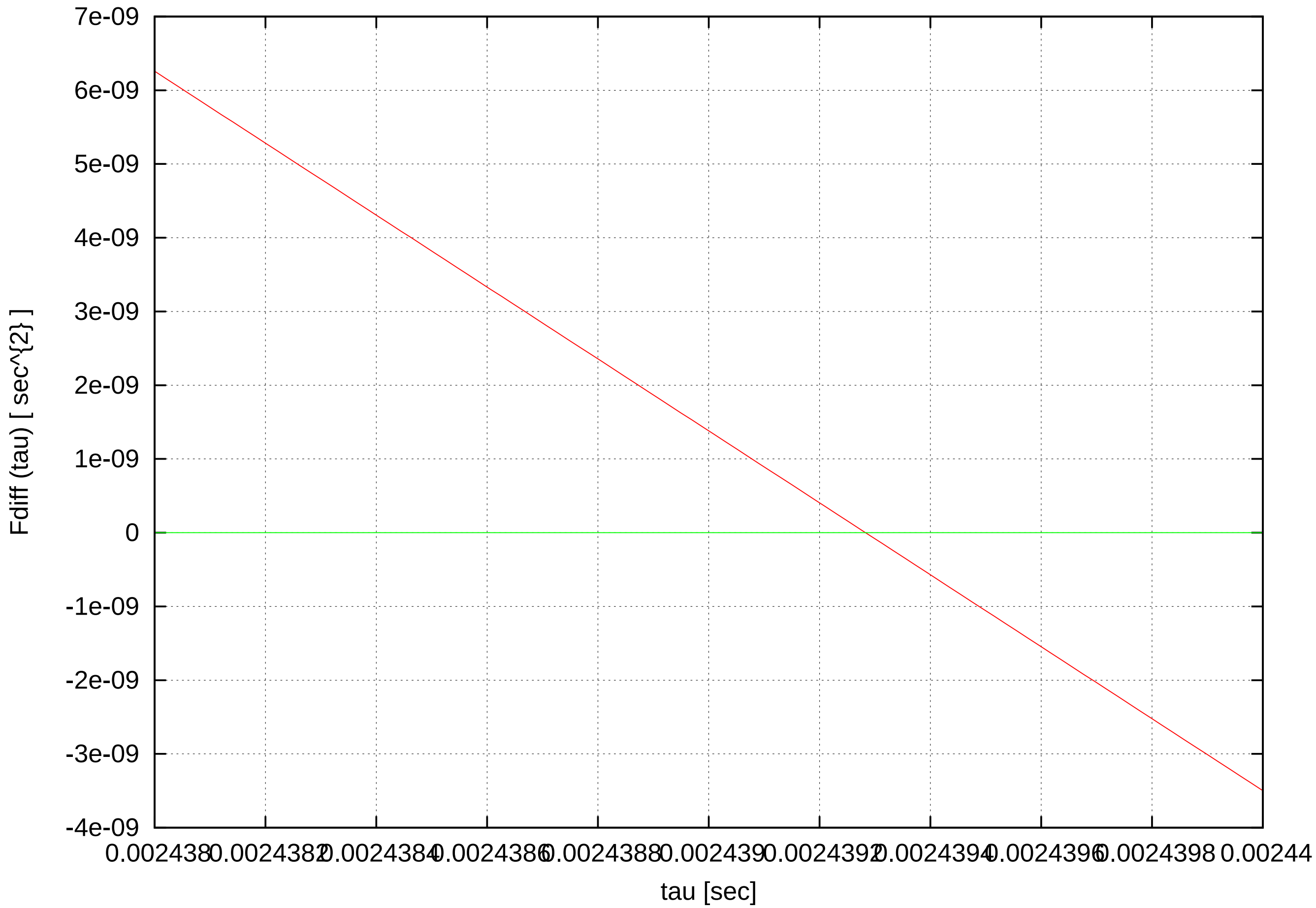


Fig Ap3-3 : $Fdiff(\tau) =$

$$(B(3)/c)^2 + (B(0)/c)^2 - 2(B(3)B(0)/c^2)\cos\varphi(\tau) + (\Delta 30^3/c)^2 - \tau^2$$

→

This shall suffice for the time beeing, followed by a few concluding remarks

1) Transforming to other inertial frames

The solution found within the center of mass of the earth bound inertial frame can be used to transform – by arbitrary inhomogeneous Lorentz transformations to an other such . A class of direct Lorentz transformations using rational units [$c_{\,limiting} \;=\; 1$] is

$$\Lambda \;=\; \left(\begin{array}{cc} \cosh\chi & \cosh\chi\,(\,v^{\,j}\,) \\ \cosh\chi\,(\,v^{\,i}\,) & \delta_{\,ij} \;+\; v^{\,i}\,v^{\,j}\;\coth^{\,2}\chi\,(\,\cosh\chi \;-\; 1\,) \end{array} \right) \qquad (17)$$

$$\vec{v} \;=\leftarrow\; \vec{v}\,/\,c \quad ; \quad i\,,\,j \;=\; 1,2,3 \quad ; \quad |\,\vec{v}\,| \;=\; \tanh\chi$$

2) GPS

There are plenty of documentations as well as very precise applications of GPS, yet crosschecks are difficult . Maybe as an exercise a group within GPS will volunteer to find otherwise blindfolded from BCT at CERN to the OPERA detector ?

3) Timing , UCT especially synchronization and TOF

It is not trivial to convincingly prove correct setting and synchronization of clocks, as an ideal monochromatic signal form with definite frequency has an infinite timing uncertainty .

4) The problem revealed by the OPERA collaboration is definitely interesting
yet I suggest further experimental and theoretical effort not tiring too soon from a psychological bias, always accompanying also physics experiments .

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→

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