On a heuristic treatment of an anomalous supercurrent in N=1 super Yang-Mills theories

Master's thesis

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So schreitet in dem engen Bretterhaus
Den ganzen Kreis der Schöpfung aus,
Und wandelt mit bedächt'ger Schnelle
Vom Himmel durch die Welt zur Hölle!

Johann Wolfgang von Goethe,
Faust. Der Tragödie erster Teil.
Vorspiel auf dem Theater.

Contents

Chapter 1

Introduction[1]

The scientific progress in Christian medieval times was guided by faith without paying much attention to the empirical phenomena of the surrounding world (except for medicine). It was believed that one could gain insights into the laws of nature by logical reasoning and studying the Bible, ignoring experimental facts. This way of scientific working has been characterized in "Leben des Galilei" by Bertold Brecht:

Galilei *am Fernrohr*: ... Ist es den Herren angenehm, mit einer Besichtigung der Jupitertrabanten zu beginnen, der Mediceischen Gestirne?

Andrea *auf dem Hocker vor dem Fernrohr zeigend*: Bitte, sich hier zu setzen.

Der Philosoph : Danke, mein Kind. Ich fürchte, dass alles ist nicht ganz so einfach. Herr Galilei, bevor wir Ihr berühmtes Rohr applizieren, möchten wir um das Vergnügen eines Disputs bitten. Thema: Können solche Planeten existieren?

It was in the 16th and 17th century that a paradigm shift took place. Scientists, especially physicists, went back to the tradition of some ancient Greek philosophers (Pythagoras, Aristoteles and many others), who believed that only by engaging in observational activity one could learn something about the underlying principles of nature and succeed in a description of them.

The second leading idea of modern physics was that of unification. Physicists have always tried to unify different theories into one overarching framework like Maxwell's work on electromagnetism and the $SU(2)_L \times U(1)_Y$ gauge theory of the electroweak force. These two principles have not been abandoned since the discovery of the law of gravitation by Newton and led to the fascinating theories of electromagnetism, general relativity, quantum mechanics and quantum fields.

In high energy physics the increasing expenses for experiments during the last 50 years drove physicists to turn away from the tradition of looking first at nature and thinking afterwards about it. Supersymmetry, supergravity and string theory are the result thereof.

[1] A detailed introduction to supersymmetry suitable also for non-physicists is found in [1].

The case of string theory will not be discussed further but there is plenty of literature like [2]. For a critical view see e.g. [3].

In the late sixties people analyzed the mass relation of the meson octet $m_\pi^2 + 3m_\eta^2 = 4m_K^2$ which holds to first order in perturbation theory and for equal quark masses $m_u = m_d$ [2]. These efforts resulted in what is nowadays viewed as the starting point of supersymmetry (for more details about the discovery of supersymmetry see [4]). A series of no-go theorems [5–7] proved that it is impossible to unify the Poincaré group and internal symmetries of a theory in a non trivial way within the context of ordinary Lie groups. The only way out was to introduce supersymmetry. It is important to note that one does not need any further assumptions for constructing a supersymmetric theory other than the ones needed for a quantum field theory. Supersymmetry is today the most promising candidate for a grand unified theory (GUT) although there are non-supersymmetric models for the physics beyond the standard model.

Supersymmetry allows one to bring together the electroweak and strong coupling constant at high energies and even to incorporate gravitation (supergravity). Furthermore, supersymmetry provides a possible solution to the hierarchy problem of the standard model. The question is why the vacuum expectation value of the Higgs field $\approx 246\ GeV$ ("weak scale") is so small compared to the Planck mass $m_P = 1.22 \cdot 10^{19}\ GeV$ or GUT scale $m_{GUT} \approx 10^{16}\ GeV$? For a review of the unsolved issues of the standard model and its possible solutions trough supersymmetry see [8].

The viable phenomenological supersymmetric model (MSSM) provides also candidates for the non-baryonic dark matter, which seems to be required by cosmology. Supersymmetric theories do also have appealing properties from a more technical point of view. Many of the divergences one has to deal with in the standard model vanish somewhat miraculously. This is what makes most of physicists community believe that we need supersymmetry to get a grand unified theory of the four fundamental forces.

But what is this supersymmetry? It is a symmetry that maps bosons onto fermions and vice versa. In the standard model these two kinds of particles play fundamentally different roles. The fermions are the building blocks of matter whereas the bosons mediate the forces between the fermions. With matter I mean hadronic particles and leptons. Our every day experience of matter only contains a small part of the rich zoo of hadrons namely protons and neutrons together with one of the three leptons, the electron. Fermions and bosons are now joined together into a supersymmetry multiplet. This means that in a theory with unbroken supersymmetry there is always a boson and fermion with the same mass. Since we do not observe such a multiplet structure in nature we have to conclude that supersymmetry is somehow broken. Despite the tremendous effort that has been done since the first appearance of a renormalizable supersymmetric model in 1974 [9], the nature of this symmetry breaking is still not fully understood.

[2] This relation is known as the Gell-Mann-Okubo relation.

Let us briefly have a look at the so called Minimal Supersymmetric Standard Model (MSSM) which is the phenomenologically most suitable model we have to date for the physics above 1 TeV. A detailed and accurate introduction is found in [10], whereas [11] provides a more phenomenological point of view of the MSSM.

The quantum corrections to one loop of the mass of the Higgs boson in the standard model are quadratically divergent i.e. $\Delta m_H^2 \propto \Lambda_{cut}^2$, where Λ_{cut} is the momentum cut-off at which new physics is expected to appear. Note that Λ_{cut} does not exist physically, it is a parameter which is introduced during the process of renormalization. This quadratic divergence means that the quantum corrections to the Higgs boson mass are infinite at every finite order. Supersymmetry solves the problem by removing this divergence. In a supersymmetric version of the standard model we would always have a bosonic and a fermionic loop contributing to Δm_H^2. Because of the opposite sign of the loops we get a cancellation of the contributions to Δm_H^2.

Therefore one puts every particle of the standard model (leptons, quarks and gauge bosons) into a supersymmetry multiplet in order to achieve the described cancellations. The super partners of fermions are called sleptons and squarks and for the boson partners one adds the suffix "-ino" e.g. gluino. Since the supersymmetry generators commute with the generators of the gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$, the super partners must be in the same representation of the gauge group. It gets somewhat trickier when one puts the Higgs boson into a supersymmetry multiplet, but these problems go beyond the scope of this short overview (see [10]).

The interesting point is that so called "soft" supersymmetry breaking terms are added to the Lagrangian ensuring that the supersymmetric partners do not have the same mass as the original particles. If in fact supersymmetry would not be broken, particles like the selectron would have been detected a long time ago. The soft breaking terms have the form of mass terms of the scale m_{soft}, which is assumed to be not much larger than $1\,TeV$. This is the energy at which physicists hope to see direct evidence of supersymmetry. It is important to note that these soft breaking terms are added ad hoc. At first there is no dynamical effect in the theory that would break supersymmetry in contrast to the super Yang-Mills theory which will be the subject of this work. In order to "explain" the terms for breaking supersymmetry the MSSM is extended. The common nomenclature is that the supersymmetry breaking origin is assigned to a hidden sector in contrast to the visible sector of the MSSM. The two sectors have only a very small or non tree-level coupling which mediates the supersymmetry breaking from the hidden sector to the visible sector. In the visible sector this mediating appears then as the mentioned soft breaking terms. The hidden sector is not yet fully understood and there are several competing possibilities for this mediation: gravitationally and gauge mediated supersymmetry breaking are the most promising ones [10].

In the MSSM there are plenty of parameters e.g. masses of the super partners etc. which are constrained by several phenomenological and theoretical reflections. A detailed list of them is found in [8] and [10]. Recently, J. Ellis et al. [12] have tried to do a statistical sampling of the CMSSM (Constrained Minimal Supersymmetric Standard Model) parameter space and therewith calculate expectation values for different masses.

Nevertheless there is still no direct experimental evidence for supersymmetry to be realized in nature. A "hint" for supersymmetry is that in the standard model the decay rate of the Z boson into charm and bottom quarks does not match the experimental values measured at LEP [13]. Also the difference between the experimental and theoretical $g-2$ of the muon can partially be explained by the MSSM (check [8] for further references).

The group of Francesco Iachello calculated molecular spectra with the help of supersymmetry and results may be interpreted as strong indication for the existence of supersymmetry. However, one has to be careful with this result since this is not a supersymmetry at a fundamental level. Major hopes of such an evidence are put into the experiments of LHC at CERN in the near future. It is believed that not only the Higgs boson but also the lightest super partners of the MSSM will be detected at LHC. If there will be a super partner at the expected 1 TeV, its existence should be proved within the first year of the LHC experiment.

Let me now outline the issue of this thesis. Yang-Mills theories were developed independently in 1953 and 1954 by W. Pauli[3] in Zurich and by C.-N.Yang and R. L. Mills [16] at Brookhaven National Laboratory. The underlying mathematical structure was derived by E. Cartan more than a decade earlier. Today such theories are a fundamental constituent of our knowledge about physics e.g. the electro-weak theory and QCD are Yang-Mills theories with matter fields and gauge group $SU(2)_L \times U(1)_Y$ and $SU(3)_C$ respectively.

The logical extension of ordinary Yang-Mills is the super Yang-Mills theory which will be the subject of investigation of this thesis. On the basis of the similarities of super Yang-Mills theory to QCD, I will assume throughout this work that the basic features of super Yang-Mills dynamics are similar to QCD: confinement of the colored degrees of freedom and spontaneous chiral symmetry breaking. Furthermore the low energy dynamics feature the realization of global chiral symmetry. I do also assume that the super Yang-Mills theory exists, i.e. there is a stable vacuum (or vacua)[4].

The question whether supersymmetry is spontaneously broken in pure (without matter fields) super Yang-Mills theory kept busy some brilliant physicists already in the early eighties. At that time it was concluded that supersymmetry is not broken in this kind of theories and hence they are not directly applicable to a phenomenological relevant model.

A few years ago it has been tried to put the simplest super Yang-Mills theory, namely the N=1, on the lattice. The DESY-Münster-Roma collaboration [17–19] did some numerical simulations but as far as I know has not yet reached a definitive result. It seems that the limit of zero lattice spacing and infinite volume, where supersymmetry is thought to be restored, is affected by systematic effects. The people of the DESY-Münster-Roma collaboration started with the hypothesis of unbroken supersymmetry, which would make the possible conclusion of broken supersymmetry even more interesting.

[3]Unfortunately there is no publication of Pauli about the subject. The only reachable documents are a letter of W. Pauli to A. Pais [14] and an article of P. Gulmanelli [15].

[4]"Hypotheses non fingo", Sir Isaac Newton.

In the mid nineties the problem of supersymmetry breaking was picked up by P. Minkowski and his collaborators. It was shown that fundamental issues have not been resolved in the past. For details see chapter 3. The aim of this thesis is to collect and summarize the work done by P. Minkowski et al. and maybe go a small step towards the correct answer of the question raised above.

The present thesis is organized as follows. First the general setup is presented. A brief review of the construction of a field theory in superspace is given in chapter 2. Especially I will investigate into pure N=1 super Yang-Mills theory and the Noether current of this theory that corresponds to a supersymmetry transformation.

In chapter 3, after a few general remarks about supersymmetry breaking, a summary of the arguments for the spontaneous breaking of supersymmetry is exposed. The possibility of the presence of a goldstino is ruled out by theoretical arguments and the conclusion of an anomalous supercurrent is drawn.

Chapter 4 deals with the perturbative confirmation of the supercurrent anomaly. Although no definitive result is presented, the constraints from dimensional analysis and transformation properties on the structure of the anomaly are discussed. Attention is then paid to the triangle graphs that could possibly contribute to the supercurrent anomaly.

In order to clarify the notations and conventions used in this thesis there is appendix A. Appendix B is ment as a short review of the mathematical fundament of supernumbers, supervectors and superanalysis. In appendix C the Feynman rules used for the diagrams of chapter 4 are presented and lastly in appendix D the general notion of the quantum effective action is reviewed.

Chapter 2

Supersymmetry in Superspace

This chapter is intended to outline the setup used for further investigation. The super Poincaré group is presented and it is shown how to build a field theory in superspace (for some mathematical definitions see appendix B). Subsequently, the super Yang-Mills theory is presented. For further reading on the subject I refer to [20].

Please note that the notation and conventions used throughout this thesis may appear unusual. In order to prevent confusions a short abstract of what is found in appendix A shall be given. The space-time metric takes the form $g_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ and the Pauli matrices are given by

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.1}$$

The index structure is $(\sigma_\mu)_{\alpha\dot{\alpha}}$ and $(\tilde{\sigma}_\mu)^{\dot{\alpha}\alpha}$ with $\tilde{\sigma}_\mu = (+\sigma_0, -\vec{\sigma})$. Spinor indices are contracted as $\psi\chi \doteq \psi^\alpha\chi_\alpha = -\psi_\alpha\chi^\alpha$ and $\bar{\psi}\bar{\chi} \doteq \bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} = -\bar{\psi}^{\dot{\alpha}}\bar{\chi}_{\dot{\alpha}}$ and the operation of complex conjugation is defined as $(\psi_\alpha)^* \equiv \bar{\psi}_{\dot{\alpha}}$. Hermitian conjugated objects are denoted with a bar, i.e. $\bar{V} \equiv V^\dagger$. The D'Alembert operator is defined as $\Box \doteq \partial_\mu\partial^\mu$ which means that it is the negative of what is historically known as the D'Alembert operator.

2.1 The super Poincaré group

Throughout this thesis I will deal with only one generation of supercharges $\mathbb{Q}_\alpha$ (N=1) and therefore the commutation relations of the N=1 Poincaré superalgebra read:

$$\begin{aligned}
[\mathbb{P}_\mu, \mathbb{P}_\nu] &= 0\,, \qquad [\mathbb{M}_{\mu\nu}, \mathbb{P}_\rho] = ig_{\mu\rho}\mathbb{P}_\nu - ig_{\nu\rho}\mathbb{P}_\mu\,,\\
[\mathbb{M}_{\mu\nu}, \mathbb{M}_{\rho\sigma}] &= ig_{\mu\rho}\mathbb{M}_{\nu\sigma} - ig_{\mu\sigma}\mathbb{M}_{\nu\rho} + ig_{\nu\sigma}\mathbb{M}_{\mu\rho} - ig_{\nu\rho}\mathbb{M}_{\mu\sigma}\,,\\
\\
[\mathbb{M}_{\mu\nu}, \mathbb{Q}_\alpha] &= i(\sigma_{\mu\nu})_\alpha{}^\beta\,\mathbb{Q}_\beta\,, \qquad [\mathbb{P}_\mu, \mathbb{Q}_\alpha] = 0\,,\\
[\mathbb{M}_{\mu\nu}, \bar{\mathbb{Q}}_{\dot{\alpha}}] &= i(\tilde{\sigma}_{\mu\nu})_{\dot{\alpha}}{}^{\dot{\beta}}\,\bar{\mathbb{Q}}_{\dot{\beta}}\,, \qquad [\mathbb{P}_\mu, \bar{\mathbb{Q}}_{\dot{\alpha}}] = 0\,,
\end{aligned} \tag{2.2}$$

$$\{\mathbb{Q}_\alpha, \mathbb{Q}_\beta\} \;=\; 0\,, \qquad\qquad \{\mathbb{Q}_\alpha, \mathbb{Q}_\beta\} \;=\; 0\,,$$
$$\{\mathbb{Q}_\alpha, \bar{\mathbb{Q}}_{\dot\beta}\} \;=\; 2(\sigma^\mu)_{\alpha\dot\beta}\,\mathbb{P}_\mu\,.$$

Here $\mathbb{M}_{\mu\nu}$ and $\mathbb{P}_\mu$ are the generators of the (ordinary) Poincaré group and $\bar{\mathbb{Q}}_{\dot\alpha}$ is the hermitian conjugate of $\mathbb{Q}_\alpha$, i.e. $(\mathbb{Q}_\alpha)^\dagger \doteq \bar{\mathbb{Q}}_{\dot\alpha}$. The N=1 super Poincaré algebra is defined as the real c-type supervectors $X \in \mathrm{S}\mathscr{P}_c(\Lambda_\infty)$ (see [20]). Every element of $\mathrm{S}\mathscr{P}_c(\Lambda_\infty)$ has the form

$$X \;=\; i\left(-b^\mu\mathbb{P}_\mu + \frac{1}{2}K^{\mu\nu}\mathbb{M}_{\mu\nu} + \epsilon^\alpha\mathbb{Q}_\alpha + \bar\epsilon_{\dot\alpha}\bar{\mathbb{Q}}^{\dot\alpha}\right)\,, \tag{2.3}$$

$$b^\mu,\; K^{\mu\nu} \;\in\; \mathbb{R}_c \qquad \epsilon^\alpha \;\in\; \mathbb{C}_a \qquad \bar\epsilon^{\dot\alpha} = (\epsilon^\alpha)^*\,.$$

where $K^{\mu\nu}$ has to be antisymmetric in its indices. The N=1 super Poincaré group SΠ is the super Lie group corresponding to the N=1 super Poincaré algebra. Every element $g \in$ SΠ is denoted as

$$g(b,K,\epsilon,\bar\epsilon) \;=\; \exp\left[i\left(-b^\mu\mathbb{P}_\mu + \frac{1}{2}K^{\mu\nu}\mathbb{M}_{\mu\nu} + \epsilon^\alpha\mathbb{Q}_\alpha + \bar\epsilon_{\dot\alpha}\bar{\mathbb{Q}}^{\dot\alpha}\right)\right]\,. \tag{2.4}$$

The special elements $g(b,0,0,0)$, $g(0,K,0,0)$ and $g(0,0,\epsilon,\bar\epsilon)$ are called space-time translations, Lorentz and supersymmetry transformations, respectively. In order to get the multiplication law of SΠ one needs only the (anti-)commutation relations of $\mathrm{S}\mathscr{P}_c(\Lambda_\infty)$ and the Campbell-Baker-Hausdorff formula.

2.2 Supersymmetric field theory

2.2.1 Superfields on $\mathbb{R}^{4|4}$

The Minkowski space-time, where ordinary (quantum) fields live on, can be defined as the coset space of the Poincaré group Π and the Lorentz group $SO(3,1)$: $\mathcal{M} \simeq \Pi/SO(3,1)$. $\mathbb{R}^{4|4}$ can be viewed in analogy to $\mathcal{M}$ as the coset space S$\Pi/SO(3,1|\mathbb{R}_c)$ where $SO(3,1|\mathbb{R}_c)$ is the Lorentz group over $\mathbb{R}_c$.

The super Poincaré group acts on $\mathbb{R}^{4|4}$ as

$$\begin{aligned} x'^\mu &= (\mathrm{e}^K)^\mu{}_\nu x^\nu + i(\theta^\alpha(\sigma^\mu)_{\alpha\dot\alpha}\bar\epsilon^{\dot\alpha} - \epsilon^\alpha(\sigma^\mu)_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}) + b^\mu\,, \\ \theta'^\alpha &= (\mathrm{e}^{-K})^\alpha{}_\beta\theta^\beta + \epsilon^\alpha\,, \\ \bar\theta'_{\dot\alpha} &= (\mathrm{e}^{\bar K})_{\dot\alpha}{}^{\dot\beta}\bar\theta_{\dot\beta} + \bar\epsilon_{\dot\alpha}\,. \end{aligned} \tag{2.5}$$

Scalar superfields are defined as superanalytic functions on $\mathbb{R}^{4|4}$. In this thesis it will be dealt only with bosonic or fermionic superfields namely

$$V : \mathbb{R}^{4|4} \to \mathbb{C}_c \ (\mathbb{R}_c) \,, \qquad V : \mathbb{R}^{4|4} \to \mathbb{C}_a \ (\mathbb{R}_a) \,. \tag{2.6}$$

Expanding $V(x,\theta,\bar\theta)$ in θ and $\bar\theta$ one gets the so called component fields and as $(\theta)^3 = (\bar\theta)^3 = 0$ this Taylor expansion is finite. For a real c-type V defined as in (2.6), the components are ordinary bosonic and fermionic fields over Minkowski space $\mathcal{M}$.

$$\begin{aligned} V(x,\theta,\bar\theta) &= A(x) + \theta^\alpha\psi_\alpha(x) + \bar\theta_{\dot\alpha}\bar\psi^{\dot\alpha}(x) + \theta^2 F(x) + \bar\theta^2 \bar F(x) \\ &\quad + \theta^\alpha(\sigma^\mu)_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}V_\mu + \bar\theta^2\theta^\alpha\lambda_\alpha(x) + \theta^2\bar\theta_{\dot\alpha}\bar\lambda^{\dot\alpha}(x) + \theta^2\bar\theta^2 D(x) \,. \end{aligned} \tag{2.7}$$

The covariant derivatives of global supersymmetry D_α and $\bar{\mathrm{D}}_{\dot\alpha}$ (see appendix A.4.1) provide a nice tool to write the component fields in terms of the superfield. One could do the same with ordinary derivatives with respect to θ and $\bar\theta$ but for convenience they are replaced by covariant derivatives.

$$\begin{aligned} A(x) &= V| \,, \qquad \psi_\alpha(x) = \mathrm{D}_\alpha V| \,, \qquad \bar\psi_{\dot\alpha}(x) = \bar{\mathrm{D}}_{\dot\alpha} V| \,, \\ F(x) &= -\frac{1}{4}\mathrm{D}^2 V| \,, \qquad \bar F(x) = -\frac{1}{4}\bar{\mathrm{D}}^2 V| \,, \qquad V_{\alpha\dot\alpha}(x) = \frac{1}{2}[\mathrm{D}_\alpha, \bar{\mathrm{D}}_{\dot\alpha}]V| \,, \\ \lambda_\alpha(x) &= -\frac{1}{4}\mathrm{D}_\alpha\bar{\mathrm{D}}^2 V| \,, \qquad \bar\lambda_{\dot\alpha}(x) = -\frac{1}{4}\bar{\mathrm{D}}_{\dot\alpha}\mathrm{D}^2 V| \,, \\ D(x) &= \frac{1}{32}\{\mathrm{D}^2, \bar{\mathrm{D}}^2\}V| \,. \end{aligned} \tag{2.8}$$

The generalization to tensor superfields is straightforward [20]. The representation of the super Poincaré group on superfields takes the form

$$\begin{aligned} \mathbb{P}_\mu &= -i\partial_\mu \,, \\ \mathbb{M}_{\mu\nu} &= i(x_\nu\partial_\mu - x_\mu\partial_\nu + (\sigma_{\mu\nu})^{\alpha\beta}\theta_\alpha\partial_\beta - (\tilde\sigma_{\mu\nu})^{\dot\alpha\dot\beta}\bar\theta_{\dot\alpha}\bar\partial_{\dot\beta}) \,, \\ \mathbb{Q}_\alpha &= i\partial_\alpha + (\sigma^\mu)_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu \,, \\ \bar{\mathbb{Q}}_{\dot\alpha} &= -i\bar\partial_{\dot\alpha} - \theta^\alpha(\sigma^\mu)_{\alpha\dot\alpha}\partial_\mu \,. \end{aligned} \tag{2.9}$$

With (2.9) we can get the infinitesimal transformation law for $i(\epsilon\mathbb{Q}+\bar\epsilon\bar{\mathbb{Q}})$ of the component fields (2.8). Here we can avoid a cumbersome calculation by noting that the supersymmetry transformation anticommutes with the covariant derivatives, i.e. $[\mathrm{D}_\alpha, \epsilon\mathbb{Q}+\bar\epsilon\bar{\mathbb{Q}}] = [\bar{\mathrm{D}}_{\dot\alpha}, \epsilon\mathbb{Q}+\bar\epsilon\bar{\mathbb{Q}}] = 0$.

$$\delta A(x) = -\epsilon\psi - \bar\epsilon\bar\psi \,,$$

$$\begin{aligned}
\delta\psi_\alpha &= -2\epsilon_\alpha F(x) - \bar{\epsilon}^{\dot{\alpha}} V_{\alpha\dot{\alpha}} - i\bar{\epsilon}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}} A(x)\,, \\
\delta F(x) &= -\bar{\epsilon}\bar{\lambda}(x)\,, \\
\delta V_{\alpha\dot{\alpha}} &= 2(\bar{\epsilon}_{\dot{\alpha}}\lambda_\alpha(x) - \epsilon_\alpha\bar{\lambda}_{\dot{\alpha}}(x)) - 2i(\epsilon_\alpha\partial_{\beta\dot{\alpha}}\psi^\beta(x) + \bar{\epsilon}_{\dot{\alpha}}\partial_{\alpha\dot{\beta}}\bar{\psi}^{\dot{\beta}}(x)) \\
&\quad + i\partial_{\alpha\dot{\alpha}}(\bar{\epsilon}\bar{\psi}(x) - \epsilon\psi(x))\,, \\
\delta\lambda_\alpha &= -2\epsilon_\alpha D(x) - i\epsilon_\alpha\partial_\mu V^\mu(x) - 2i\bar{\epsilon}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}} F(x)\,, \\
\delta D(x) &= -\frac{i}{2}\partial_{\alpha\dot{\alpha}}(\epsilon^\alpha\bar{\lambda}^{\dot{\alpha}}(x) + \bar{\epsilon}^{\dot{\alpha}}\lambda^\alpha(x))\,.
\end{aligned} \tag{2.10}$$

An important notion in supersymmetric field theories is that of chiral superfields. The representation on superfields introduced in (2.9) is not irreducible. Imposing constraints on the superfields leads to irreducible representations. The most common is the chiral and anti-chiral constraint

$$\begin{aligned}
\bar{\mathrm{D}}_{\dot{\alpha}}\Phi(x,\theta,\bar{\theta}) &= 0\,, \\
\mathrm{D}_\alpha\bar{\Phi}(x,\theta,\bar{\theta}) &= 0\,,
\end{aligned} \tag{2.11}$$

where D_α is the (supersymmetric) covariant derivative and $\bar{\Phi} \equiv \Phi^\dagger$. A superfield obeying (2.11) is called chiral or antichiral superfield and takes the form

$$\begin{aligned}
\Phi(x,\theta,\bar{\theta}) &= \exp(i\theta\sigma^\mu\bar{\theta}\partial_\mu)\Phi(x,\theta) \\
&= A(x) + \theta^\alpha\psi_\alpha + \theta^2 F(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu A(x) \\
&\quad + \frac{i}{2}\theta^2\bar{\theta}\tilde{\sigma}^\mu\partial_\mu\psi(x) + \frac{1}{4}\theta^2\bar{\theta}^2\Box A(x)\,,
\end{aligned} \tag{2.12}$$

where we used the expansion of $\Phi(x,\theta)$ in θ. The components of $\Phi(x,\theta)$ expressed through the covariant derivatives read

$$A(x) = \Phi|\,, \qquad \psi_\alpha(x) = \mathrm{D}_\alpha\Phi|\,, \qquad F(x) = -\frac{1}{4}\mathrm{D}^2\Phi|\,. \tag{2.13}$$

Using again the fact that the covariant derivative commutes with the supersymmetry generators, we find that the component fields of $\Phi(x,\theta,\bar{\theta})$ transform under the infinitesimal supersymmetry transformation $i(\epsilon\mathbb{Q} + \bar{\epsilon}\bar{\mathbb{Q}})$ as

$$\begin{aligned}
\delta A(x) &= -\epsilon\psi(x)\,, \\
\delta\psi_\alpha(x) &= -2\epsilon_\alpha F(x) - 2i\bar{\epsilon}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}} A(x)\,, \\
\delta F(x) &= -i\bar{\epsilon}\sigma^\mu\partial_\mu\psi(x)\,.
\end{aligned} \tag{2.14}$$

The same relations are obtained in complete analogy for the anti-chiral super field $\bar{\Phi}$.

2.2.2 Field theory of superfields

The starting point for a theory is a specific Lagrangian $\mathcal{L}$ (or Hamiltonian) which is a function of the dynamical variables of the theory. Through the action principle one obtains the (classical) equations of motion. In this master's thesis the following constraints are assumed to be fulfilled:

(i) **Locality:** $S = \int d^d x \, \mathcal{L}$

(ii) **Causality:** $\mathcal{L}$ contains no derivatives higher than second order

(iii) **Unitarity:** $S \in \mathbb{R}$

(iv) **Poincaré invariance:** There is no breaking of space-time symmetries

(v) **Renormalizability**

Of course the discussion of the principles of a (quantum) field theory is not concluded with the points raised above.

Note that when integrating the Lagrangian over the spacial dimensions we get

$$L = \int d^{d-1} x \, \mathcal{L} = T - V \,, \tag{2.15}$$

where T is the kinetic and V the potential energy[5]. The Hamiltonian then reads $\mathbb{H} = T + V$.

In order to write down a Lagrangian composed of (chiral) superfields we need a mathematical statement about transformation of integrals.

Theorem 2.1. *Let $\mathscr{L}(z)$ be a scalar and $\mathscr{L}_c(z)$ a chiral superfield on $\mathbb{R}^{4|4}$. Then the integrals*

$$I = \int d^8 z \, \mathscr{L}(z) \qquad \text{and} \qquad I_c = \int d^6 z \, \mathscr{L}_c(z) \tag{2.16}$$

are invariant under super Poincaré transformations.

Of course the theorem is true also for $(I_c)^\dagger = \bar{I}_c = \int d^6 \bar{z} \, \bar{\mathscr{L}}_c(z)$. For the proof one needs only to show that the Berezinian ("super Jacobi determinant", see appendix B) of a super Poincaré transformation on $\mathbb{R}^{4|4}$ is equal one. This is carefully done in [20]. In what follows I will use the notation $\mathscr{L}$ for a Lagrangian over $\mathbb{R}^{4|4}$ and $\mathcal{L}$ for a Lagrangian over space-time $\mathcal{M}$. $\mathscr{L}$ and $\mathcal{L}$ are related to each other by the following relation:

$$S = \int d^8 z \, \mathscr{L} + \int d^6 z \, \mathscr{L}_c + \int d^6 \bar{z} \, \bar{\mathscr{L}}_c = \int d^4 x \, \mathcal{L}$$

[5]More precisely: L is the Lagrangian and $\mathcal{L}$ is the Lagrangian density. However, I will also refer to $\mathcal{L}$ as Lagrangian.

$$\text{with} \qquad \bar{\mathrm{D}}_{\dot{\alpha}} \mathscr{L}_c \;=\; 0$$

$$\Rightarrow \;\mathcal{L} \;=\; \frac{1}{32}\{\bar{\mathrm{D}}^2, \mathrm{D}^2\}\mathscr{L}| - \frac{1}{4}\mathrm{D}^2\mathscr{L}_c| - \frac{1}{4}\bar{\mathrm{D}}^2\bar{\mathscr{L}}_c|\,. \tag{2.17}$$

2.2.3 Wess-Zumino model

To illustrate the general assumptions made in the previous section, the simplest renormalizable supersymmetric model shall be presented. The Wess-Zumino model is for supersymmetric field theories what ϕ^4 theory is for quantum field theories. In [9] J. Wess and B. Zumino presented a model with the following action:

$$S_{WZ} \;=\; \int d^8z\,\bar{\Phi}\Phi + \left\{\int d^6z\left[\frac{m}{2}\Phi^2 + \frac{\lambda}{3!}\Phi^3\right] + h.c.\right\} \tag{2.18}$$

$$\text{with} \qquad \bar{\mathrm{D}}_{\dot{\alpha}}\Phi \;=\; 0\,,$$

where m and λ are real parameters. In component form this action reads

$$\begin{aligned} S_{WZ} \;=\; & \int d^4x\left[A\Box A - \frac{i}{2}\psi^\alpha\partial_{\alpha\dot{\alpha}}\bar{\psi}^{\dot{\alpha}} + F\bar{F}\right] \\ & + \int d^4x\left[mFA - \frac{m}{4}\bar{\psi}\bar{\psi} + \frac{\lambda}{2}F|A|^2 + \frac{\lambda}{4}\bar{\psi}\bar{\psi}A + h.c.\right]\,, \end{aligned} \tag{2.19}$$

where for convenience I dropped the space-time argument. The component fields of Φ transform as shown in equation (2.14) but in order to see that S_{WZ} is supersymmetric it is more convenient to work with Φ. Just recall theorem 2.1 and note that $\bar{\Phi}\Phi$ is a scalar and $\frac{m}{2}\Phi^2 + \frac{\lambda}{3!}\Phi^3$ a chiral superfield. It is even possible to do perturbative calculation with superfields and go to the "physical" component fields only at the end of the day.

The above action does not depend on derivatives of F which means that I can eliminate F with its equation of motion. Such a field is then called "auxiliary". This procedure is often referred to as "going on-shell":

$$\bar{F} + mA + \frac{\lambda}{2}|A|^2 \;=\; 0\,. \tag{2.20}$$

Inserting equation (2.20) in (2.19) I get the on-shell action of the Wess-Zumino model:

$$S_{WZ}\Big|_{\text{on-shell}} \;=\; \int d^4x\left[\bar{A}(\Box - m^2)A - \frac{m\lambda}{2}(A+\bar{A})|A|^2 + \frac{\lambda^2}{2}|A|^4\right.$$

$$-\frac{i}{2}\psi^\alpha\partial_{\alpha\dot\alpha}\bar\psi^{\dot\alpha} - \frac{m}{4}\bar\psi\bar\psi - \frac{m}{4}\psi\psi + \frac{\lambda}{4}\bar\psi\bar\psi A + \frac{\lambda}{4}\psi\psi\bar A\Big] . \qquad (2.21)$$

Thus I end up with a model that describes a massive complex scalar field with quadratic and cubic self-interaction and a massive fermion field with Yukawa interaction with the scalar field.

2.3 Super Yang-Mills theory

Now I would like to construct a supersymmetric Yang-Mills theory, i.e. an action which has a Yang-Mills structure at component level. I do not discuss the problem of giving mass to a gauge particle but rather mass terms for (anti)chiral scalar superfields are incorporated.

2.3.1 Matter multiplets

The supersymmetric extension of a Yang-Mills theory can be constructed in complete analogy to the non-supersymmetric case. The gauge field V is introduced as a compensating field when one localizes a global symmetry of the action.

Let us again have a look at the Wess-Zumino model (2.18). The term $\int d^8z\,\bar\Phi\Phi$, which at component level gives the kinetic terms, is invariant under the global $U(1)$ symmetry

$$\Phi \to \Phi' \;=\; e^{iq\xi}\Phi \qquad \text{with} \qquad \xi \;=\; \xi^* , \qquad (2.22)$$

where q is the transformation charge. The parameter ξ does not preserve the chirality of the field Φ when it is "upgraded" to a real superfield $\xi(z) = \bar\xi(z)$ depending on the local coordinates:

$$\bar{\mathrm{D}}_{\dot\alpha}\Phi \;=\; 0 \qquad \text{but} \qquad \bar{\mathrm{D}}_{\dot\alpha}(e^{iq\xi(z)}\Phi) \;\neq\; 0 . \qquad (2.23)$$

Taking instead of a real superfield $\xi(z)$ a chiral superfield $\Lambda(z)$, the chirality of Φ' is preserved but we spoil the invariance of the term $\int d^8z\,\bar\Phi\Phi$ since now $\Lambda \neq \bar\Lambda$. We can save the invariance of $\int d^8z\,\bar\Phi\Phi$ if we introduce a new superfield V

$$\int d^8z\,\bar\Phi\Phi \to \int d^8z\,\bar\Phi e^{2qV}\Phi , \qquad (2.24)$$

which transforms as

$$V \to V' = V + \frac{i}{2}(\bar\Lambda(z) - \Lambda(z)) . \qquad (2.25)$$

This makes the new action of equation (2.24) invariant under the transformation $\Phi \rightarrow \Phi' = e^{iq\Lambda(z)}\Phi$. The field Φ is called a matter multiplet in analogy to the Wess-Zumino model (2.18), where the mass term of the action is of the form $\int d^6z\, \frac{m}{2}\Phi^2 + h.c.$. In fact if one wants to add gauge invariant matter terms to the pure super Yang-Mills theory, these terms have exactly this form. The next task is to find kinetic and self-interaction terms for the superfield V.

2.3.2 Non-Abelian gauge transformations

So far we only looked at an Abelian gauge transformation and how it can be localized. However, the focus of this thesis lies on the non-Abelian case which shall now be investigated more carefully.

The gauge group G is a compact and connected Lie group with a finite-dimensional unitary representation. The generators T^A, $A = 1...dim(G)$ are therefore hermitian and satisfy the commutation relation

$$[T^A, T^B] \;=\; if^{ABC}T^C\,, \tag{2.26}$$

where it is understood that a sum runs over repeated group indices. The generators T^A are $N \times N$ matrices where N is the dimension of the chosen representation of the gauge group. For convenience the matrix indices $(T^A)_m{}^n$ of the generators will not be written explicitly and I adopt the conventional normalization

$$\mathrm{Tr}\left[T^A T^B\right] \;=\; \delta^{AB}\,. \tag{2.27}$$

The role of the gauge field is played by a Lie-algebra valued real scalar superfield $V^A(z)$.

$$\begin{aligned} V^A(x,\theta,\bar\theta) \;=\;& A^A(x) + \theta^\alpha \psi_\alpha{}^A(x) + \bar\theta_{\dot\alpha}\bar\psi^{\dot\alpha A}(x) + \theta^2 F^A(x) + \bar\theta^2 F^A(x) \\ & \theta^\alpha(\sigma^\mu)_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}V_\mu{}^A + \bar\theta^2\theta^\alpha\lambda_\alpha{}^A(x) + \theta^2\bar\theta_{\dot\alpha}\bar\lambda^{\dot\alpha A}(x) + \theta^2\bar\theta^2 D^A(x)\,. \end{aligned} \tag{2.28}$$

For notational simplicity, I will accompany the Lie algebra valued fields with the generators of the gauge group and drop the coordinate argument

$$V \;\doteq\; V^A T^A\,, \qquad \Lambda \;\doteq\; \Lambda^A T^A\,, \quad \text{etc.} \tag{2.29}$$

As in section 2.3.1, I start from an action of the form $\int d^8z\, \bar\Phi_m \Phi^m$, where now a set of chiral fields Φ^m, $m = 1...N$ is taken instead of a single one. The dimension N of the generators T^A is now the same as the number of different Φ^m. The transformations of the matter multiplets and the gauge fields now read

$$\Phi'^m \;=\; (e^{i\Lambda})^m{}_n\Phi^n\,, \qquad \bar\Phi'_m \;=\; (e^{-i\bar\Lambda})_m{}^n\bar\Phi_n\,, \tag{2.30}$$

$$e^{2V'} \;=\; e^{i\bar{\Lambda}} e^{2V} e^{-i\Lambda}\,, \tag{2.31}$$

where in the first equation the indices of the matter multiplets where explicitly written. Note that equation (2.31) leads to (2.25) only in the case of an Abelian gauge group. The analogon to equation (2.25) for a non-Abelian group will be derived later on. Finally we get an action for the matter multiplets

$$S_{free}[\Phi, V] \;=\; \int d^8z\, \bar{\Phi}_n (e^{2V})^n{}_m \Phi^m \;=\; \int d^8z\, \bar{\Phi} e^{2V} \Phi\,, \tag{2.32}$$

which contains the massless part of Φ and $\bar{\Phi}$ as in the Wess-Zumino model (2.18) and an interaction term of the matter multiplets with the gauge field V. We can now add mass and interaction terms for the matter multiplets in complete analogy to equation (2.18):

$$S[\Phi, V] \;=\; \int d^8z\, \bar{\Phi} e^{2V} \Phi + \left\{ \int d^6z \left[\frac{1}{2} m_{mn} \Phi^m \Phi^n + \frac{1}{3!} \lambda_{lmn} \Phi^l \Phi^m \Phi^n \right] + h.c. \right\}. \tag{2.33}$$

What remains to construct is the pure super Yang-Mills term e.g. the free and self interacting action for V. But let us first have a closer look at the infinitesimal gauge transformation of V.

Infinitesimal gauge transformation

Equation (2.31) yields the infinitesimal transformation

$$\delta e^{2V} \;=\; e^{2(V+\delta V)} - e^{2V} \;=\; i\bar{\Lambda} e^{2V} - i e^{2V} \Lambda\,. \tag{2.34}$$

It is now a highly non trivial task to express δV in terms of V, Λ and $\bar{\Lambda}$. First we need an identity for two arbitrary operators $\mathbb{A}$, $\mathbb{B}$ and an infinitesimal parameter ϵ:

$$e^{\mathbb{A}+\epsilon\mathbb{B}} \;=\; e^{\mathbb{A}} \left[1 + \int_0^1 d\tau\, e^{-\tau\mathbb{A}}\, \epsilon\mathbb{B}\, e^{\tau\mathbb{A}} \right]. \tag{2.35}$$

Proof. Define $K(\tau, \epsilon) \doteq e^{-\tau\mathbb{A}} e^{\tau(\mathbb{A}+\epsilon\mathbb{B})}$ and solve the approximative differential equation for $K(\tau, \epsilon)$.

$$\begin{aligned} \frac{\mathrm{d}K(\tau,\epsilon)}{\mathrm{d}\tau} \;&=\; e^{-\tau\mathbb{A}}\, \epsilon\mathbb{B}\, e^{\tau(\mathbb{A}+\epsilon\mathbb{B})} \;\approx\; e^{-\tau\mathbb{A}}\, \epsilon\mathbb{B}\, e^{\tau\mathbb{A}} \\ \Rightarrow K(\tau, \epsilon) \;&=\; \int_0^\tau d\tau' e^{-\tau'\mathbb{A}}\, \epsilon\mathbb{B}\, e^{\tau'\mathbb{A}} + c \end{aligned} \tag{2.36}$$

$$\begin{aligned} K(\tau=0,\epsilon) &\doteq \mathbb{1} \\ \Rightarrow K(\tau,\epsilon)\big|_{\tau=1} &= e^{-\mathbb{A}}e^{\mathbb{A}+\epsilon\mathbb{B}} = 1+\int_0^1 d\tau\, e^{-\tau\mathbb{A}}\,\epsilon\mathbb{B}\, e^{\tau\mathbb{A}} \end{aligned} \quad (2.37)$$

□

We need the identity (2.35) in order to write (with $C_{\mathbb{A}}\cdot = [\mathbb{A}, .]$)

$$\begin{aligned} e^{\mathbb{A}+\epsilon\mathbb{B}} &= e^{\mathbb{A}}\left[\mathbb{1}+\int_0^1 d\tau\, e^{-\tau\mathbb{A}}\,\epsilon\mathbb{B}\, e^{\tau\mathbb{A}}\right] \\ &= e^{\mathbb{A}}\left[\mathbb{1}+\int_0^1 d\tau\, e^{-\tau C_{\mathbb{A}}}\,\epsilon\mathbb{B}\right] \\ &= e^{\mathbb{A}}\left[\mathbb{1}+(C_{\mathbb{A}})^{-1}\left(\mathbb{1}-e^{-C_{\mathbb{A}}}\right)\epsilon\mathbb{B}\right] , \end{aligned} \quad (2.38)$$

where we used the identity $e^{-\tau\mathbb{A}}\,\epsilon\mathbb{B}\, e^{\tau\mathbb{A}} = e^{-\tau C_{\mathbb{A}}}\epsilon\mathbb{B}$ which follows directly from the differential equation satisfied by $e^{-\tau\mathbb{A}}\,\epsilon\mathbb{B}\, e^{\tau\mathbb{A}}$ and $e^{-\tau C_{\mathbb{A}}}\epsilon\mathbb{B}$:

$$\frac{\mathrm{d}}{\mathrm{d}\tau} f_1(\tau) \doteq \frac{\mathrm{d}}{\mathrm{d}\tau} e^{-\tau\mathbb{A}}\,\epsilon\mathbb{B}\, e^{\tau\mathbb{A}} = -e^{-\tau\mathbb{A}}\,\mathbb{A}\epsilon\mathbb{B}\, e^{\tau\mathbb{A}} + e^{-\tau\mathbb{A}}\,\epsilon\mathbb{B}\mathbb{A}\, e^{\tau\mathbb{A}} = -C_{\mathbb{A}} f_1(\tau)\,, \quad (2.39)$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau} f_2(\tau) \doteq \frac{\mathrm{d}}{\mathrm{d}\tau} e^{-\tau C_{\mathbb{A}}}\epsilon\mathbb{B} = -C_{\mathbb{A}} e^{-\tau C_{\mathbb{A}}}\epsilon\mathbb{B} = -C_{\mathbb{A}} f_2(\tau)\,. \quad (2.40)$$

The two functions $e^{-\tau\mathbb{A}}\,\epsilon\mathbb{B}\, e^{\tau\mathbb{A}}$ and $e^{-\tau C_{\mathbb{A}}}\epsilon\mathbb{B}$ do further have the same initial condition namely $f_1(0) = f_2(0) = \epsilon\mathbb{B}$.

Now we can put together equation (2.34) and (2.38), identifying $\mathbb{A} \equiv 2V$ and $\epsilon\mathbb{B} \equiv 2\delta V$. When multiplying both sides by $C_V e^{C_V}$ we get

$$\begin{aligned} e^{2V}\left[1+(C_{2V})^{-1}\left(1-e^{2C_V}\right)2\delta V\right]-e^{2V} &= i\bar{\Lambda}e^{2V}-ie^{2V}\Lambda \\ e^{2V}\left[(C_V)^{-1}\left(1-e^{2C_V}\right)\delta V\right] &= e^{2V}\left[ie^{-2C_V}\bar{\Lambda}-i\Lambda\right] \\ \left(e^{C_V}-e^{-C_V}\right)\delta V &= iC_V\left(e^{-C_V}\bar{\Lambda}-e^{C_V}\Lambda\right) \\ 2\sinh(C_V)\delta V &= iC_V\left[\cosh(C_V)\left(\bar{\Lambda}-\Lambda\right)\right. \\ &\quad \left.-\sinh(C_V)\left(\bar{\Lambda}+\Lambda\right)\right] . \end{aligned} \quad (2.41)$$

The final result for δV now reads

$$\delta V = \frac{i}{2} C_V\left[\coth(C_V)\left(\bar{\Lambda}-\Lambda\right)-\bar{\Lambda}-\Lambda\right] . \quad (2.42)$$

The $\coth(C_V)$ can now be expanded to first order, resulting in

$$\delta V = \frac{i}{2}\left(\bar{\Lambda}-\Lambda\right) - \frac{i}{2}[V,\bar{\Lambda}+\Lambda] + \frac{i}{6}[V,[V,\bar{\Lambda}-\Lambda]] + \mathcal{O}(V^4)\,. \tag{2.43}$$

Obviously we recover equation (2.25) from (2.43) when the gauge group is Abelian.

Let me now stop for a moment the discussion of constructing a super Yang-Mills theory and say a few words about the gauge fixing that will be used throughout this thesis.

2.3.3 Wess-Zumino gauge

To start the discussion of the Wess-Zumino gauge fixing, we go back to an Abelian gauge transformation. Rewrite the chiral field $\Lambda(x,\theta,\bar{\theta})$ as was done in equation (2.12) with arbitrary complex scalar fields $u(x)$, $f(x)$ and spinor ρ_α:

$$\Lambda(x,\theta,\bar{\theta}) = e^{i\theta\sigma^\mu\bar{\theta}\partial_\mu}\big(u(x)+\theta^\alpha\rho_\alpha(x)+\theta^2 f(x)\big)\,. \tag{2.44}$$

The transformation (2.25) then reads

$$\begin{aligned}\delta V &= \frac{i}{2}\big(\bar{u}(x)-u(x)\big) + \frac{i}{2}\big(\bar{\theta}_{\dot{\alpha}}\bar{\rho}^{\dot{\alpha}}(x)-\theta^\alpha\rho_\alpha(x)\big)\\ &\quad + \frac{i}{2}\big(\bar{\theta}^2\bar{f}(x)-\theta^2 f(x)\big) + \frac{1}{2}\theta\sigma^\mu\bar{\theta}\partial_\mu(\bar{u}(x)-u(x)) + \ldots\end{aligned} \tag{2.45}$$

The dots indicate terms with space-time derivatives or at least third order in θ and $\bar{\theta}$. This shows us that the lowest components of V, i.e. A, ${\psi_\alpha}^A$ and F, suffer an arbitrary displacement by the transformation (2.25) and can therefore be gauged away. This means that in what follows we will set V to

$$V = \theta\sigma\bar{\theta}\,V_\mu + \bar{\theta}^2\theta^\alpha\lambda_\alpha + \theta^2\bar{\theta}_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}} + \theta^2\bar{\theta}^2 D\,, \tag{2.46}$$

which is equivalent to the constraint

$$V| = \mathrm{D}_\alpha V| = \mathrm{D}^2 V| = 0\,. \tag{2.47}$$

The components of equation (2.46) can be extracted from V as done in equation (2.8)

$$\begin{aligned}V_{\alpha\dot{\alpha}} &= \frac{1}{2}[\mathrm{D}_\alpha,\bar{\mathrm{D}}_{\dot{\alpha}}]V|\,, \qquad \lambda_\alpha = -\frac{1}{4}\mathrm{D}_\alpha\bar{\mathrm{D}}^2V|\,, \qquad \bar{\lambda}_{\dot{\alpha}} = -\frac{1}{4}\bar{\mathrm{D}}_{\dot{\alpha}}\mathrm{D}^2V|\,,\\ D &= \frac{1}{32}\{\mathrm{D}^2,\bar{\mathrm{D}}^2\}V|\,.\end{aligned} \tag{2.48}$$

However, the gauge fixation of equation (2.46) does not completely fix the transformation (2.25). There is still the freedom of the transformation

$$\Lambda_\xi \ = \ e^{i\theta\sigma^\mu\bar{\theta}\partial_\mu}\xi(x) \qquad \text{with} \qquad \xi \ = \ \xi^* \,. \tag{2.49}$$

which preserves the Wess-Zumino gauge. At component level this transformation reads

$$V'_\mu \ = \ V_\mu + \partial_\mu\xi \,, \qquad \lambda'_\alpha \ = \ \lambda_\alpha \,, \qquad D' \ = \ D \,. \tag{2.50}$$

Probably you have already noticed that the Wess-Zumino gauge is not supersymmetric. The supersymmetry transformation

$$\delta_\epsilon V = i(\epsilon\mathbb{Q} + \bar{\epsilon}\bar{\mathbb{Q}})\, V \tag{2.51}$$

generates, according to the transformation laws of equation (2.10), component fields ψ_α and F breaking supersymmetry:

$$\psi'_\alpha \ = \ -\bar{\epsilon}^{\dot{\alpha}}V_{\alpha\dot{\alpha}} \,, \qquad F' \ = \ -\epsilon\lambda \,. \tag{2.52}$$

One may redefine the supersymmetry transformation (2.51) by adding a gauge transformation which depends on ϵ such that the Wess-Zumino gauge becomes supersymmetric:

$$\tilde{\delta}_\epsilon V \ = \ i(\epsilon\mathbb{Q} + \bar{\epsilon}\bar{\mathbb{Q}})V + \frac{i}{2}(\bar{\Lambda}(\epsilon) - \Lambda(\epsilon)) \tag{2.53}$$

with

$$\Lambda(\epsilon) \ = \ e^{i\theta\sigma^\mu\bar{\theta}\partial_\mu}\big(2i\theta\sigma^\nu\bar{\epsilon}\, V_\nu + 2i\theta^2\bar{\epsilon}\bar{\lambda}\big) \,. \tag{2.54}$$

This means that despite the fact that the Wess-Zumino gauge is very useful when dealing with components of superfields, it is not the right choice when one wants to do covariant and explicitly supersymmetric superfield calculations. In this master's thesis I will mainly work with components and therefore the Wess-Zumino gauge fixing condition is used in what follows.

Although there exist covariant gauge fixing conditions such as the super Lorentz gauge, which is characterised by the linearity conditions

$$\mathrm{D}^2 V \ = \ \bar{\mathrm{D}}^2 V \ = \ 0 \,, \tag{2.55}$$

there is still a discussion going on on the topic of local supersymmetric gauges (see e.g. [21]).

Infinitesimal gauge transformations in the Wess-Zumino gauge

In analogy to equation (2.49) for the Abelian case, we find gauge transformations that preserve the Wess-Zumino gauge:

$$\Lambda_\xi \;=\; e^{i\theta\sigma^\mu\bar{\theta}\partial_\mu}\xi(x) \qquad \text{with} \qquad \xi \;\doteq \xi^A T^A \;=\; \xi^\dagger\,. \tag{2.56}$$

On V this transformation acts only as

$$\delta V \;=\; \frac{i}{2}\left(\bar{\Lambda}_\xi - \Lambda_\xi\right) - \frac{i}{2}[V, \bar{\Lambda}_\xi + \Lambda_\xi]\,, \tag{2.57}$$

since the higher commutator $[V,[V,\bar{\Lambda}-\Lambda]]$ vanishes. In components this transformation then reads

$$\begin{aligned} \delta V_\mu^A \;&=\; \partial_\mu \xi^A - f^{ABC}\xi^B V_\mu^C\,,\\ \delta\lambda_\alpha^A \;&=\; -f^{ABC}\xi^B\lambda_\alpha^C\,,\\ \delta D^A \;&=\; -f^{ABC}\xi^B D^C\,. \end{aligned} \tag{2.58}$$

In the transformation law for V_μ we encounter the transformation of an ordinary Yang-Mills field and the two other component fields transform in the adjoint representation of the gauge group. This means that we found a gauge transformation for superfields that at component level reproduces the transformation structure of a Yang-Mills theory. But still we do not have the the pure Yang-Mills Lagrangian.

2.3.4 Pure super Yang-Mills action

I do not know if there is a constructive way to get the pure super Yang-Mills action, thus I will just show that the proposed ansatz leads to the ordinary Yang-Mills structure at component level. Let us investigate the properties of the so called Yang-Mills superfield strengths

$$W_\alpha \;=\; -\frac{1}{8}\bar{\mathrm{D}}^2\left(e^{-2V}\mathrm{D}_\alpha e^{2V}\right)\,, \tag{2.59}$$

$$\bar{W}_{\dot\alpha} \;=\; \frac{1}{8}\mathrm{D}^2\left(e^{2V}\bar{\mathrm{D}}_{\dot\alpha}e^{-2V}\right)\,, \tag{2.60}$$

where W_α is a Lie algebra valued object $W_\alpha = W_\alpha^A T^A$ and $\bar{W}_{\dot\alpha} = (W_\alpha)^\dagger$. Using the relations of appendix A.4.1 it is obvious that W_α and $\bar{W}_{\dot\alpha}$ are chiral and antichiral, respectively. The superfield strengths do further transform covariantly under supergauge transformations (2.31):

$$
\begin{aligned}
W'_\alpha &= -\frac{1}{8}\bar{\mathrm{D}}^2\left[e^{i\Lambda}e^{-2V}e^{-i\bar\Lambda}\mathrm{D}_\alpha e^{i\bar\Lambda}e^{2V}e^{-i\Lambda}\right] \\
&= -\frac{1}{8}\bar{\mathrm{D}}^2\left[e^{i\Lambda}e^{-2V}e^{-i\bar\Lambda}e^{i\bar\Lambda}\left(i\mathrm{D}_\alpha\bar\Lambda\right)e^{2V}e^{-i\Lambda}+e^{i\Lambda}e^{-2V}\mathrm{D}_\alpha\left(e^{2V}e^{-i\Lambda}\right)\right] \\
&= -\frac{1}{8}\bar{\mathrm{D}}^2\left[e^{i\Lambda}e^{-2V}\left(\mathrm{D}_\alpha e^{2V}\right)e^{-i\Lambda}+e^{i\Lambda}\mathrm{D}_\alpha e^{-i\Lambda}\right] \\
&= -\frac{1}{8}\bar{\mathrm{D}}_{\dot\alpha}\left[e^{i\Lambda}\left(\bar{\mathrm{D}}^{\dot\alpha}e^{-2V}\mathrm{D}_\alpha e^{2V}\right)e^{-i\Lambda}+e^{i\Lambda}\bar{\mathrm{D}}^{\dot\alpha}\mathrm{D}_\alpha e^{-i\Lambda}\right] \\
&= -\frac{1}{8}e^{i\Lambda}\bar{\mathrm{D}}^2\left(e^{-2V}\mathrm{D}_\alpha e^{2V}\right)e^{-i\Lambda}-\frac{1}{8}e^{i\Lambda}\bar{\mathrm{D}}^2\mathrm{D}_\alpha e^{-i\Lambda} \\
&= e^{i\Lambda}W_\alpha e^{-i\Lambda}\,.
\end{aligned} \tag{2.61}
$$

Here relation (A.49) is used in order to get rid of the second term. Equation (2.61) tells us that $\mathrm{Tr}\left[W_\alpha W^\alpha\right]$ and $\mathrm{Tr}\left[\bar W^{\dot\alpha}\bar W_{\dot\alpha}\right]$ are gauge invariant. Therefore we can take the gauge invariant action

$$
S_{SYM}[V] = \frac{1}{4g^2}\int d^6z\,\mathrm{Tr}\left[W_\alpha W^\alpha\right]+h.c. \tag{2.62}
$$

that leads to a Yang-Mills structure at component level as we shall see in a moment. g is a coupling constant which will be absorbed in redefined component fields later on. Using the condition (2.47) W_α is rewritten as

$$
\begin{aligned}
W_\alpha &= -\frac{1}{8}\bar{\mathrm{D}}^2\left[e^{-2V}\mathrm{D}_\alpha e^{2V}\right] \\
&= -\frac{1}{8}\bar{\mathrm{D}}^2\left[(1-2V+2V^2)\mathrm{D}_\alpha(1+2V+2V^2)\right] \\
&= -\frac{1}{8}\bar{\mathrm{D}}^2\left[\left(1-2V+2V^2\right)\left(2\mathrm{D}_\alpha V+2(\mathrm{D}_\alpha V)V+2V(\mathrm{D}_\alpha)\right)\right] \\
&= -\frac{1}{8}\bar{\mathrm{D}}^2\Big[2\mathrm{D}_\alpha V+2(\mathrm{D}_\alpha V)V+2V(\mathrm{D}_\alpha)+2V\mathrm{D}_\alpha V-4V\mathrm{D}_\alpha V \\
&\quad -4V(\mathrm{D}_\alpha V)V-4V^2\mathrm{D}_\alpha V+4V^2\mathrm{D}_\alpha V+4V^2(\mathrm{D}_\alpha V)V+4V^3\mathrm{D}_\alpha V\Big] \\
&= -\frac{1}{8}\bar{\mathrm{D}}^2\Big[2\mathrm{D}_\alpha V-2[V,\mathrm{D}_\alpha V]-4V(\mathrm{D}_\alpha V)V+4V^2(\mathrm{D}_\alpha V)V\Big] \\
&= -\frac{1}{4}\bar{\mathrm{D}}^2\mathrm{D}_\alpha V+\frac{1}{4}\bar{\mathrm{D}}^2[V,\mathrm{D}_\alpha V]\,.
\end{aligned} \tag{2.63}
$$

In the calculation above there are terms that dropped out because of the Wess-Zumino gauge condition or because they are of $\mathcal{O}(\bar\theta^3)$ or higher. The form (2.63) is now more convenient for calculating the components of W_α. However, it is a rather cumbersome calculation and therefore it will not be carried out here. After some work, the nice result is that the components of W_α are

$$\begin{aligned} W_\alpha| &= \lambda_\alpha\,, \\ \mathrm{D}^\alpha W_\alpha| &= -4D\,, \\ \mathrm{D}_{[\alpha} W_{\beta]}| &= iG_{\alpha\beta}\,, \\ -\frac{1}{4}\mathrm{D}^2 W_\alpha| &= -i\mathrm{D}_{\alpha\dot\alpha}\bar\lambda^{\dot\alpha}\,. \end{aligned} \tag{2.64}$$

The Yang-Mills field strength $G_{\alpha\beta}$ and the Yang-Mills covariant derivative are defined as

$$G_{\mu\nu} = \partial_\mu V_\nu - \partial_\mu V_\nu - i[V_\mu, V_\nu]\,, \qquad G_{\alpha\beta} = (\sigma^{\mu\nu})_{\alpha\beta} G_{\mu\nu}\,, \tag{2.65}$$

$$\mathrm{D}_\mu \lambda^\alpha = \partial_\mu \lambda^\alpha - i[V_\mu, \lambda^\alpha]\,. \tag{2.66}$$

Thus we found a supersymmetric extension of a pure Yang-Mills theory:

$$\frac{1}{4g^2}\int d^6z\,\mathrm{Tr}\,[W_\alpha W^\alpha] = \frac{1}{g^2}\int d^4x\,\mathrm{Tr}\left[-\frac{1}{8}G^{\alpha\beta}G_{\alpha\beta} - \frac{1}{2}i\lambda^\alpha(\sigma^\mu)_{\alpha\dot\alpha}\mathrm{D}_\mu\bar\lambda^{\dot\alpha} + D^2\right]\,. \tag{2.67}$$

From the component form of $\frac{1}{4g^2}\int d^6z\,\mathrm{Tr}\,[W_\alpha W^\alpha]$ we can conclude that in the Wess-Zumino gauge we have $\int d^6z\,\mathrm{Tr}\,[W_\alpha W^\alpha] = \int d^6\bar z\,\mathrm{Tr}\left[\bar W^{\dot\alpha}\bar W_{\dot\alpha}\right]$, and due to gauge invariance of the expression above it holds also for a different gauge fixing i.e. the general case. In addition, I rescale the component fields as

$$V_\mu \to \frac{1}{g}V_\mu\,, \qquad \lambda_\alpha \to \frac{1}{g}\lambda_\alpha\,, \qquad D \to \frac{1}{g}D\,, \tag{2.68}$$

yielding the final expression for the super Yang-Mills action in component form

$$\boxed{S_{SYM}[V_\mu, \lambda_\alpha, D] = \int d^4x\,\mathrm{Tr}\left[-\frac{1}{4}G^{\mu\nu}G_{\mu\nu} - i\lambda^\alpha\mathrm{D}_{\alpha\dot\alpha}\bar\lambda^{\dot\alpha} + 2D^2\right]}\,, \tag{2.69}$$

with

$$\begin{aligned} G_{\mu\nu} &= \partial_\mu V_\nu - \partial_\mu V_\nu - ig[V_\mu, V_\nu]^{\;6}\,, \\ \mathrm{D}_{\alpha\dot\alpha}\bar\lambda^{\dot\alpha} &= (\sigma^\mu)_{\alpha\dot\alpha}\mathrm{D}_\mu\bar\lambda^{\dot\alpha} = (\sigma^\mu)_{\alpha\dot\alpha}\Big(\partial_\mu\bar\lambda^{\dot\alpha} - ig[V_\mu, \bar\lambda^{\dot\alpha}]\Big)\,. \end{aligned} \tag{2.70}$$

The auxiliary field D does not propagate and can be eliminated by its equation of motion. Please note the affinity of the above action to one flavor QCD in the chiral limit when taking $G \equiv SU(3)_c$. It is also important to see that due to the gauge fixing condition the action (2.69) is not explicitly supersymmetric. However, I will exclusively work with this action throughout my thesis. Later on I might also refer to (2.69) as the super Yang-Mills Lagrangian defined as $S_{SYM} \doteq \int d^4x \, \mathcal{L}_{SYM}$.

In order to get back to the starting point of our construction of the super Yang-Mills action, I lastly give the most general form of a super Yang-Mills action which couples to matter multiplets

$$S = \int d^8z \, \bar{\Phi} e^{2V} \Phi + \left\{ \frac{1}{4} \int d^6z \, \mathrm{Tr} \left[W^\alpha W_\alpha \right] + \int d^6z \, \mathscr{L}_c(\Phi_i) + h.c. \right\} , \quad (2.71)$$

where $\mathscr{L}_c(\Phi_i)$ is an arbitrary chiral function of the matter multiplets and has to be invariant under gauge transformations.

$$\mathscr{L}_c(\Phi_i) = \int d^6z \left[\frac{1}{2} m_{mn} \Phi^m \Phi^n + \frac{1}{3!} \lambda_{lmn} \Phi^l \Phi^m \Phi^n \right] \quad (2.72)$$

is an example for it.

2.3.5 Supercurrent

In order to construct the Noether current belonging to supersymmetry transformations, some knowledge about the Noether theorem is needed. In the literature you may find the notion "supercurrent" for a supersymmetry multiplet of currents but in this thesis I will call supercurrent the Noether current related to supersymmetry transformations.

Noether theorem

Theorem 2.2. *For any continuous symmetry transformation of a classical action there is a conserved current and a conserved charge.*

Proof. Let us take a general action S depending on a set of fields $\varphi_i(x)$. The index i runs over all fields present in the theory no matter if bosonic or fermionic (suppressing spinor indices) and the field variable x is arbitrary (space time coordinates, superspace, ...). According to locality, S can be written as an integral over x of the Lagrangian $\mathcal{L}$: $S = \int d^dx \, \mathcal{L}$. The infinitesimal transformation

[6] The term $-\frac{1}{4} G^{\mu\nu} G_{\mu\nu}$ can also be expressed in terms of the "electric" and "magnetic" field: $-\frac{1}{4} G^{\mu\nu} G_{\mu\nu} = \frac{1}{2} \left(\vec{E}^2 - \vec{B}^2 \right)$.

$$\begin{aligned}\varphi_i(x) \;\to\; \varphi_i'(x) &= \varphi_i(x) + \epsilon\,\delta\varphi_i(x) \\ \mathcal{L}(x) \;\to\; \mathcal{L}'(x) &= \mathcal{L}(x) + \epsilon\,\partial_\mu X^\mu \;=\; \mathcal{L}(x) + \epsilon\,\delta\mathcal{L}(x)\end{aligned} \tag{2.73}$$

leaves S invariant for an X^μ that vanishes at infinity as the premise of the theorem requires. The index μ here numerates all components of x. Varying the fields we can calculate $\delta\mathcal{L}$ (suppressing the x dependency):

$$\begin{aligned}\delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\varphi_i}\,\delta\varphi_i + \frac{\partial\mathcal{L}}{\partial\,\partial_\mu\varphi_i}\,\delta\,\partial_\mu\varphi_i \\ &= \left[\frac{\partial\mathcal{L}}{\partial\varphi_i} - \left(\partial_\mu\frac{\partial\mathcal{L}}{\partial\,\partial_\mu\varphi_i}\right)\right]\delta\varphi_i + \frac{\partial\mathcal{L}}{\partial\,\partial_\mu\varphi_i}\,\delta\,\partial_\mu\varphi_i + \left(\partial_\mu\frac{\partial\mathcal{L}}{\partial\,\partial_\mu\varphi_i}\right)\delta\varphi_i \\ &= \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\,\partial_\mu\varphi_i}\delta\varphi_i\right]\end{aligned} \tag{2.74}$$

With the definition $\delta\mathcal{L} = \partial_\mu X^\mu$ we get the result that the current j^μ is conserved

$$\partial_\mu j^\mu \;\doteq\; \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial\,\partial_\mu\varphi_i}\delta\varphi_i - X^\mu\right] \;=\; 0\,. \tag{2.75}$$

Integrating the "time" component of $j^\mu(x)$ over all coordinates but the time axis we get the conserved charge

$$Q \;=\; \int d^{d-1}x\; j_0(x)\,, \tag{2.76}$$

which proves the Noether theorem 2.2. □

Some important notes about the Noether theorem have to be added. The above proof relies crucially on the validity of the Euler-Lagrange equations, meaning that the constructed Noether current is only conserved modulo the equations of motion.

This leads to the problem of a quantized theory. In this case one can not be sure whether the equations of motion hold and therefore there could be a non vanishing term in the derivative of j^μ. This is then called an anomaly. The strategy for a proof of the existence of an anomaly is that one assumes the equations of motion to be valid and only when an inconsistency arises one is forced to abandon the the conservation of the Noether current $\partial_\mu j^\mu \neq 0$.

But let us turn back to the classical case and compute the Noether current belonging to the supersymmetry transformation in a super Yang-Mills theory.

Supercurrent

For the case of supersymmetry the parameter ϵ and therefore $\delta\phi_i$ and X_μ in (2.73) become spinors. Rewrite the transformation (2.73) as

$$\begin{aligned} \varphi_i \;\rightarrow\; \varphi_i' &= \varphi_i + \epsilon^\alpha\,\delta\varphi_{i\alpha} + \bar\epsilon_{\dot\alpha}\,\delta\bar\varphi_i^{\dot\alpha} \\ \mathcal{L}_{SYM}(x) \;\rightarrow\; \mathcal{L}'_{SYM}(x) &= \mathcal{L}_{SYM}(x) + \epsilon^\alpha\,\partial_\mu X^\mu{}_\alpha + \bar\epsilon_{\dot\alpha}\,\partial_\mu \bar X^{\mu\dot\alpha}\,, \end{aligned} \tag{2.77}$$

where φ_i now stands for the fields in $\mathcal{L}_{SYM}$ defined in equation (2.69) namely the gauge and gaugino fields V_μ and λ_α. It is also clear that the auxiliary field D of $\mathcal{L}_{SYM}$ does not contribute to the supercurrent since $\mathcal{L}_{SYM}$ does not contain a term with derivatives acting on D (see also the first remark to the Noether theorem).

With the transformation properties of the fields V_μ and λ_α given in equation (2.10) and collecting the terms in ε, the Noether current $S_{\mu\alpha}$ reads

$$\boxed{S^\mu{}_\alpha \;=\; -2(\sigma_\nu)_{\alpha\dot\alpha}\mathrm{Tr}\left[G^{\mu\nu}\bar\lambda^{\dot\alpha}\right]}\quad . \tag{2.78}$$

The $\bar\varepsilon$-terms give the the hermitian conjugated supercurrent $\bar{\mathcal{S}}^\mu{}_{\dot\alpha} = (\mathcal{S}^\mu{}_\alpha)^\dagger$. This current and a possible anomaly of it is the main subject of this work.

Chapter 3

Spontaneous Supersymmetry Breaking of N=1 Super Yang-Mills Theories

In this Chapter I try to answer the question whether in the pure N=1 Super Yang-Mills theory a spontaneous breaking of supersymmetry occurs. This question has already been discussed in the early eighties by [22] and [23]. I allow myself to doubt the answer given by E. Witten, G. Veneziano and S. Yankielovicz and follow the arguments initiated by G. Shore and carried out by P. Minkowski and his collaborators.

3.1 Remarks on spontaneous supersymmetry breaking

As for every symmetry of a field theory one talks of a "spontaneous symmetry breaking" when the vacuum state is not invariant under the symmetry transformation but the action still is. Let again T^A be the generators of some symmetry group and α^A the corresponding local or global transformation parameters. Having a non-invariant vacuum state means

$$e^{i\alpha^A T^A} |0\rangle \neq |0\rangle \qquad \Longleftrightarrow \qquad T^A |0\rangle \neq 0\,. \tag{3.1}$$

Thus one has to check if the vacuum is annihilated by the symmetry generators or not. Unfortunately there is no general recipe for dealing with a system where a spontaneous symmetry breaking occurs. One runs into trouble because of the break down of the symmetry algebra and some objects that were well-defined in the case of unbroken symmetry lose their mathematical fundament.

Let us have a look at some general properties of supersymmetry breaking. The supersymmetry generators are the conserved charges of the Noether theorem 2.2 and are therefore defined as

$$\mathbb{Q}_\alpha = \int d^3x\, \mathcal{S}_{0\alpha}(x)\,, \tag{3.2}$$

where $\mathcal{S}_{\mu\alpha}$ is the supercurrent defined in equation (2.78). As is shown later in this chapter (section 3.3.1), $\mathbb{Q}_\alpha$ is ill-defined when the corresponding symmetry is spontaneously broken. However, the (anti-) commutator of the supersymmetry generator with an operator $\mathbb{O}(y)$ exists for space-like separated regions:

$$[\mathbb{Q}_\alpha, \mathbb{O}(y)\} = \int d^3x\, [\mathcal{S}_{0\alpha}(x), \mathbb{O}(y)\}\,. \tag{3.3}$$

Thus the criterion for a spontaneous supersymmetry breaking is

$$\langle 0|\, [\mathbb{Q}_\alpha, \mathbb{O}(y)\}\, |0\rangle \neq 0\,. \tag{3.4}$$

I will base the analysis of this chapter on the energy density ε:

$$\langle 0|\, \Theta_{\mu\nu}\, |0\rangle = -\varepsilon g_{\mu\nu}\,. \tag{3.5}$$

The energy momentum tensor is denoted as $\Theta_{\mu\nu}$. The crucial point here is that we can write $\Theta_{\mu\nu}$ as

$$\Theta_{\mu\nu} = -\frac{1}{4}(\tilde{\sigma}_\mu)^{\alpha\dot{\alpha}}\left\{\mathbb{Q}_\alpha, \bar{\mathcal{S}}_{\nu\dot{\alpha}}\right\}\,. \tag{3.6}$$

This relation can be seen easily from the relation (2.2) of the Poincaré superalgebra:

$$\begin{aligned}\mathbb{P}^\mu &= -\frac{1}{4}(\tilde{\sigma}^\mu)^{\dot{\alpha}\alpha}\left\{\mathbb{Q}_\alpha, \bar{\mathbb{Q}}_{\dot{\alpha}}\right\}\\ &= -\frac{1}{4}(\tilde{\sigma}^\mu)^{\dot{\alpha}\alpha}\int d^3x\, \left\{\mathbb{Q}_\alpha, \bar{\mathcal{S}}_{0\dot{\alpha}}\right\}\,.\end{aligned} \tag{3.7}$$

Recalling the definition of the four momentum $\mathbb{P}^\mu \doteq \int d^3x\, \Theta^{0\mu}$ gives equation (3.6). Note that further investigation is needed for the generalization from $\Theta^{\mu 0}$ to $\Theta^{\mu\nu}$. Hence the energy-momentum tensor $\Theta^{\mu\nu}$ can be written in the form of equation (3.4) which means that it can be taken as an order parameter for the spontaneous breakdown of supersymmetry.

For further reading about supersymmetry breaking see e.g. [1, 24] or a textbook like [25].

3.2 Veneziano-Yankelowicz effective action

This subject has already been discussed in great detail in [4, 26–28] thus only a brief sketch of the problems shall be given here.

In [23] the two authors constructed a low-energy effective action for the N=1 super Yang Mills theory, meaning that they did not take the fundamental fields as basic degrees of freedom (as has been done by H. Leutwyler and J. Gasser with pion fields for QCD). Please do not confuse the quantum effective action with an effective theory. When talking about an effective theory one usually means a low-energy approximation of a fundamental theory. The "effective" in the quantum effective action has nothing to do with this. Veneziano and Yankielowicz start off with the anomaly multiplet[7]

$$\begin{aligned} \phi &= \frac{1}{8}\,\mathrm{Tr}\,[W^\alpha W_\alpha] \\ &= \varphi + \theta^\alpha\,\psi_\alpha + \theta^2 F\,, \end{aligned} \tag{3.8}$$

where φ is the gaugino condensate, ψ_α the would-be goldstino and F the classical Lagrangian. It is not discussed here why a low-energy description can be obtained from ϕ (see [4, 26, 28] and the references therein). As can be seen in appendix D, one cannot in general calculate an explicit form of the effective action. Hence one makes the following (local) ansatz

$$\Gamma[\Phi,\bar\Phi] \;=\; \int d^4x \left[\int d^4\theta\; K(\Phi,\bar\Phi) - \left(\int d^2\theta\; W(\Phi) + \int d^2\bar\theta\; W(\bar\Phi)\right)\right] + \mathcal{O}(\partial_\mu)\,. \tag{3.9}$$

The term $\mathcal{O}(\partial_\mu)$ designates terms with space-time derivatives. From the construction of $\Gamma[\Phi,\bar\Phi]$ it is clear that it can not be a local form and therefore the expression above is to be understood as an approximation. Please do not confuse the classical field Φ i.e. the vacuum expectation value of ϕ with its associated operator field ϕ (see appendix D for the general construction of Γ). Getting through all constraints of symmetries and anomaly cancellation, which is done very carefully in [28], the Veneziano-Yankielowicz effective action reads

$$\Gamma_{VY} \;=\; \int d^4x \left\{\int d^4\theta\; 9a(\bar\Phi\Phi)^{\frac{1}{3}} - \left[\int d^2\theta\; \frac{2\beta(g)}{9C(G)g^3}\left(\Phi\log\frac{z\Phi}{\Lambda^3} - \Phi\right) + h.c.\right]\right\}, \tag{3.10}$$

which is again a local approximation of the "true" effective action. The dimensionless constants a, z and the renormalized coupling constant g of the fundamental theory remain undetermined. From Γ_{VY} one deduces in a cumbersome but straightforward calculation that the potential for the component F takes the form

[7]This multiplet is also known as the Lagrangian or glue-ball multiplet.

$$V = -g_{\varphi\bar{\varphi}} F\bar{F} + cF + \bar{c}\bar{F} \,, \tag{3.11}$$

where $\bar{F}$ is the hermitian conjugate of F, i.e. $\bar{F} = F^\dagger$. $g_{\varphi\bar{\varphi}}$ is the same coefficient as the kinetic term for φ and therefore has to be positive and c is some function of φ and ψ_α that will not be important for the following arguments. The form (3.11) means that V is not bounded from below and is thus unphysical. Veneziano and Yankelovicz saved the day declaring F to be an auxiliary field. In (3.10) there is no term with a spacetime derivative acting on F, allowing one to eliminate it with its equation of motion. But this is where the problems arise: Γ_{VY} is a low-energy approximation, but declaring F as auxiliary is an exact statement about the full theory [26]. Φ is a real supersymmetry multiplet meaning that by introducing derivatives for φ or ψ_α there will automatically appear derivatives acting on F. The fact that such derivatives are present at some higher order makes it impossible to declare F as an auxiliary field. But if F is not auxiliary we have a real problem with the potential V being unbounded from below. Hence Γ_{VY} can impossibly be meaningful as a low-energy approximation for the effective action of super Yang-Mills theory.

Taking a more general ansatz for Γ as done in [26, 28] including spacetime derivatives of F, different dynamics than the one of Veneziano and Yankielowicz arise. The conclusion of conserved supersymmetry is not reliable anymore. In fact the opposite result is obtained from more accurate investigations [26, 27, 29, 30].

3.3 Witten index

The supersymmetry algebra (2.2) contains the anticommutator $\{\mathbb{Q}_\alpha, \bar{\mathbb{Q}}_{\dot\alpha}\} = 2\,(\sigma^\mu)_{\alpha\dot\alpha}\,\mathbb{P}_\mu$ which implies that the Hamiltonian is

$$\begin{aligned} \mathbb{H} &\doteq -\mathbb{P}_0 \\ &= \frac{1}{4}\,(\tilde\sigma_0)^{\dot\alpha\alpha}\,\{\mathbb{Q}_\alpha, \bar{\mathbb{Q}}_{\dot\alpha}\} \\ &= \frac{1}{4}\,\left(\mathbb{Q}_1\bar{\mathbb{Q}}_{\dot1} + \mathbb{Q}_2\bar{\mathbb{Q}}_{\dot2} + \bar{\mathbb{Q}}_{\dot1}\mathbb{Q}_1 + \bar{\mathbb{Q}}_{\dot2}\mathbb{Q}_2\right) \end{aligned} \tag{3.12}$$

This means that for any unitary representation I get a positive energy representation. Note that for the (non-super) Poincaré algebra there are principally positive and negative energy unitary representations but only the positive ones are physically meaningful.

$$\begin{aligned} \langle\psi|\,\mathbb{H}\,|\psi\rangle &= \frac{1}{4}\,\left(\langle\psi|\,\mathbb{Q}_1\bar{\mathbb{Q}}_{\dot1}\,|\psi\rangle + \langle\psi|\,\mathbb{Q}_2\bar{\mathbb{Q}}_{\dot2}\,|\psi\rangle + \langle\psi|\,\bar{\mathbb{Q}}_{\dot1}\mathbb{Q}_1\,|\psi\rangle + \langle\psi|\,\bar{\mathbb{Q}}_{\dot2}\mathbb{Q}_2\,|\psi\rangle\right) \\ &= \frac{1}{4}\,\left(\|\bar{\mathbb{Q}}_{\dot1}\,|\psi\rangle\,\|^2 + \|\bar{\mathbb{Q}}_{\dot2}\,|\psi\rangle\,\|^2 + \|\mathbb{Q}_1\,|\psi\rangle\,\|^2 + \|\mathbb{Q}_2\,|\psi\rangle\,\|^2\right) \\ &\geq 0\,, \end{aligned} \tag{3.13}$$

where $|\psi\rangle$ is an arbitrary state of the Hilbert space of the representation.

To construct the Witten Index I_W two observations are needed: If supersymmetry is not spontaneously broken, i.e. $\mathbb{Q}_\alpha |0\rangle = 0$, there exists at least one state with zero energy. But if $\mathbb{Q}_\alpha |0\rangle \neq 0$ there is no state for which the matrix element (3.13) vanishes. The second observation is that in supersymmetric theories every state $|\psi\rangle$ with non-zero energy E_ψ is paired with a state $|\psi'\rangle$ obeying opposite statistics. $|\psi'\rangle$ is the supersymmetric partner of $|\psi\rangle$ with energy E_ψ and it exists also if supersymmetry is spontaneously broken.

The crucial point is that the pairing applies only for states with non-zero energy. For zero energy states the supersymmetry transformation that maps bosons into fermions is ill defined and the uniqueness of the pairing $|\psi\rangle$ and $|\psi'\rangle$ is spoiled. This means that zero energy states do not have a supersymmetric partner whereas states with non-zero energy do.

The Witten index I_W is now defined as the difference of zero energy bosonic states and zero energy fermionic states.

$$I_W \doteq n_b^{E=0} - n_f^{E=0} \tag{3.14}$$

$I_W \neq 0$ implies that there are always states with zero energy because such a state can only gain energy through varying the parameters of the theory if it gets also a supersymmetric partner. But this means that I_W does not change its value. Hence:

If $I_W \neq 0$ supersymmetry is not spontaneously broken.

Note that if $I_W = 0$ there is no conclusion of this kind because the two cases $n_b^{E=0} = n_f^{E=0} = 0$ and $n_b^{E=0} = n_f^{E=0} \neq 0$ are not distinguishable.

In [22] I_W is then identified with the trace of the operator of 2π rotations $\mathrm{Tr}\left[(-1)^F\right] = \mathrm{Tr}\left[\exp(2\pi i J_z)\right]$ in order to calculate it. The calculations in [22] and the correction thereof in the appendix of [31] lead to the result that for any compact simple Lie group taken as gauge group in super Yang-Mills theories I_W does not vanish and therefore supersymmetry is not spontaneously broken.

3.3.1 Discussion of I_W

Mathematical Background

First of all we need to clarify some mathematical definitions and statements about unbounded operators acting on a Hilbert space $\mathcal{H}$. A detailed and pedagogical introduction to the subject well suited for physicists is found in [32]. In this book you will find further references for a deeper insight and for the proofs that are omitted here.

Let $\mathbb{T} : \mathcal{D} \to \mathcal{H}$ be a linear operator acting on a Hilbert space $\mathcal{H}$. It is assumed that $\mathcal{H}$ is infinite dimensional. $\mathcal{D} \subseteq \mathcal{H}$ is the domain of $\mathbb{T}$.

Definition 3.1. $\mathbb{T}$ *is a bounded operator if it has a finite norm.*

$$\|\mathbb{T}\| \doteq max\left\{ \frac{\|\mathbb{T}\,|u\rangle\,\|}{\|\,|u\rangle\,\|}\,;\,|u\rangle \neq 0\,,\ |u\rangle \in \mathcal{H} \right\} \,<\, \infty$$

Definition 3.2. $\{|u_n\rangle\} \subset \mathcal{D}$ *and* $\{\mathbb{T}\,|u_n\rangle\}$ *are both convergent series in* $\mathcal{H}$*.*

$$\lim_{n\to\infty} |u_n\rangle = |u\rangle \qquad\qquad \lim_{n\to\infty} \mathbb{T}\,|u_n\rangle = |v\rangle$$

$\mathbb{T}$ *is called **closed** if* $|v\rangle \in \mathcal{D}$ *and* $\mathbb{T}\,|u\rangle = |v\rangle$*.*

The following theorem shows the necessity of restricting the domain of an unbounded operator to a true subspace of the Hilbert space $\mathcal{H}$. Acting with an unbounded operator on an arbitrary state, in general one falls out of the Hilbert space i.e. the new state is not defined anymore.

Theorem 3.1. *A closed linear operator* $\mathbb{T}$ *in* $\mathcal{H}$ *that is defined at every point of* $\mathcal{H}$ *(*$\mathcal{D}(\mathbb{T}) \equiv \mathcal{H}$*) is bounded.*

As is known, the hermitian conjugate of an operator in a Hilbert space is defined as $\left\langle \mathbb{T}^\dagger u | v \right\rangle = \langle u | \mathbb{T} v\rangle$.

Definition 3.3. *The operator* $\mathbb{T}$ *is called **hermitian** if*

$$\mathcal{D}(\mathbb{T}) \subset \mathcal{D}(\mathbb{T}^\dagger) \qquad \text{and} \qquad \mathbb{T}^\dagger\,|u\rangle = \mathbb{T}\,|u\rangle$$

Definition 3.4. $\mathbb{T}$ *is called **self-adjoint** if*

$$\mathcal{D}(\mathbb{T}) \equiv \mathcal{D}(\mathbb{T}^\dagger) \qquad \text{and} \qquad \mathbb{T}^\dagger\,|u\rangle = \mathbb{T}\,|u\rangle$$

Please note the subtle difference between "hermitian" and "self-adjoint". This difference is only important for unbounded operators since for bounded ones $\mathcal{D} \equiv \mathcal{H}$. The basis of the present discussion of the Witten index will be the falling out of $\mathcal{H}$ when defining an unbounded operator on the whole Hilbert space.

In some applications one can avoid the problems mentioned above if $\mathcal{D}(\mathbb{T})$ is dense in $\mathcal{H}$. Think about the Fourier series of an $L^2(a,b)$ function. The derivative is an unbounded operator on this Hilbert space. To see this, check the definition of the operator norm with the function $f(x) = \sqrt{x-a}$. The domain of the derivative is now the subspace spanned by $\{\exp(2\pi i n x/L)\}$ with $L = b - a$ which is dense in $\mathcal{H}$.

Problems with I_W

First I show that $\mathbb{Q}_\alpha$ is not a bounded operator. The supersymmetry generators $\mathbb{Q}_\alpha$ can be written as a spatial integral over the corresponding supercurrent $\mathcal{S}_{\mu\,\alpha}$:

$$\mathbb{Q}_\alpha \;=\; \int d^3x\; \mathcal{S}_{0\,\alpha}(x)\,. \tag{3.15}$$

Equation (3.15) and the translation invariance of $\langle 0|\,\mathbb{Q}_\alpha\,|0\rangle$ imply that for $\mathbb{Q}_\alpha\,|0\rangle \neq 0$ the local current operator does not vanish i.e. $\mathcal{S}_{0\,\alpha}\,|0\rangle \neq 0$. But if we try to calculate the norm of $\mathbb{Q}_\alpha\,|0\rangle$ we run into trouble:

$$\begin{aligned}\langle 0|\,\bar{\mathbb{Q}}_{\dot\alpha}\mathbb{Q}_\alpha\,|0\rangle &= \int d^3x\ \langle 0|\,\bar{\mathbb{Q}}_{\dot\alpha}\,\mathcal{S}_{0\,\alpha}(x)\,|0\rangle \\ &= \int d^3x\ \langle 0|\,\bar{\mathbb{Q}}_{\dot\alpha}\,\mathcal{S}_{0\,\alpha}(x^0,\vec{x}=0)\,|0\rangle\ ^8 \\ &\propto \int d^3x\,. \end{aligned} \tag{3.16}$$

Thus $\mathbb{Q}_\alpha$ is an unbounded operator and equation (3.13) is not well defined.

Conclusion Take $|\varphi\rangle \in \mathcal{H}$ such that $\mathbb{Q}_\alpha\,|\varphi\rangle \notin \mathcal{H}$. Since $\mathbb{Q}_\alpha$ is unbounded, such states do exist. $|\varphi\rangle$ has no supersymmetric partner but not necessarily zero energy. This makes the conclusion that if $I_W \neq 0$ supersymmetry is unbroken impossible, since in the derivation of I_W it was crucial that there are no such states. Thus Witten's argument for the non-breaking of supersymmetry in super Yang-Mills theories breaks down.

An alternative argument was found by B. Scheuner. In [33] it is nicely shown that one cannot calculate I_W in a finite volume, as done in [22, 31], and then take the infinite volume limit assuming that no field develops an infinite vacuum expectation value. This assumption is based on the argument that in this limit the theory does not depend on the boundary conditions of the finite volume anymore. The Schwinger model provides an example where this is not true.

3.4 Anomalous supersymmetry breaking

In this section I want to summarize the arguments of [27]. My analysis will be based on the possible values of the energy density ε as discussed in section 3.1.

3.4.1 Goldstino

In [34] it has been shown that when a spontaneous breaking of supersymmetry occurs a particle with spin 1/2 and zero mass emerges. This particle is called, in analogy to the bosonic case, goldstino or Goldstone fermion. It is important to note that the premises for a goldstino are the same as for an ordinary Goldstone boson. The presence of a goldstino implies an interesting and unexpected property of ε which is the main subject of this section.

[8] The matrix element inside the integral must be translationally (only for spacial translations) invariant and non-vanishing. I can therefore evaluate it at $\vec{x} = 0$ which means that the integrand is constant and non-zero.

As seen in section 3.1, the supersymmetry algebra allows one to express the energy momentum tensor $\Theta_{\mu\nu}$ in terms of supercharges and the supercurrent. Here I state again equation (3.6):

$$\Theta_{\mu\nu} \;=\; -\frac{1}{4}(\tilde{\sigma}_\mu)^{\dot{\alpha}\alpha}\left\{\mathbb{Q}_\alpha, \bar{\mathcal{S}}_{\nu\dot{\alpha}}\right\} . \tag{3.17}$$

The goldstino decay constant f_g is defined as

$$\langle 0|\, \mathcal{S}_{\mu\alpha} \left|\bar{\psi}\right\rangle = f_g \,(\sigma_\mu)_{\alpha\dot{\beta}}\, \bar{\psi}^{\dot{\beta}} , \tag{3.18}$$

where $\left|\bar{\psi}\right\rangle$ is a one fermion state. Inserting relation (3.6) in $\langle 0|\,\Theta_{\mu\nu}\,|0\rangle = -\varepsilon g_{\mu\nu}$ and using the definition of f_g, we get the important result:

$$\varepsilon \;=\; \frac{1}{4}\,|f_g|^2 . \tag{3.19}$$

Thus we found that in the presence of a goldstino the vacuum energy density is $\varepsilon \geq 0$!

3.4.2 Trace anomaly

The trace anomaly is a well known phenomenon in gauge theories and signals the breakdown of scale invariance at quantum level. It can be treated in complete analogy to the axial anomaly which is the prototype of an anomaly (see section 4.1). For gauge theories the problem of the trace anomaly was discussed in the seventies in [35, 36] and many others. The trace anomaly takes the form

$$-\,\theta^{\mu}{}_{\mu} \;=\; -\frac{2\beta(g)}{g}\,\mathcal{L}_{SYM} . \tag{3.20}$$

This means that the energy density is then given by

$$\varepsilon \;=\; -\frac{1}{4}\,\langle 0|\,\theta^{\mu}{}_{\mu}\,|0\rangle \;=\; -\frac{\beta(g)}{2g}\,\langle 0|\,\mathrm{Tr}\left[-\frac{1}{4}G^{\mu\nu}G_{\mu\nu} - i\lambda^\alpha \mathrm{D}_{\alpha\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}} + 2D^2\right]|0\rangle . \tag{3.21}$$

If the term of the auxiliary field D gets a vacuum expectation value different from zero it becomes physical and would therefore not be auxiliary anymore. The gaugino term $\langle 0|\,\lambda^\alpha \mathrm{D}_{\alpha\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}\,|0\rangle$ is also equal to zero by analogy to QCD[9] and therefore we are left with

$$\varepsilon \;=\; -\frac{\beta(g)}{2g}\,\langle 0|\,\mathrm{Tr}\left[-\frac{1}{4}G^{\mu\nu}G_{\mu\nu}\right]|0\rangle . \tag{3.22}$$

[9]Please note that also in QCD this fact is not strictly derivable.

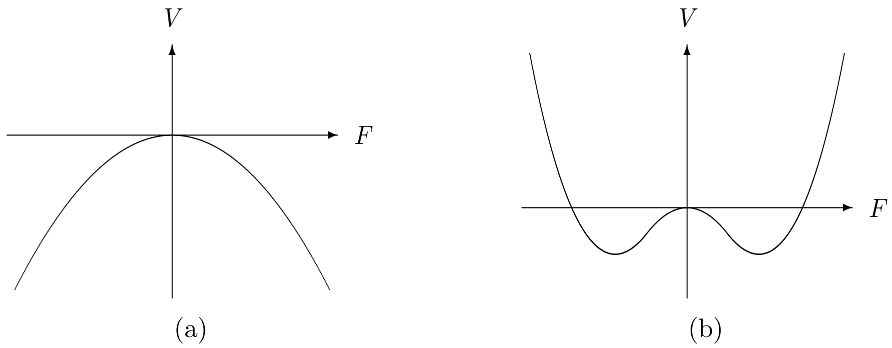


Figure 3.1: The potential of the "auxiliary" field.

Assuming now that the N=1 super Yang-Mills theory features a negative β-function as is found in QCD, we need to have a negative sign of the energy density:

$$\varepsilon \leq 0\,. \tag{3.23}$$

3.4.3 Excluding $\varepsilon = 0$

The case of $\varepsilon = 0$ implies the conservation of supersymmetry. It is interesting to note that even in the form of equation (3.10) the effective action would imply a supersymmetry breaking.

The leading term in the potential of F is proportional to $-\bar{F}F$ as seen in (3.11) and displayed in figure 3.1(a). For the investigation of the stability of the theory the linear terms in equation (3.11) can be neglected. The unstable maximum is at the supersymmetry preserving point $F = F_0$ and the only way to get a stable theory is to add higher order terms to Γ_{VY} in order to achieve a minimum somewhere. Such terms may be suppressed by some coefficient $\propto 1/\Lambda^n_{cut}$ when assuming that Γ_{VY} is the complete approximation up to some order. In figure 3.1(b) the situation is sketched for a higher order term of the form $+(F\bar{F})^2$ and there are now two stable minima that determine the vacuum state of the theory.

It is obvious that by adding higher order terms proportional to $(F\bar{F})^n$, $n > 2$ one can never turn the maximum of $F = F_0$ into a minimum. All possible minima of the corrected Γ_{VY} have to be at $F \neq F_0$ which means that supersymmetry is not conserved anymore.

Anyway, when investigating a more appropriate effective action than Γ_{VY}, the conclusion that supersymmetry is spontaneously broken is evident. This has been discussed in detail in [26, 27, 29, 30, 37]. In [37] it is shown that with a breakdown of chiral symmetry (as in QCD) we get a supersymmetry breaking condensate.

3.4.4 Conclusions

What do we draw out of this pile of shards I have just produced? The goldstino implies $\varepsilon \geq 0$ but the trace anomaly forces $\varepsilon \leq 0$. Thus one would think that $\varepsilon = 0$ and supersymmetry is unbroken as can easily be seen from equation (3.6). But since we excluded the latter case we have a serious problem. The only point where we could start thinking about an inconsistency is in section 3.4.1. In the derivation of equation (3.19) I relied on the fact that the super current $\mathcal{S}_{\mu\alpha}$ is conserved. If $\mathcal{S}_{\mu\alpha}$ would suffer an anomaly the arguments of section 3.4.1 would break down and we could conclude that supersymmetry is spontaneously broken, $\varepsilon < 0$, the supercurrent is anomalous and that there is no goldstino. In chapter 4 I will investigate the anomaly of the supercurrent in a heuristic way.

Chapter 4

Supercurrent Anomaly

In this chapter the anomaly of the supercurrent is investigated. Unfortunately I am not able to give a direct proof of the anomaly but I can qualitatively guess its structure and exclude a contribution of the triangle graphs.

4.1 Remarks on anomalies

The world of modern physics is fundamentally based on symmetries, no matter whether they are exact, approximate or broken by some dynamical effect. In section 2.3.5 we saw that the Noether theorem 2.2 provides a strong tool for dealing with symmetries of a classical theory. At first sight one would expect that the Noether theorem is valid also in a quantized theory, but since the discovery of the axial anomaly in the late sixties [38–40] it is known that this is not necessarily true. An anomaly is a term $\mathcal{A}$ that breaks the conservation of the Noether current:

$$\partial_\mu j^\mu \;=\; \mathcal{A} \neq 0\,. \tag{4.1}$$

Since in a quantized theory it is more common to express symmetry properties through Ward identities, the non-conservation of the Noether current is usually expressed in terms of an anomalous Ward identity. Obviously, the presence of an anomaly questions all results one can relay on in a classical theory such as the equations of motion.

However, not every symmetry of a classical theory is spoiled by quantum effects. In order to prove an anomaly one starts off with a semi-classical analysis. First, one assumes that everything works as in the classical case. If one finds the operator $\mathcal{A}$, one tries to redefine the current j^μ in order to regain the classical conservation law. Only when there is no chance to get rid of the anomalous term one can be sure that there is indeed an anomaly. The proof of an anomalous supercurrent I gave in chapter 3 is therefore indirect. The aim is now to prove directly the presence of a supercurrent anomaly.

Basically there are four different ways to see the anomaly arising in a quantized theory:

(i) As a variation of the "anomalous" symmetry. There can be a non-vanishing surface term.

(ii) As the index of an operator (non-triviality of the fibre bundle).

(iii) As a non-invariance of the path integral measure. The anomaly $\mathcal{A}$ then arises in the Jacobi determinant.

(iv) Calculating perturbatively the anomalous Ward identities.

There has been a rather technical and unaccessible analysis by A. Casher and Y. Shamir [41–43] where the anomaly of the supercurrent is shown with method (i). There is further an attempt by T. Wyder [44] to get the supercurrent anomaly from a topological analysis (ii), which hopefully will give a result in the near future. The case (iii) was discovered for the axial anomaly by K. Fujikawa [45, 46], but for the supercurrent this method has not yet been explored. Lastly there is the perturbative case (iv) which is the standard way to prove an anomaly and has led to the discovery of it. Since in this work I will focus on the perturbative analysis of the supercurrent anomaly I would like to have a closer look how this works for the primal anomaly: the axial anomaly.

For a detailed treatment of the manifold facets of anomalies in quantum field theory check the book by R. A. Bertlmann [47].

4.1.1 Axial anomaly

In 1967 D. G. Sutherland [48] and M. Veltman [49] noted that the π^0 should not decay into two photons. But exactly this process has been observed. The solution to this problem has been provided by S. Adler [38] and independently by J. S. Bell and R. Jackiw [39]. They calculated the triangle diagrams[10] for the process $\pi^0 \to \gamma\gamma$ of figure 4.1 and found that the divergence of the axial current j^5_μ picks up an anomalous term. With the axial current anomaly one can perfectly reproduce the experimental value for the decay rate.

When evaluating the loop integral that belongs to the graphs 4.1 (as done e.g. in [47]) one runs into an integral of the form

$$\int \frac{d^4 l}{(2\pi)^4} \left[f(l-p) - f(l) \right] , \tag{4.2}$$

where f is a function of momenta and p is a finite external momentum. Naively, one would think that for a finite p this integrals vanishes and the axial current is conserved. But since this is a (linearly) divergent integral we have something like $\infty - \infty$ and one has to be very careful. In fact it is exactly this expression that gives us a finite contribution which manifests itself in terms of an anomaly. The final result is then

[10] The triangle graphs were already calculated by L. Rosenberg [50] for the reaction $\nu\bar{\nu} \leftrightarrow \gamma\gamma$. This is of great importance for the understanding of fusion in stars. See also M. Gell-Mann [51].

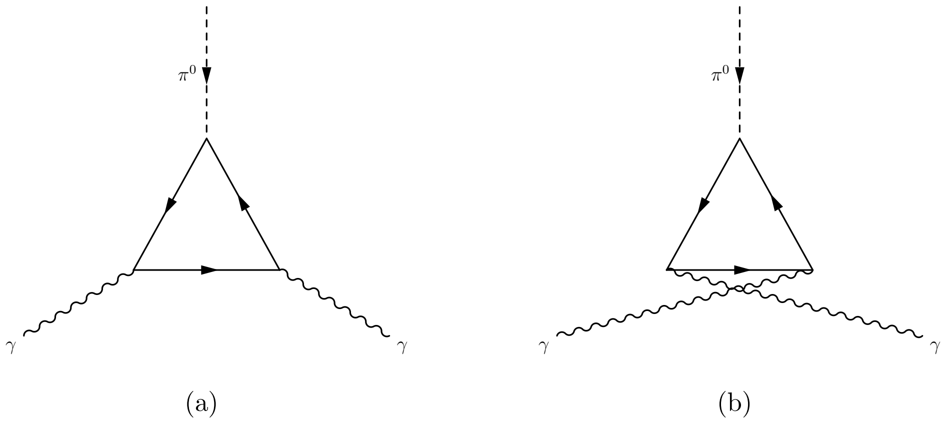


Figure 4.1: Triangle graphs for $\pi^0 \to \gamma\gamma$ with quarks running inside the loop.

$$\partial^\mu j^5_\mu \;=\; f_\pi m_\pi^2 \phi_\pi + \frac{\alpha}{8\pi}\varepsilon^{\mu\nu\rho\lambda}F_{\mu\nu}F_{\rho\lambda}\,. \tag{4.3}$$

$F_{\mu\nu}$ is the QED field strength, ϕ_π the pion field, m_π the physical pion mass and f_π is the physical pion decay constant. The anomalous term is $\frac{\alpha}{8\pi}\varepsilon^{\mu\nu\rho\lambda}F_{\mu\nu}F_{\rho\lambda}$ since anyway for massive pions the axial symmetry is broken by the quark mass terms in the Lagrangian. $f_\pi m_\pi^2 \phi_\pi$ follows from the classical equations of motion and corresponds to the term $2im\bar{q}\gamma_5 q$, where q are the quarks that run inside the loop of figure 4.1 (see appendix A.2.3 for the notion of γ_5).

When calculating the triangle graphs of 4.1 there is an ambiguity that arises. One can a priori choose whether to have a conserved vector current and an anomalous axial current, an anomalous vector current and conserved axial current or an anomaly for both currents.

For an Abelian gauge theory the whole structure of the anomaly is given by the triangle graphs of 4.1. There are no higher order contributions, contributions from other one-loop diagrams or radiative corrections. This has been found by S. Adler and W. Bardeen [52]. The non-Abelian case has been treated by W. Bardeen [40] where more than only the triangle diagrams contribute to the anomaly.

Anomalies have not to be understood as a failure or inconsistency of the quantized theory. They naturally arise when one takes into account quantum effects and, as you can see, they are needed in order to explain experimental data.

4.2 Dimensional analysis of the supercurrent divergence

Before starting the perturbative analysis I would like to show what the candidates for the supercurrent anomaly are. Dimensional analysis and the constraints of Lorentz and gauge invariance give us already a deep insight of what we have to expect [27]. Every dimension will be displayed in mass powers.

The supercurrent $\mathcal{S}_{\mu\alpha}$ is given by equation (2.78). The dimensions of the relevant quantities we have in our theory (2.69) are

$$\begin{aligned} \left[\mathcal{L}_{SYM}\right] &= 4\,, \qquad [g] = 0\,, \\ \left[V_\mu\right] &= 1\,, \qquad [\lambda_\alpha] = \frac{3}{2}\,, \\ \left[G^{\mu\nu}\right] &= 2\,. \end{aligned} \tag{4.4}$$

Trivially we get

$$\begin{aligned} \left[\mathcal{S}_{\mu\alpha}\right] &= \left[G^{\mu\nu}\right] + \left[\bar{\lambda}_{\dot{\alpha}}\right] = \frac{7}{2}\,, \\ \left[\partial^\mu \mathcal{S}_{\mu\alpha}\right] &= \frac{9}{2}\,. \end{aligned} \tag{4.5}$$

Since an anomaly has to be non-zero modulo the equations of motion, there will never be an anomalous term containing the auxiliary field D.

4.2.1 The necessity of a D'Alembert operator

Let us now try to find an operator $\mathcal{A}_\alpha \neq 0$ such that

$$\partial^\mu \mathcal{S}_{\mu\alpha} = \mathcal{A}_\alpha \qquad \text{and} \qquad [\mathcal{A}_\alpha] = \frac{9}{2}\,. \tag{4.6}$$

The components for constructing $\mathcal{A}_\alpha$ are $G^{\mu\nu}$, λ_α, $\bar{\lambda}_{\dot{\alpha}}$, D_μ. Using an ordinary derivative instead of the covariant derivative would spoil the gauge transformation property of the supercurrent. Since I must get a quantity with half integral dimension and the gaugino is the only operator with this property, there must be an odd number of gauginos. Let me first leave the possibility of using covariant derivatives apart and focus on $G^{\mu\nu}$, λ_α and $\bar{\lambda}_{\dot{\alpha}}$. Starting off with one $G^{\mu\nu}$, 5/2 of mass powers remains. But it is impossible to get 5/2 out of the 3/2 of the gauginos. Thus we can only make the ansatz

$$\mathcal{A}_\alpha = c\,\mathrm{Tr}\left[\lambda_\alpha \lambda^\beta \lambda_\beta\right] \qquad \text{or} \tag{4.7}$$

$$\mathcal{A}_\alpha = c\,\mathrm{Tr}\left[\lambda_\alpha \bar{\lambda}_{\dot{\beta}} \bar{\lambda}^{\dot{\beta}}\right]\,. \tag{4.8}$$

(4.7) vanishes since there are three anticommuting λ. In order to see that equation (4.8) vanishes we have to write explicitly the group indices.

$$\begin{aligned} \mathrm{Tr}\left[\lambda_\alpha\,\bar\lambda_{\dot\beta}\,\bar\lambda^{\dot\beta}\right] &= \lambda^A_\alpha\,\bar\lambda^B_{\dot\beta}\,\bar\lambda^{C\dot\beta}\,\mathrm{Tr}\left[T^AT^BT^C\right] \\ &= \lambda^A_\alpha\,\bar\lambda^B_{\dot\beta}\,\bar\lambda^{C\dot\beta}\left\{if^{BCD}\mathrm{Tr}\left[T^AT^D\right]+\mathrm{Tr}\left[T^AT^CT^B\right]\right\}. \end{aligned} \tag{4.9}$$

And therefore

$$\mathrm{Tr}\left[\lambda_\alpha\,\bar\lambda_{\dot\beta}\,\bar\lambda^{\dot\beta}\right] = \frac{i}{2}f^{ABC}\,\lambda^A_\alpha\,\bar\lambda^B_{\dot\beta}\,\bar\lambda^{C\dot\beta}\,. \tag{4.10}$$

But since $\bar\lambda^B_{\dot\beta}\,\bar\lambda^{C\dot\beta}$ is symmetric in B and C and f^{ABC} is totally antisymmetric, the possibility for $\mathcal{A}_\alpha$ of equation (4.8) is excluded.

For $\mathcal{A}_\alpha$ linear in λ_α or $\bar\lambda_{\dot\alpha}$ we have to involve also covariant derivatives. There are two possibilities to achieve the remaining dimension of 3: one D_μ and one $G^{\mu\nu}$ or three D_μ's. The latter case gives the two possibilities

$$\mathcal{A}_\alpha = c\,\mathrm{Tr}\left[\mathrm{D}_{\alpha\dot\alpha}\mathrm{D}^{\beta\dot\alpha}\mathrm{D}_{\beta\dot\beta}\,\bar\lambda^{\dot\beta}\right] \qquad \text{and} \tag{4.11}$$

$$\mathcal{A}_\alpha = c\,\mathrm{Tr}\left[\mathrm{D}_{\beta\dot\beta}\mathrm{D}^{\beta\dot\beta}\mathrm{D}_{\alpha\dot\alpha}\,\bar\lambda^{\dot\alpha}\right]. \tag{4.12}$$

Note that there is no chance to have $\mathcal{A}_\alpha$ with three covariant derivatives and linear in λ_α since we would be left with a dotted index instead of an undotted. Equation (4.11) and (4.12) are no valuable forms for the anomaly since they are zero when one inserts the equation of motion for $\bar\lambda_{\dot\alpha}$.

The next possibility is to take one gaugino, one covariant derivative and one field strength. Thereof we can build the ansatz

$$\mathcal{A}_\alpha = c\,\mathrm{Tr}\left[G_\alpha{}^\beta\,\mathrm{D}_{\beta\dot\alpha}\,\bar\lambda^{\dot\alpha}\right] \qquad \text{and} \tag{4.13}$$

$$\mathcal{A}_\alpha = c\,\mathrm{Tr}\left[\bar\lambda_{\dot\alpha}\,\mathrm{D}^{\beta\dot\alpha}\,G_{\alpha\beta}\right] \qquad \text{and} \tag{4.14}$$

$$\mathcal{A}_\alpha = c\,\mathrm{Tr}\left[\bar\lambda_{\dot\beta}\,\mathrm{D}_{\alpha\dot\alpha}\,G^{\dot\alpha\dot\beta}\right]. \tag{4.15}$$

Again, because of the equation of motion the ansatz (4.13) vanishes. The two equations (4.14) and (4.15) do not provide valid candidates for $\mathcal{A}_\alpha$ too. Inserting $\mathrm{D}_\mu G^{\mu\nu} = \lambda\sigma^\nu\bar\lambda$ into (4.14) and (4.15) and after some algebra (appendix A.3) we come back again to equation (4.7) and (4.8).

Up to now we just took into account terms where the covariant derivative has a contracted index with the object that it acts on. But one could imagine also a term like

$$\mathcal{A}_\alpha = c\,\mathrm{Tr}\left[G_{\dot\alpha\dot\beta}\mathrm{D}_\alpha{}^{\dot\alpha}\bar\lambda^{\dot\beta}\right]. \tag{4.16}$$

Note that it is not possible to have three covariant derivatives that have all indices contracted with each other. In the ansatz (4.16) we can turn the covariant derivative over to the $G^{\dot{\alpha}\dot{\beta}}$ modulo a total derivative. We then end up with one term like (4.14) or (4.15) and one of the form $\partial_{\alpha\dot{\alpha}}(G^{\dot{\alpha}\dot{\beta}}\bar{\lambda}_{\dot{\beta}})$. But this is nothing else than the divergence of the supercurrent $\mathcal{S}_{\mu\alpha}$ itself.

Therefore we cannot find a local form for the anomaly term $\mathcal{A}_{\mu}$. But does this mean that there is no supercurrent anomaly? No, in fact the operator $\mathcal{A}_{\alpha}$ does not need to be local. The form of the axial anomaly suggests that an anomaly is a local non-vanishing operator, but there is nothing that would imply locality. The above argumentation just means that we have to look for a non-local $\mathcal{A}_{\alpha}$. This can be done by searching local forms of higher order derivatives of the supercurrent, i.e.

$$\Box^{n}\,\partial^{\mu}\mathcal{S}_{\mu\alpha} \;=\; \mathcal{A}^{(n)}_{\alpha}\,. \tag{4.17}$$

For some n the anomaly will become a local composite operator. Unfortunately, there is no possibility to restrict n and therefore we start with the simplest case.

4.2.2 The case $n = 1$

For $n = 1$ we now have to look for an operator $\mathcal{A}^{(1)}_{\alpha}$ with mass dimension $\left[\mathcal{A}^{(1)}_{\alpha}\right] = 13/2$. Again it is not possible to have an $\mathcal{A}^{(1)}_{\alpha}$ linear in λ_{α} or $\bar{\lambda}_{\dot{\alpha}}$ without a covariant derivative: $13/2 - 3/2 = 5$ but $G^{\mu\nu}$ has dimension 2. Hence we can only build

$$\mathcal{A}^{(1)}_{\alpha} \;=\; c\,\mathrm{Tr}\left[\bar{\lambda}_{\dot{\alpha}}\,\bar{\lambda}^{\dot{\alpha}}\lambda^{\beta}G_{\alpha\beta}\right] \qquad \text{and} \tag{4.18}$$

$$\mathcal{A}^{(1)}_{\alpha} \;=\; c\,\mathrm{Tr}\left[\lambda^{\beta}\lambda_{\beta}\lambda^{\gamma}G_{\alpha\gamma}\right]\,. \tag{4.19}$$

Obviously (4.19) is zero, whereas equation (4.18) does not seem to vanish a priori and represents the first candidate for the anomaly term.

As in section 4.2.1 the possibility of including covariant derivatives in the ansatz for $\mathcal{A}^{(1)}_{\alpha}$ will give terms that vanish when inserting the Euler-Lagrange equations. For $n = 1$ there are now more possibilities:

(i) D_{μ} acts on λ_{α} or $\bar{\lambda}_{\dot{\alpha}}$ with contracted indices $\Rightarrow$ vanishes.

(ii) D_{μ} acts on $G^{\mu\nu}$ with contracted indices $\Rightarrow$ can be brought to the form of (4.18) or (4.19).

(iii) D_{μ} acts on λ_{α} or $\bar{\lambda}_{\dot{\alpha}}$ *without* contracted indices.

(iv) D_{μ} acts on $G^{\mu\nu}$ *without* contracted indices.

The cases (i) (ii) do not need any further discussion whereas (iii) (iv) are more interesting. The possibility (iii) allows the two terms

$$\mathcal{A}^{(1)}_\alpha = c\,\mathrm{Tr}\left[G_\alpha{}^\beta\, G_{\dot\alpha\dot\beta}\, \mathrm{D}_\beta{}^{\dot\alpha}\, \bar\lambda^{\dot\beta}\right] \qquad \text{and} \tag{4.20}$$

$$\mathcal{A}^{(1)}_\alpha = c\,\mathrm{Tr}\left[G_{\dot\alpha\dot\beta}\, G^{\dot\alpha}{}_{\dot\gamma}\, \mathrm{D}_\alpha{}^{\dot\beta}\, \bar\lambda^{\dot\gamma}\right]. \tag{4.21}$$

Both equations (4.20) and (4.21) do not vanish a priori and it is also not possible to bring them to one of the forms for the anomaly we saw earlier. For (iv), after excluding terms which are proportional to $G^\alpha{}_\alpha = G^{\dot\alpha}{}_{\dot\alpha} = 0$, we get the following two possible terms:

$$\mathcal{A}^{(1)}_\alpha = c\,\mathrm{Tr}\left[\bar\lambda_{\dot\alpha}\, G_{\dot\beta\dot\gamma}\, \mathrm{D}_\alpha{}^{\dot\alpha}\, G^{\dot\beta\dot\gamma}\right] \qquad \text{and} \tag{4.22}$$

$$\mathcal{A}^{(1)}_\alpha = c\,\mathrm{Tr}\left[\bar\lambda_{\dot\alpha}\, G^{\beta\gamma} \mathrm{D}_\beta{}^{\dot\alpha} G_{\alpha\gamma}\right]. \tag{4.23}$$

These two anomaly terms do also not vanish and thus by collecting all non vanishing terms we end up with the final ansatz for the anomaly term

$$\boxed{\begin{aligned}\mathcal{A}^{(1)}_\alpha = {} & c_1\,\mathrm{Tr}\left[\bar\lambda_{\dot\alpha}\, \bar\lambda^{\dot\alpha} \lambda^\beta G_{\alpha\beta}\right] \\ & + c_2\,\mathrm{Tr}\left[G_\alpha{}^\beta\, G_{\dot\alpha\dot\beta}\, \mathrm{D}_\beta{}^{\dot\alpha}\, \bar\lambda^{\dot\beta}\right] + c_3\,\mathrm{Tr}\left[G_{\dot\alpha\dot\beta}\, G^{\dot\alpha}{}_{\dot\gamma}\, \mathrm{D}_\alpha{}^{\dot\beta}\, \bar\lambda^{\dot\gamma}\right] \\ & + c_4\,\mathrm{Tr}\left[\bar\lambda_{\dot\alpha}\, G_{\dot\beta\dot\gamma}\, \mathrm{D}_\alpha{}^{\dot\alpha}\, G^{\dot\beta\dot\gamma}\right] + c_5\,\mathrm{Tr}\left[\bar\lambda_{\dot\alpha}\, G^{\beta\gamma} \mathrm{D}_\beta{}^{\dot\alpha} G_{\alpha\gamma}\right].\end{aligned}} \tag{4.24}$$

However this is just an ansatz. I do not know if all of the five terms in (4.24) are really part of the anomaly term and if the supercurrent anomaly is already local for one D'Alembert operator i.e. $n = 1$. The above expression for the anomaly has to be understood as the most general form of the anomaly for a single case out of infinitely many.

4.3 Perturbative analysis of the supercurrent anomaly

4.3.1 Diagram jungle

For the axial anomaly there was an experimental hint which matrix element suffers an anomalous Ward identity, namely the matrix element of $\pi^0 \to \gamma\gamma$. Such a helping hand from outside does not exist in the case of supersymmetry and we are left with a bunch of matrix elements and the corresponding Feynman graphs that could all possibly contain the anomaly we are looking for. Here, a list of some diagrams that have to be calculated

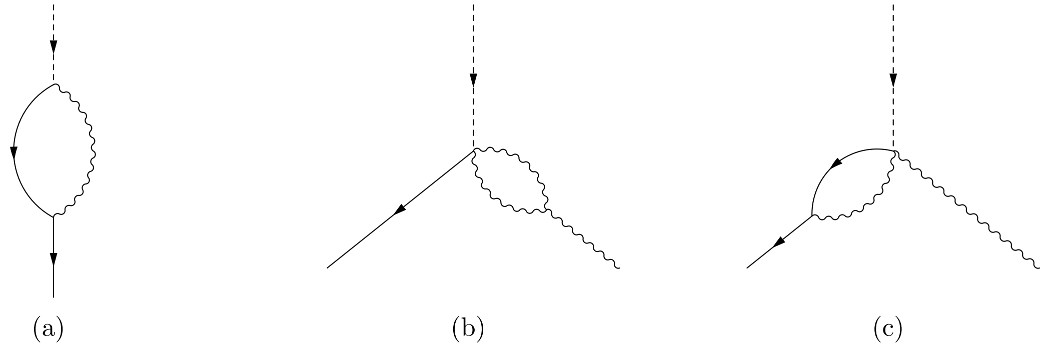


Figure 4.2: Feynman graphs with two propagators inside the loop.

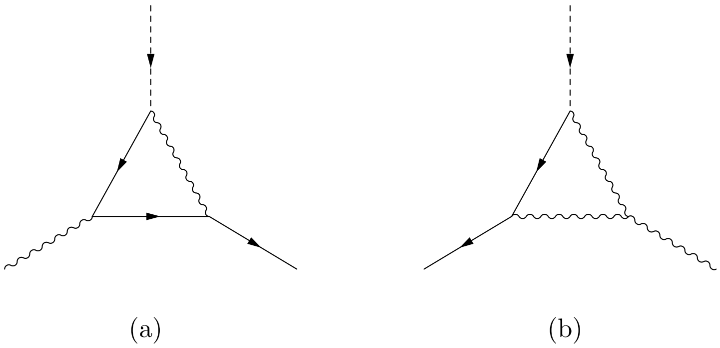
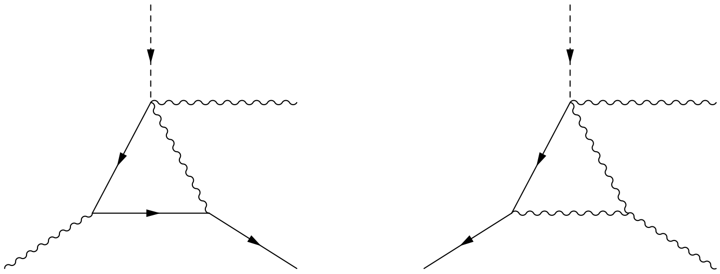


Figure 4.3: Triangle graphs.

in order to prove a supercurrent anomaly shall be given. The dashed lines denote the supercurrent, the wavy lines the gauge boson and the straight lines represent the gaugino. The Feynman rules are found in appendix C

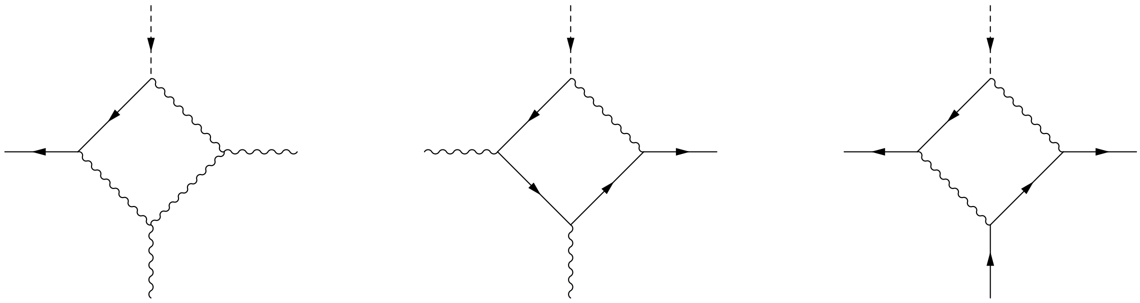
Of course tree diagrams do not contribute directly since the anomaly is a quantum effect and has nothing to do with the classical tree process. However, in order to calculate the complete matrix element, the contributions form the tree graphs must also be included. The simplest Feynman graphs with only two propagators inside the loop are displayed in figure 4.2. The superficial degree of divergence of 4.2(a) is 2, meaning that the corresponding integral is quadratically divergent ($\sim \Lambda^2_{cut}$). The other two diagrams 4.2(b) and 4.2(c) are only linearly divergent since the vertex (C.9) has no derivative.

The next level has three propagators in the loop (figure 4.3). This two diagrams are investigated with more detail in section 4.3.2. As one expects, the triangle graphs have one degree of divergence less than the diagram of figure 4.2(a) i.e. they diverge linearly just as 4.2(b) and 4.2(c). By exchanging the supercurrent vertex (C.8) with the second possible vertex (C.9) we get the diagrams depicted in figure 4.4. Since the vertex with two bosons has no derivative acting on the external lines, the diagrams of figure 4.4 are only



(a) (b)

Figure 4.4: Triangle graphs with the vertex (C.9) instead of (C.8).



(a) (b) (c)

Figure 4.5: Quadrangle graphs.

logarithmically divergent.

The graphs of figure 4.5(a) and 4.5(b) also have one power of loop momentum less than the triangle graphs and are therefore logarithmically divergent, whereas the diagram 4.5(c) is not divergent anymore. The integral is proportional to Λ_{cut}^{-1}. It is a sophism to think that the amplitude of a Feynman graph does not contribute to the anomaly if the diagram is not superficially divergent. This is because when differentiating the amplitude in order to calculate the would-be anomalous Ward identity one augments the degree of divergence by one. At first sight this seems a bit weird since the amplitude is multiplied (differentiated) with the external momenta and not with the loop momentum. However, when inserting the momentum conservation law of the supercurrent vertex one “replaces” the external momentum with a loop momentum. Therefore we have to include also formally convergent Feynman graphs in our list of possibly contributing diagrams.

Note that there are two different versions of the graph 4.5(c) since I can flip the gauginos from $\bar{\lambda}_{\dot{\alpha}}$ to λ_α and vice versa. Only the gaugino $\bar{\lambda}_{\dot{\alpha}}$ from the supercurrent vertex C.8 is fixed. As for the triangle graphs I can take the vertex (C.9) instead of (C.8) which gives us the quadrangle graphs with one more external boson line as in figure 4.6. Since in this

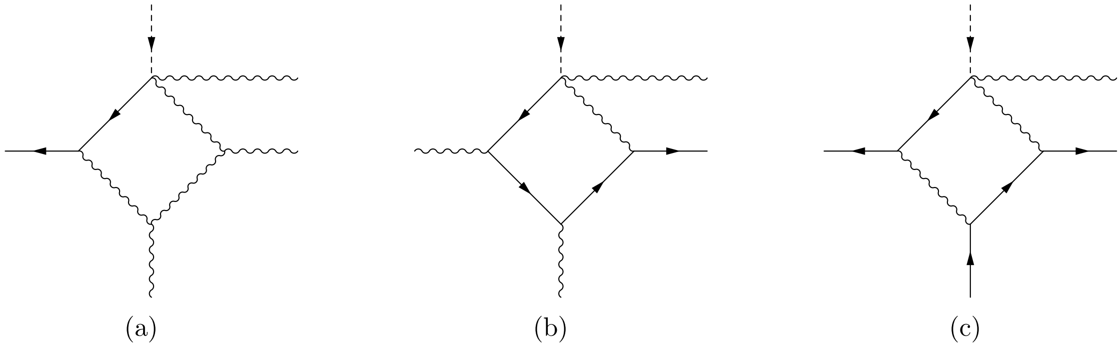


Figure 4.6: Quadrangle graphs with the second supercurrent vertex (C.9).

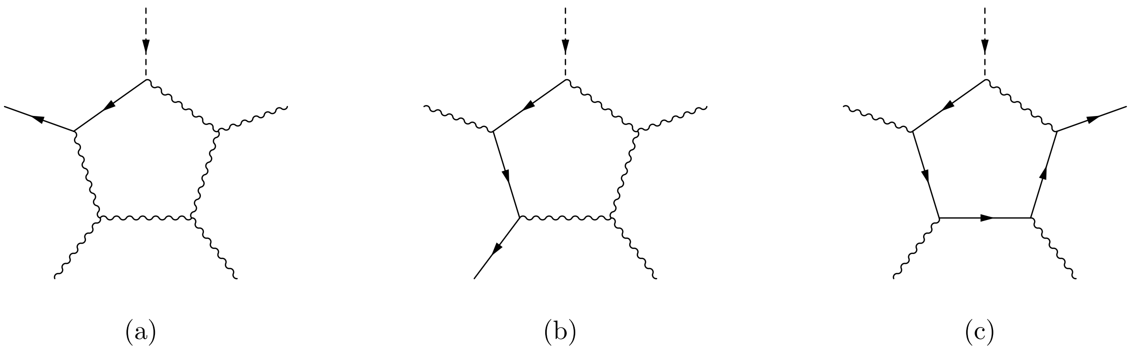


Figure 4.7: Pentagon graphs that converge as Λ_{cut}^{-1}.

other vertex there is no derivative, the superficial degree of divergence drops by one unit i.e. the graphs in figure 4.6 are proportional to Λ_{cut}^{-1} and Λ_{cut}^{-2} respectively.

After the quadrangle graphs of figure 4.5 and 4.6 there are the pentagon diagrams 4.7, 4.8 and 4.9. The diagrams 4.7(a), 4.7(b) and 4.7(c) are $\sim \Lambda_{cut}^{-1}$ whereas all graphs of 4.8 are converging as Λ_{cut}^{-2}. Note that 4.8(e), 4.8(f) and 4.8(g) are just the graphs of figure 4.7 with the alternative supercurrent vertex. Figure 4.9 shows the last pentagon graphs. They have again one power of Λ_{cut} less than the previous ones, i.e Λ_{cut}^{-3}.

Since the non-Abelian axial anomaly discussed by W. Bardeen [40] receives only contributions up to pentagon graphs, I stop the list at this point. However, bear in mind that I have no idea how many D'Alembert operators are needed in order go get a local supercurrent anomaly. There could be infinitely many Feynman graphs that give a contribution since acting the D'Alembert operator on the supercurrent I can make every possible amplitude formally divergent.

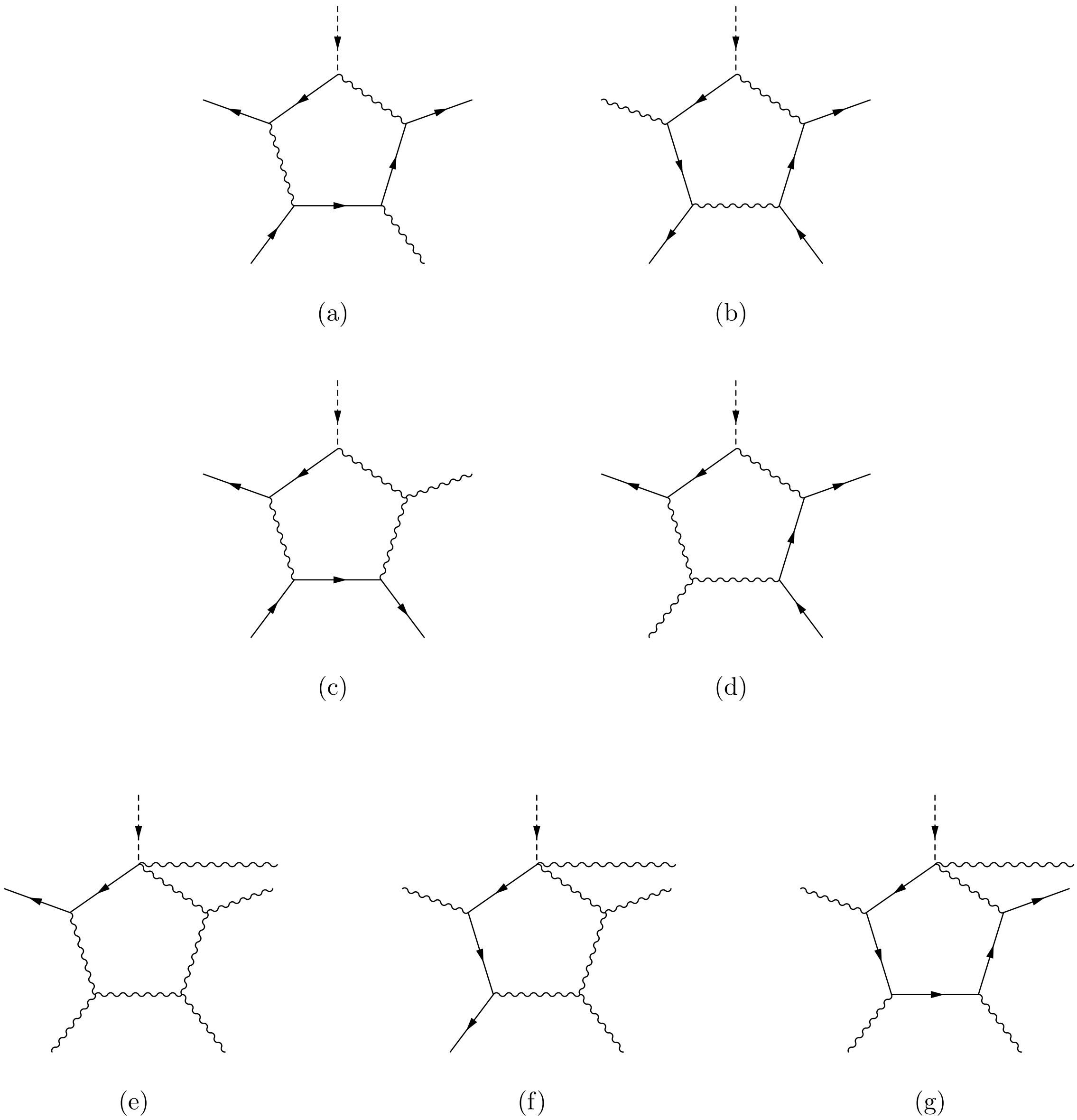


Figure 4.8: Pentagon graphs $\sim\ \Lambda_{cut}^{-2}$.

4.3.2 The Triangle graphs

As already mentioned, for supersymmetry there exists no experimental input and I am left with no idea which one of the diagrams of section 4.3.1 is contributing to the anomaly. To calculate all diagrams would take some time and would go far beyond the scope of this thesis.

However, by analogy to the axial anomaly I try to look at the triangle graphs depicted in 4.3. The graphs of the axial anomaly of figure 4.1 are linearly divergent and therefore

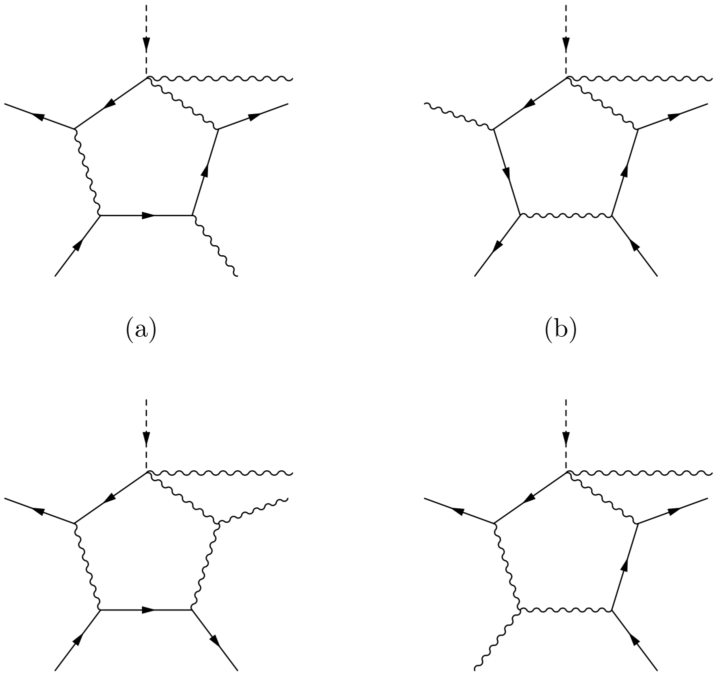

(a) (b) (c) (d)

Figure 4.9: Pentagon graphs converging as Λ_{cut}^{-3}.

it might be that there is a contribution to the supercurrent anomaly.

To my astonishment this was already done in 1977 by L. F. Abbot, M. T. Grisaru and H. J. Schnitzer [53] before the discussions of Witten, Veneziano, Yankielowicz and others. Without the knowledge of the arguments of chapter 3 and the analysis of section 4.2 they “proved” the supercurrent anomaly by calculating the diagrams 4.2(b), 4.2(c) and the triangle graphs of figure 4.3. Let me now investigate the result they obtained.

Apparently, in the Feynman-'t Hooft gauge the diagrams 4.2(b) and 4.2(c) do not contribute to the anomaly and do therefore not need any further discussion. The result for the supercurrent anomaly of [53] reads

$$\partial_\mu \mathcal{S}^\mu \;=\; -\frac{3ig^2}{8\pi^2}\Big(\partial_\nu V_\rho^A - \partial_\rho V_\nu^A\Big)\Big(\gamma^\rho \partial^\nu \lambda^A\Big)\,. \tag{4.25}$$

Note that here the notation of Dirac spinors (see appendix A.2.3) is used and the spinor indices are suppressed. As it is readily seen the above result is brought to

$$\partial_\mu \mathcal{S}^\mu \;=\; -\frac{3ig^2}{8\pi^2}\,G_{\nu\rho}^A\,\gamma^\rho\,\mathrm{D}^\nu\lambda^{\mathrm{A}} \tag{4.26}$$

when "covariantizing" equation (4.25). The crucial point now is that this result does not represent a proper anomaly. This can be seen by rewriting (4.26) with the notation adopted throughout this work (appendix A).

$$G^A_{\nu\rho}\,\gamma^\rho\,\mathrm{D}^\nu\,\lambda^\mathrm{A} \;\equiv\; \mathrm{G}^\mathrm{A}_{\nu\rho}\,(\sigma^\rho)_{\alpha\dot\alpha}\,\mathrm{D}^\nu\,\bar\lambda^{\mathrm{A}\dot\alpha} + \mathrm{G}^\mathrm{A}_{\nu\rho}\,(\tilde\sigma^\rho)^{\dot\alpha\alpha}\,\mathrm{D}^\nu\,\lambda^\mathrm{A}_\alpha\,. \tag{4.27}$$

Looking at the first term and writing again $G^A_{\nu\rho}T^A = G_{\nu\rho}$ and $\bar\lambda^{A\dot\alpha}T^A = \bar\lambda^{\dot\alpha}$ we get

$$\begin{aligned}
\mathrm{Tr}\left[G_{\nu\rho}(\sigma^\rho)_{\alpha\dot\alpha}\mathrm{D}^\nu\bar\lambda^{\dot\alpha}\right] &= \mathrm{Tr}\left[(\sigma_{\nu\rho})^{\beta\gamma}(\sigma^\rho)_{\alpha\dot\alpha}(\sigma^\nu)_{\delta\dot\delta}\,G_{\beta\gamma}\,\mathrm{D}^{\delta\dot\delta}\,\bar\lambda^{\dot\alpha}\right]\\
&= \frac{1}{2}\mathrm{Tr}\left[\left((\sigma_\nu)_{\beta\dot\alpha}(\sigma^\nu)_{\delta\dot\delta}\,G^\beta{}_\alpha - (\sigma_\rho)_{\beta\dot\delta}(\sigma^\rho)_{\alpha\dot\alpha}\,G^\beta{}_\delta\right)\mathrm{D}^{\delta\dot\delta}\,\bar\lambda^{\dot\alpha}\right]\\
&= \frac{1}{2}\mathrm{Tr}\left[G_{\delta\alpha}\,\mathrm{D}^\delta{}_{\dot\alpha}\,\bar\lambda^{\dot\alpha} - \mathrm{G}_{\alpha\delta}\,\mathrm{D}^\delta{}_{\dot\alpha}\,\bar\lambda^{\dot\alpha}\right]\\
&= -\mathrm{Tr}\left[G_{\alpha\delta}\,\mathrm{D}^\delta{}_{\dot\alpha}\,\bar\lambda^{\dot\alpha}\right]\,,
\end{aligned} \tag{4.28}$$

where we find again the equation of motion of the gaugino. This is exactly the case we had in equation (4.13). The second term is brought in complete analogy to the same point as the first one. Thus what L. F. Abbot, M. T. Grisaru and H. J. Schnitzer found is not an anomaly of the supercurrent. I can thereof conclude that the diagrams 4.2(b), 4.2(c) and 4.3 do not contribute to the anomalous divergence of the supercurrent.

Chapter 5

Summary and Conclusions

Lastly, I would like to summarize the assumptions and arguments of this master's thesis and give a conclusion and outlook.

5.1 Summary

5.1.1 Assumptions

The assumptions of the present master's thesis are not further commented and changing only one of them would most probably lead to a breakdown of arguments and the conclusions could differ heavily.

The framework of a quantum field theory is never left, meaning that the following five properties of the theory are always true:

(i) Locality

(ii) Causality

(iii) Unitarity

(iv) Poincaré invariance

(v) Renormalizability

Supersymmetry is always assumed to be a global symmetry, no matter if it is broken or not. The super Yang-Mills theory is assumed to have basically the same features as QCD:

(vi) Stability of the ground state i.e. the theory exists.

(vii) Confinement.

(viii) Mass gap.

(ix) Global chiral symmetry in the low energy dynamics.

(x) The gauge group is compact and connected but arbitrary within these restrictions.

5.1.2 Spontaneous supersymmetry breaking and anomaly

In chapter 3 I first showed that important issues were not covered in the discussions of the early eighties (section 3.2 and 3.3). Going through the possible values of the energy density ε, which is an order parameter for the spontaneous breaking of supersymmetry, I showed indirectly that there has to be an anomalous divergence of the supercurrent. Based on work done by the group of P. Minkowski, it was shown in section 3.4 that the presence of a goldstino implies $\varepsilon \geq 0$ whereas the trace anomaly requires $\varepsilon \leq 0$. A technically demanding analysis of the effective action by P. Minkowski et al. revealed that $\varepsilon \neq 0$. If the supercurrent is affected by an anomaly there would be no goldstino and one would be left with $\varepsilon > 0$. Of course this would mean that supersymmetry is spontaneously (and dynamically) *broken* in a super Yang-Mills theory.

A detailed analysis of the structure of the supercurrent anomaly is then found in chapter 4. The correct dimensions and transformation properties of the possible anomaly operator give tight constraints to the form of the supercurrent anomaly. It becomes clear that the divergence of the supercurrent has to be a non-local operator and an extra D'Alembert operator is needed in order to get a possible local expression for the anomaly. In section 4.3 we have had a closer look at the possibility of proving the supercurrent anomaly perturbatively. Due to the lack of experimental input it remains an open question which matrix element suffers an anomalous Ward identity. From a rather long list of Feynman diagrams which could possibly contribute to the anomaly, the triangle graphs are picked out. By analogy to the triangle graphs of the axial anomaly it seems sensible to look for the supercurrent anomaly in the analogous graphs. Moreover, the supercurrent triangle graphs have the same degree of divergence as those belonging to the axial anomaly. The calculation of the triangle diagrams has already been carried out in 1977 by L. F. Abbott, M. T. Grisaru and H. J. Schnitzer [53], but the result does not lead to the anomaly I am looking for. Although the authors of [53] thought to have found a supercurrent anomaly when inserting the equation of motion for the gaugino, the anomalous term vanishes.

5.1.3 Conclusions

Global supersymmetry in a pure N=1 super Yang-Mills theory breaks itself and an anomaly of the supercurrent arises. Unfortunately it is not yet possible to give a direct proof of the supercurrent anomaly with perturbative methods. It would take much more time and routine to calculate all possibly contributing Feynman graphs than was provided for this thesis. The only sure thing is that there is an anomaly and that the triangle graphs do not contribute to it.

5.2 Outlook

The problem of the anomalous supercurrent remains unsolved, but the fundamental importance of this question outlined in the introduction does not allow giving up. The perturbative way to prove the anomaly seems to be time-consuming and it is not clear if the anomaly arises already with one D'Alembert operator i.e. $n = 1$.

It is possible that a different approach to the supercurrent anomaly could lead to a definitive determination of the supercurrent anomaly. So far, it has not been tried to calculate the supercurrent anomaly through the Jacobi determinant of the path integral as was done by K. Fujikawa for the axial anomaly. Even though this approach might be very elegant in the case of the axial anomaly, for supersymmetry it would involve a local variation with respect to the superspace coordinates θ and $\bar{\theta}$ which leads directly to supergravity. Furthermore, there is still some hope to understand the supercurrent anomaly in terms of the fibre bundle structure of the N=1 super Yang-Mills theory.

There is also the possibility that G. Girardi, R. Grimm and R. Stora [54] found some indication to the supercurrent anomaly in their investigation on chiral anomalies. Unfortunately this paper is an account of very long and cumbersome calculations making it impossible for me to find such a hint in it.

Given that in the near future a successful accomplishment of the N=1 case is achieved, it would be interesting to have a look at theories with more than one supercharge i.e. N=2 and N=4.

There is still a lot of work to do on the questions raised in this thesis; not only on the physical and mathematical part of the problem but also on a social level.

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Appendix A

Notation, Conventions and Useful Formulae

A listing of the notations and conventions used throughout this thesis shall be given together with the basic formulae and helpful identities. For a more detailed account on the conventions used here see [20].

A.1 General nomenclature and conventions

Group indices are denoted as capital letters

$$A,\, B,\, C,\, \ldots \tag{A.1}$$

Lorentz indices as small Greek letters starting from the middle of the Greek alphabet

$$\lambda,\, \mu,\, \nu,\, \ldots \tag{A.2}$$

Of course I use the Einstein summation convention, meaning that where no other specifications are given

$$A_\mu B^\mu \qquad \text{means} \qquad \sum_{\mu=0}^{3} A_\mu B^\mu \,, \tag{A.3}$$

where the space-time metric is chosen as

$$g_{\mu\nu} \;=\; \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{A.4}$$

The Pauli matrices σ_μ read

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad \text{(A.5)}$$

They have one Lorenz, one dotted and one undotted spinor index (the notion of spinor indices is introduced in the next section). I therefore denote

$$(\sigma_\mu)_{\alpha\dot{\alpha}} \qquad \text{and} \qquad (\tilde{\sigma}_\mu)^{\dot{\alpha}\alpha} \quad \text{where} \quad \tilde{\sigma}_\mu = (+\sigma_0, -\vec{\sigma}) \,. \quad \text{(A.6)}$$

The Levi-Civita tensor $\varepsilon_{\mu\nu\rho\sigma}$ is chosen to have $\varepsilon_{0123} = -1$. For antisymmetric indices I write $[\mu\nu]$ and $\{\mu\nu\}$ for symmetric ones respectively, e.g. $T_{[\mu\nu]}$ is a antisymmetric second rank tensor. A "$*$" stands for the complex conjugation and hermitian conjugated objects are denoted as $\bar{V} = V^\dagger$.

The one-dimensional Fourier transformation and its inverse of a function $f(x)$ are defined as

$$f(x) = \frac{1}{2\pi} \int dp\, \mathrm{e}^{-ipx} \tilde{f}(p) \qquad \text{and} \qquad \tilde{f}(k) = \int dx\, \mathrm{e}^{ipx} f(x) \,. \quad \text{(A.7)}$$

A.2 Spinors

A.2.1 Weyl spinors

This work makes extensive use of the notion of Weyl spinors and therefore it is important to say a few words about it.

The proper orthochronous Lorentz group $SO(1,3)^\uparrow$ has $SL(2,\mathbb{C})$ as universal covering group, i.e. $SO(1,3)^\uparrow \simeq SL(2,\mathbb{C})/\mathbb{Z}_2$. The representations thereof are labeled by (l^+, l^-) where $l^\pm$ is the label of one $SU(2)$ representation. The two non-trivial (non-equivalent) irreducible representations with $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ are called left- and right-chiral. The elements of this representation spaces are Weyl spinors[11].

Spinor indices are also written as Greek letters but from the beginning of the alphabet and I use dotted and undotted indices according to the representation of the Lorentz group:

$$\alpha,\, \beta,\, \gamma,\, ... \quad \text{for} \quad (\frac{1}{2}, 0) \qquad \text{and} \qquad \dot{\alpha},\, \dot{\beta},\, \dot{\gamma},\, ... \quad \text{for} \quad (0, \frac{1}{2}) \,. \quad \text{(A.8)}$$

When taking the complex conjugate of a left-handed spinor ψ_α one gets the corresponding right-handed spinor:

$$(\psi_\alpha)^* = \bar{\psi}_{\dot{\alpha}} \,. \quad \text{(A.9)}$$

[11]Elements of the $(\frac{1}{2}, \frac{1}{2})$ representation are Lorentz vectors.

The Einstein summation convention is also used for spinors:

$$\psi^\alpha \chi_\alpha \,, \quad \bar\psi_{\dot\alpha} \bar\chi^{\dot\alpha} \qquad \text{means} \qquad \sum_{\alpha=1}^{2} \psi^\alpha \chi_\alpha \,, \quad \sum_{\dot\alpha=1}^{2} \bar\psi_{\dot\alpha} \bar\chi^{\dot\alpha} \,. \tag{A.10}$$

I do not use different Greek letters for dotted and undotted indices meaning that when writing $\sigma_{\alpha\dot\alpha}$ there is no summation (if nothing else is specified). The transition between a representation of $SL(2,\mathbb{C})$ and its contragradient representation ($M \in SL(2,\mathbb{C}) \to (M^T)^{-1}$) is given by the antisymmetric matrix

$$\varepsilon_{\alpha\beta} \;=\; \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{with} \qquad \varepsilon_{\alpha\beta} \;=\; \varepsilon_{\dot\alpha\dot\beta} \;=\; -\varepsilon^{\alpha\beta} \;=\; -\varepsilon^{\dot\alpha\dot\beta} \,. \tag{A.11}$$

This simply means that with the ε matrices defined above I can rise and lower spinor indices

$$\psi_\alpha \;=\; \varepsilon_{\alpha\beta}\psi^\beta \qquad \bar\psi^{\dot\alpha} \;=\; \varepsilon^{\dot\alpha\dot\beta}\bar\psi_{\dot\beta} \,. \tag{A.12}$$

Obviously the simple relation $\varepsilon^{\alpha\beta}\varepsilon_{\alpha\beta} = \varepsilon^{\dot\alpha\dot\beta}\varepsilon_{\dot\alpha\dot\beta} = -2$ holds. According to the connection between spin and statistics all spinors are odd elements of some Grassmann algebra:

$$\psi_\alpha\chi_\beta \;=\; -\chi_\beta\psi_\alpha \qquad \bar\psi_{\dot\alpha}\bar\chi_{\dot\beta} \;=\; -\bar\chi_{\dot\beta}\bar\psi_{\dot\alpha} \qquad \psi_\alpha\bar\chi_{\dot\beta} \;=\; -\bar\chi_{\dot\beta}\psi_\alpha \,. \tag{A.13}$$

Whenever there is no possibility of confusion I suppress the summed spinor indices i.e.

$$\psi\chi \;\doteq\; \psi^\alpha\chi_\alpha \;=\; -\psi_\alpha\chi^\alpha \,, \qquad \bar\psi\bar\chi \;\doteq\; \bar\psi_{\dot\alpha}\bar\chi^{\dot\alpha} \;=\; -\bar\psi^{\dot\alpha}\bar\chi_{\dot\alpha} \,, \tag{A.14}$$

$$\psi^2 \;\doteq\; \psi\psi \,, \qquad \bar\psi^2 \;\doteq\; \bar\psi\bar\psi \,, \tag{A.15}$$

$$\chi\sigma^\mu\bar\psi \;\doteq\; \chi^\alpha(\sigma^\mu)_{\alpha\dot\alpha}\bar\psi^{\dot\alpha} \,, \qquad \bar\psi\tilde\sigma_\mu\chi \;\doteq\; \bar\psi_{\dot\alpha}(\tilde\sigma_\mu)^{\dot\alpha\alpha}\chi_\alpha \,. \tag{A.16}$$

Using equation (A.13) one gets the relations

$$\psi\chi \;=\; \chi\psi \,, \qquad \bar\psi\bar\chi \;=\; \bar\chi\bar\psi \,, \qquad \psi\sigma_\mu\bar\chi \;=\; -\bar\chi\tilde\sigma_\mu\psi \,, \tag{A.17}$$

$$(\chi\sigma^\mu\bar\psi)^\dagger \;=\; \psi\sigma^\mu\bar\chi \,, \qquad (\bar\psi\tilde\sigma^\mu\chi)^\dagger \;=\; \bar\chi\tilde\sigma^\mu\psi \,. \tag{A.18}$$

A.2.2 Fierz identities

The Fierz[12] identities are useful when dealing with products of spinors.

$$\begin{aligned} \psi_\alpha\psi_\beta &= \frac{1}{2}\varepsilon_{\alpha\beta}\psi^2 & \psi^\alpha\psi^\beta &= -\frac{1}{2}\varepsilon^{\alpha\beta}\psi^2 \\ \bar\psi_{\dot\alpha}\bar\psi_{\dot\beta} &= -\frac{1}{2}\varepsilon_{\dot\alpha\dot\beta}\bar\psi^2 & \bar\psi^{\dot\alpha}\bar\psi^{\dot\beta} &= \frac{1}{2}\varepsilon^{\dot\alpha\dot\beta}\bar\psi^2 \end{aligned} \tag{A.19}$$

$$\psi_\alpha\bar\psi_{\dot\alpha} = -\frac{1}{2}(\sigma_\mu)_{\alpha\dot\alpha}\,\psi\sigma^\mu\bar\psi \tag{A.20}$$

$$\begin{aligned} (\chi\psi_1)(\chi\psi_2) &= -\chi^2(\psi_1\psi_2) \\ (\bar\chi\psi_1)(\bar\chi\psi_2) &= -\bar\chi^2(\psi_1\psi_2) \end{aligned} \tag{A.21}$$

$$\begin{aligned} (\chi\sigma^\mu\bar\psi_1)(\chi\psi_2) &= -\chi^2(\psi_2\sigma^\mu\bar\psi_1) \\ (\psi_1\sigma^\mu\bar\chi)(\bar\chi\bar\psi_2) &= -\bar\chi^2(\psi_1\sigma^\mu\bar\psi_2) \end{aligned} \tag{A.22}$$

$$(\chi\sigma^\mu\bar\chi)(\chi\sigma^\nu\bar\chi) = -\frac{1}{2}g^{\mu\nu}(\chi\chi)(\bar\chi\bar\chi) \tag{A.23}$$

$$\begin{aligned} (\psi_1\psi_2)(\psi_3\psi_4) &= -(\psi_1\psi_3)(\psi_2\psi_4) - (\psi_1\psi_4)(\psi_2\psi_3) \\ (\psi_1\psi_2)(\bar\psi_3\bar\psi_4) &= -\frac{1}{2}(\psi_1\sigma^\mu\bar\psi_4)(\psi_2\sigma_\mu\bar\psi_3) \end{aligned} \tag{A.24}$$

A.2.3 From Dirac to Weyl spinors

In contrast to textbooks on supersymmetry, in the literature of non-supersymmetric quantum field theory the common notation for spinors is the Dirac spinor. Here I want to briefly show how the Dirac and Weyl formalism are connected to each other.

Consider a dotted and a undotted Weyl spinor $\bar\chi^{\dot\alpha}$ and ψ_α. The Dirac spinor and its conjugate are then four component spinors of the form

$$\Psi \doteq \begin{pmatrix} \psi_\alpha \\ \bar\chi^{\dot\alpha} \end{pmatrix}, \qquad \bar\Psi \doteq (\chi^\alpha, \bar\psi_{\dot\alpha})\,. \tag{A.25}$$

$\bar\Psi$ is obtained by $\bar\Psi = \Psi^\dagger\gamma_0$. Note that in general one does not write the spinor indices anymore when dealing with Dirac spinors. The role of the Pauli matrices is now played by a set of 4×4 matrices

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \tilde\sigma_\mu & 0 \end{pmatrix} \tag{A.26}$$

[12] I sadly regret the recent loss of Markus Fierz. The necrology is found on www.mink.itp.unibe.ch

that obeys the Dirac algebra

$$\{\gamma_\mu, \gamma_\nu\} \;=\; -2g_{\mu\nu}\;. \tag{A.27}$$

An important notion is that of the "charge conjugated" Dirac spinor:

$$\Psi_C \;\doteq\; \begin{pmatrix} \chi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix} \;=\; C\bar{\Psi}^{\mathrm{T}}\,. \tag{A.28}$$

The so called "charge conjugation" matrix C is unitary and antisymmetric.

$$C \;\doteq\; \begin{pmatrix} \varepsilon_{\alpha\beta} & 0 \\ 0 & \varepsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix} \tag{A.29}$$

It satisfies the identity $C^{-1}\gamma_\mu C = -\gamma_\mu^{\mathrm{T}}$. After introducing the matrix γ_5 and the operators P_L and P_R we can project Dirac spinors on Weyl spinors.

$$\gamma_5 \;\doteq\; -i\gamma_0\gamma_1\gamma_2\gamma_3 \;=\; \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \tag{A.30}$$

$$P_L \;\doteq\; \frac{1}{2}(1+\gamma_5)\,, \qquad\qquad P_R \;\doteq\; \frac{1}{2}(1-\gamma_5)\,. \tag{A.31}$$

Applying P_L and P_R on a Dirac spinor Ψ gives us the left- and right-handed Dirac spinors

$$\Psi_L \;=\; P_L\Psi = \begin{pmatrix} \psi_\alpha \\ 0 \end{pmatrix}, \qquad \Psi_R \;=\; P_R\Psi = \begin{pmatrix} 0 \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}, \tag{A.32}$$

which represent the four component (Dirac) forms of the two Weyl spinors ψ_α and $\bar{\chi}^{\dot{\alpha}}$. A special case of Dirac spinors are Majorana spinors. A Majorana spinor is defined as the four component Dirac spinor where the left and right handed component are the complex conjugate of each other:

$$\Psi_M \;\doteq\; \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix}. \tag{A.33}$$

A Majorana spinor has an interesting "reality" property, namely $\Psi_M = C\bar{\Psi}_m^{\mathrm{T}}$.

Please note that when dealing with Dirac spinors there is a doubling that is not a priori. Just think about the neutrino.

A.3 Properties of Pauli matrices

Because of the index structure of the Pauli matrices (A.6) I can always write a Lorentz vector V_μ with spinor indices. This allows me to switch between the two notations depending on which one is more convenient:

$$V_\mu = -\frac{1}{2}(\tilde{\sigma}_\mu)^{\dot{\alpha}\alpha}V_{\alpha\dot{\alpha}}\,, \qquad V_{\alpha\dot{\alpha}} = (\sigma^\mu)_{\alpha\dot{\alpha}}V_\mu\,. \tag{A.34}$$

The same can be done with a second rank tensor $T_{\mu\nu}$,

$$T_{\alpha\beta\dot{\alpha}\dot{\beta}} = (\sigma^\mu)_{\alpha\dot{\alpha}}(\sigma^\nu)_{\beta\dot{\beta}}T_{\mu\nu}\,, \tag{A.35}$$

which can be decomposed as

$$\begin{aligned} T_{\alpha\beta\dot{\alpha}\dot{\beta}} &= T_{[\alpha\beta]\{\dot{\alpha}\dot{\beta}\}} + T_{\{\alpha\beta\}[\dot{\alpha}\dot{\beta}]} + T_{[\alpha\beta][\dot{\alpha}\dot{\beta}]} + T_{\{\alpha\beta\}\{\dot{\alpha}\dot{\beta}\}}\,. \\ &= \varepsilon_{\alpha\beta}T_{\{\dot{\alpha}\dot{\beta}\}} + \varepsilon_{\dot{\alpha}\dot{\beta}}T_{\{\alpha\beta\}} + \varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}} + T_{\{\alpha\beta\}\{\dot{\alpha}\dot{\beta}\}} \end{aligned} \tag{A.36}$$

If $T_{\mu\nu}$ is symmetric, the first and second term vanish (if it is traceless also the third term drops out) whereas if $T_{\mu\nu}$ is antisymmetric, the third and forth term are zero. In order to write the second case somewhat simpler and to illustrate explicitly the correspondence between an antisymmetric pair of Lorentz indices and two spinor indices I introduce new matrices:

$$(\sigma_{\mu\nu})_\alpha{}^\beta \doteq -\frac{1}{4}(\sigma_\mu\tilde{\sigma}_\nu - \sigma_\nu\tilde{\sigma}_\mu)_\alpha{}^\beta\,, \tag{A.37}$$

$$(\tilde{\sigma}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} \doteq -\frac{1}{4}(\tilde{\sigma}_\mu\sigma_\nu - \tilde{\sigma}_\nu\sigma_\mu)^{\dot{\alpha}}{}_{\dot{\beta}}\,. \tag{A.38}$$

There are some useful identities of the Pauli matrices. Of course there are many other relations of this kind, but I will just list the most important ones that have been used for the calculations in this thesis.

$$\begin{aligned} (\sigma_\mu\tilde{\sigma}_\nu + \sigma_\nu\tilde{\sigma}_\mu)_\alpha{}^\beta &= -2g_{\mu\nu}\delta_\alpha{}^\beta \\ (\tilde{\sigma}_\mu\sigma_\nu + \tilde{\sigma}_\nu\sigma_\mu)^{\dot{\alpha}}{}_{\dot{\beta}} &= -2g_{\mu\nu}\delta^{\dot{\alpha}}{}_{\dot{\beta}} \end{aligned} \tag{A.39}$$

$$\begin{aligned} \mathrm{Tr}\,[\sigma_\mu\tilde{\sigma}_\nu] &= -2g_{\mu\nu} \\ (\sigma^\mu)_{\alpha\dot{\alpha}}(\tilde{\sigma}_\mu)^{\beta\dot{\beta}} &= -2\delta_\alpha{}^\beta\delta_{\dot{\alpha}}{}^{\dot{\beta}} \end{aligned} \tag{A.40}$$

$$\sigma_\mu\tilde{\sigma}_\nu\sigma_\rho = g_{\mu\rho}\sigma_\nu - g_{\nu\rho}\sigma_\mu - g_{\mu\nu}\sigma_\rho + i\varepsilon_{\mu\nu\rho\lambda}\sigma^\lambda$$

$$\tilde{\sigma}_\mu \sigma_\nu \tilde{\sigma}_\rho = g_{\mu\rho}\tilde{\sigma}_\nu - g_{\nu\rho}\tilde{\sigma}_\mu - g_{\mu\nu}\tilde{\sigma}_\rho - i\varepsilon_{\mu\nu\rho\lambda}\tilde{\sigma}^\lambda \tag{A.41}$$

$$\begin{aligned}
(\sigma_\mu)_{\alpha\dot{\alpha}}(\sigma^{\mu\nu})_{\beta\gamma} &= \frac{1}{2}\{\varepsilon_{\alpha\beta}(\sigma^\nu)_{\gamma\dot{\alpha}} - \varepsilon_{\alpha\gamma}(\sigma^\nu)_{\beta\dot{\alpha}}\} \\
(\sigma_\mu)_{\alpha\dot{\alpha}}(\tilde{\sigma}^{\mu\nu})_{\dot{\beta}\dot{\gamma}} &= \frac{1}{2}\{\varepsilon_{\dot{\alpha}\dot{\beta}}(\sigma^\nu)_{\alpha\dot{\gamma}} - \varepsilon_{\dot{\alpha}\dot{\gamma}}(\sigma^\nu)_{\alpha\dot{\beta}}\} \\
(\sigma_{\mu\nu})_{\alpha\beta}(\sigma^{\mu\nu})_{\gamma\delta} &= \varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta} - \varepsilon_{\alpha\delta}\varepsilon_{\beta\gamma} \\
(\tilde{\sigma}_{\mu\nu})_{\dot{\alpha}\dot{\beta}}(\tilde{\sigma}^{\mu\nu})_{\dot{\gamma}\dot{\delta}} &= \varepsilon_{\dot{\alpha}\dot{\gamma}}\varepsilon_{\dot{\beta}\dot{\delta}} - \varepsilon_{\dot{\alpha}\dot{\delta}}\varepsilon_{\dot{\beta}\dot{\gamma}} \\
(\tilde{\sigma}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}(\sigma^{\mu\nu})_\alpha{}^\beta &= 0
\end{aligned} \tag{A.42}$$

$$\begin{aligned}
\sigma_{\mu\nu}\sigma_\rho &= -\frac{1}{2}\{g_{\mu\rho}\sigma_\nu - g_{\nu\rho}\sigma_\mu + i\varepsilon_{\mu\nu\rho\lambda}\sigma^\lambda\} \\
\sigma_\rho\tilde{\sigma}_{\mu\nu} &= -\frac{1}{2}\{g_{\rho\nu}\sigma_\mu - g_{\rho\mu}\sigma_\nu + i\varepsilon_{\mu\nu\rho\lambda}\sigma^\lambda\} \\
\tilde{\sigma}_{\mu\nu}\tilde{\sigma}_\rho &= -\frac{1}{2}\{g_{\mu\rho}\tilde{\sigma}_\nu - g_{\nu\rho}\tilde{\sigma}_\mu + i\varepsilon_{\mu\nu\rho\lambda}\tilde{\sigma}^\lambda\} \\
\tilde{\sigma}_\rho\sigma_{\mu\nu} &= -\frac{1}{2}\{g_{\rho\nu}\tilde{\sigma}_\mu - g_{\rho\mu}\tilde{\sigma}_\nu + i\varepsilon_{\mu\nu\rho\lambda}\tilde{\sigma}^\lambda\}
\end{aligned} \tag{A.43}$$

A.4 Supersymmetry

A.4.1 The covariant derivatives

The ordinary derivative with respect to a Grassmann variable is defined in appendix B.3.2. Through out this diploma work I adopt the shorthand notation

$$\frac{\partial}{\partial\theta_\alpha} \doteq \partial^\alpha\,, \qquad \frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} \doteq \partial^{\dot{\alpha}}\,. \tag{A.44}$$

Sometimes I also use the notation $\partial_M \doteq (\partial_\mu, \partial_\alpha, \bar{\partial}_{\dot{\alpha}})$. The covariant derivative of global supersymmetry is defined as

$$\mathrm{D}_\alpha \doteq \partial_\alpha + i\bar{\theta}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}\,, \qquad \bar{\mathrm{D}}_{\dot{\alpha}} \doteq -\bar{\partial}_{\dot{\alpha}} - i\theta^\alpha\partial_{\alpha\dot{\alpha}}\,. \tag{A.45}$$

One can construct the covariant derivatives D_α and $\bar{\mathrm{D}}_{\dot{\alpha}}$ out of the left and right shifts of elements of $\mathrm{S\Pi}/SO(3,1|\mathbb{R}_c)$ [20, 44]. It then follows that the covariant derivatives, as the name suggests, commute with supersymmetry transformations:

$$\begin{aligned}
\{\mathrm{D}_\alpha, \mathbb{Q}_\beta\} &= 0\,, & \{\mathrm{D}_\alpha, \bar{\mathbb{Q}}_{\dot{\beta}}\} &= 0\,, \\
\{\bar{\mathrm{D}}_{\dot{\alpha}}, \mathbb{Q}_\beta\} &= 0\,, & \{\bar{\mathrm{D}}_{\dot{\alpha}}, \bar{\mathbb{Q}}_{\dot{\beta}}\} &= 0\,, \\
[\mathrm{D}_\alpha, \epsilon\mathbb{Q} + \bar{\epsilon}\bar{\mathbb{Q}}] &= 0\,, & [\bar{\mathrm{D}}_{\dot{\alpha}}, \epsilon\mathbb{Q} + \bar{\epsilon}\bar{\mathbb{Q}}] &= 0\,.
\end{aligned} \tag{A.46}$$

D_α and $\bar{\mathrm{D}}_{\dot\alpha}$ do further have an interesting anticommutation relation as can be easily checked.

$$\{\mathrm{D}_\alpha,\mathrm{D}_\beta\} \;=\; 0\,, \qquad \{\bar{\mathrm{D}}_{\dot\alpha}\bar{\mathrm{D}}_{\dot\beta}\} \;=\; 0\,,$$
$$\{\mathrm{D}_\alpha,\bar{\mathrm{D}}_{\dot\alpha}\} \;=\; -2i\,\partial_{\alpha\dot\alpha}\,. \tag{A.47}$$

Equation (A.47) looks like the algebra of the supersymmetry generators of equation (2.2). Furthermore we have the trivial relations

$$[\mathrm{D}_\alpha,\partial_\mu] \;=\; 0\,, \qquad [\bar{\mathrm{D}}_{\dot\alpha},\partial_\mu] \;=\; 0\,. \tag{A.48}$$

When dealing with integrals over superspace one often makes use of the following relations. The proofs thereof are straight forward but rather cumbersome.

$$[\mathrm{D}^2,\bar{\mathrm{D}}_{\dot\alpha}] \;=\; -4\mathrm{i}\,\partial_{\alpha\dot\alpha}\mathrm{D}^\alpha \qquad [\bar{\mathrm{D}}^2,\mathrm{D}_\alpha] \;=\; 4\mathrm{i}\,\partial_{\alpha\dot\alpha}\bar{\mathrm{D}}^{\dot\alpha} \tag{A.49}$$

$$\mathrm{D}^\alpha\bar{\mathrm{D}}^2\mathrm{D}_\alpha \;=\; \bar{\mathrm{D}}_{\dot\alpha}\mathrm{D}^2\bar{\mathrm{D}}^{\dot\alpha}$$
$$\mathrm{D}^2\bar{\mathrm{D}}^2+\bar{\mathrm{D}}^2\mathrm{D}^2-2\mathrm{D}^\alpha\bar{\mathrm{D}}^2\mathrm{D}_\alpha \;=\; 16\,\Box \tag{A.50}$$

$$\mathrm{D}^2\bar{\mathrm{D}}_{\dot\alpha}\mathrm{D}^2 \;=\; 0 \qquad \bar{\mathrm{D}}^2\mathrm{D}_\alpha\bar{\mathrm{D}}^2 \;=\; 0$$
$$\mathrm{D}^2\bar{\mathrm{D}}^2\mathrm{D}^2 \;=\; 16\mathrm{D}^2\Box \qquad \bar{\mathrm{D}}^2\mathrm{D}^2\bar{\mathrm{D}}^2 \;=\; 16\bar{\mathrm{D}}^2\Box \tag{A.51}$$

As for the normal derivatives I will sometimes use $\mathrm{D}_\mathrm{M} \doteq (\partial_\mu,\mathrm{D}_\alpha,\bar{\mathrm{D}}_{\dot\alpha})$. Note that the box operator is defined as $\Box \doteq \partial_\mu\partial^\mu$ which means that we have $\Box = -\partial_t^2+\triangle_x^2$.

A.4.2 Superfields

A general superfield on $\mathbb{R}^{4|4}$ reads in component form

$$\Sigma(x,\theta,\bar\theta) \;\doteq\; A(x)+\theta^\alpha\psi_\alpha(x)+\bar\theta_{\dot\alpha}\bar\chi^{\dot\alpha}(x)+\theta^2F(x)+\bar\theta^2G(x) \tag{A.52}$$
$$+\theta\sigma^\mu\bar\theta V_\mu(x)+\bar\theta^2\theta^\alpha\lambda_\alpha(x)+\theta^2\bar\theta_{\dot\alpha}\bar\eta^{\dot\alpha}(x)+\theta^2\bar\theta^2D(x)\,. \tag{A.53}$$

Imposing the reality condition

$$\bar A \;=\; A\,,\; (\bar\chi^{\dot\alpha})^* \;=\; \psi^\alpha\,,\; \bar F \;=\; G\,,\; \bar V_\mu \;=\; V_\mu\,,\; (\bar\eta^{\dot\alpha})^* \;=\; \lambda^\alpha\,,\; \bar D \;=\; D\,, \tag{A.54}$$

we get the real superfield

$$V(x,\theta,\bar\theta) \;=\; A(x)+\theta^\alpha\psi_\alpha(x)+\bar\theta_{\dot\alpha}\bar\psi^{\dot\alpha}(x)+\theta^2F(x)+\bar\theta^2\bar F(x)$$

$$+\theta\sigma^\mu\bar{\theta}V_\mu + \bar{\theta}^2\theta^\alpha\lambda_\alpha(x) + \theta^2\bar{\theta}_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}(x) + \theta^2\bar{\theta}^2 D(x)\,. \tag{A.55}$$

How to write the components of Σ or V respectively in terms of Σ or V has already been shown in section 2.2. I state it here again for completeness:

$$\begin{aligned}
A(x) &= V|\,, \qquad \psi_\alpha(x) = \mathrm{D}_\alpha V|\,, \qquad \bar{\psi}_{\dot{\alpha}}(x) = \bar{\mathrm{D}}_{\dot{\alpha}} V|\,,\\
F(x) &= -\frac{1}{4}\mathrm{D}^2 V|\,, \qquad \bar{F}(x) = -\frac{1}{4}\bar{\mathrm{D}}^2 V|\,, \qquad V_{\alpha\dot{\alpha}}(x) = \frac{1}{2}[\mathrm{D}_\alpha, \bar{\mathrm{D}}_{\dot{\alpha}}]V|\,,\\
\lambda_\alpha(x) &= -\frac{1}{4}\mathrm{D}_\alpha\bar{\mathrm{D}}^2 V|\,, \qquad \bar{\lambda}_{\dot{\alpha}}(x) = -\frac{1}{4}\bar{\mathrm{D}}_{\dot{\alpha}}\mathrm{D}^2 V|\,,\\
D(x) &= \frac{1}{32}\{\mathrm{D}^2, \bar{\mathrm{D}}^2\}V|\,.
\end{aligned} \tag{A.56}$$

Here I used the notation

$$V|\,, \quad \mathrm{D}_\alpha V|\,, \quad \ldots\,, \tag{A.57}$$

which means that the whole expression has to be evaluated at $\theta = \bar{\theta} = 0$ i.e. one projects onto ordinary space-time. It is more convenient to write the components of a superfield with the covariant derivatives instead of the normal θ-derivatives ∂_α and $\partial_{\dot{\alpha}}$. This is possible without any complication since we have

$$\mathrm{D}_\alpha V| = \left(\partial_\alpha + i\bar{\theta}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}\right)V| = \partial_\alpha V|\,. \tag{A.58}$$

The θ-expansion of a chiral superfield, i.e $\bar{\mathrm{D}}_{\dot{\alpha}}\Phi(x,\theta,\bar{\theta}) = 0$, is

$$\begin{aligned}
\Phi(x,\theta,\bar{\theta}) &= \exp(i\theta\sigma^\mu\bar{\theta}\partial_\mu)\Phi(x,\theta)\\
&= A(x) + \theta^\alpha\psi_\alpha + \theta^2 F(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu A(x)\\
&\quad + \frac{i}{2}\theta^2\bar{\theta}\tilde{\sigma}^\mu\partial_\mu\psi(x) + \frac{1}{4}\theta^2\bar{\theta}^2\Box A(x)
\end{aligned} \tag{A.59}$$

and its components read

$$A(x) = \Phi|\,, \qquad \psi_\alpha(x) = \mathrm{D}_\alpha\Phi|\,, \qquad F(x) = -\frac{1}{4}\mathrm{D}^2\Phi|\,. \tag{A.60}$$

A.4.3 Superspace integration

In appendix B.3.3 the integration of Grassmann variables is introduced. In supersymmetric field theories one generally deals with integrals of the form

$$\int d^8z\, V(x,\theta,\bar\theta) \doteq \int d^4x\, d^2\theta\, d^2\bar\theta\, V(x,\theta,\bar\theta)\,, \tag{A.61}$$

where $d^2\theta \doteq \frac{1}{4}\varepsilon^{\alpha\beta}\, d\theta_\alpha d\theta_\beta$ and $d^2\bar\theta \doteq \frac{1}{4}\varepsilon_{\dot\alpha\dot\beta}\, d\bar\theta^{\dot\alpha} d\bar\theta^{\dot\beta}$. The difference in the definition of the measure $d^2\theta$ to what is defined in appendix B.3.3 is because I want to have the property

$$\int d^8z\, V(z) = \int d^4x\, D(x)\,, \tag{A.62}$$

where $D(x)$ is the highest component of the multiplet $V(z)$. This can be shown by using the fact that the integration over a Grassmann variable corresponds to the derivative with respect to the integration variable (appendix B.3.3). This fact allows us to write

$$\int d^8z\, V(z) = -\frac{1}{4}\int d^4x\, d^2\theta\, \bar{\mathrm{D}}^2 V(z) = \int d^4x\, \left(\frac{1}{16}\mathrm{D}^2\bar{\mathrm{D}}^2\mathrm{V(z)}|\right)\,. \tag{A.63}$$

Please note that if we have an action in terms of superfields, i.e $S = \int d^8z\, \mathscr{L}$, the Lagrangian $S = \int d^4x\, \mathcal{L}$ is the highest component of the multiplet $\mathscr{L}$.

It is always assumed that a superfield vanishes at spacetime infinity i.e. for $x_\mu \to \infty$. This means that $\int d^8z\, \partial_M f(z) = 0$.

Proof.

$$\begin{aligned} \int d^8z\, \partial_\mu V(z) &= \int d^4x\, \partial_\mu D(x) = \text{integral over the surface of } \mathbb{R}^4 = 0 \\ \int d^8z\, \partial_\alpha V(z) &= \int d^4x\, d^2\theta d^2\,\bar\theta\, \left[\text{const.} + \text{terms less than } \mathcal{O}(\theta^2\bar\theta^2)\right] = 0 \end{aligned} \tag{A.64}$$

and analogously for $\dot\alpha$. □

This relation, which gives us a rule for integrating by parts, can be extended to covariant derivatives.

$$\int d^8z\, \mathrm{D}_M V(z) = \left\{ \begin{array}{l} \int d^8z\, \partial_\mu V(z) = 0 \\ \int d^8z\, \partial_\alpha V(z) + i\int d^4z\, \partial_\mu(\sigma^\mu\bar\theta)_\alpha V(z) = 0 \\ \text{analogously for } \dot\alpha \end{array} \right\} = 0 \tag{A.65}$$

I further define the "3/4" superspace integration:

$$\int d^6z\, V_c(x,\theta) \doteq \int d^4x\, d^2\theta\, V_c(x,\theta)\,, \tag{A.66}$$

$$\int d^6\bar{z}\, V_c(x,\bar{\theta}) \;\doteq\; \int d^4x\, d^2\bar{\theta}\, V_c(x,\bar{\theta})\,. \tag{A.67}$$

Lastly, the superspace delta function $\delta^8(z)$ is defined as

$$\begin{aligned} \delta^8(z) \;&\doteq\; \delta^4(x)\,\delta^2(\theta)\,\delta^2(\bar{\theta})\,, \\ \delta^2(\theta) \;&\doteq\; \theta^2\,, \qquad\qquad \delta^2(\bar{\theta}) \;\doteq\; \bar{\theta}^2\,. \end{aligned} \tag{A.68}$$

Appendix B

Supernumbers and Superanalysis

In this appendix a brief discussion of the dealing with anticommutating numbers shall be given. Especially, I would like to give the exact definitions of the tools used in this work such as derivatives and integration with respect to anticommutating objects.

For a detailed and pedagogical treatment of supernumbers and basic superanalysis well suited for physicists see [20]. For those interested in a rigorous mathematical construction of superanalysis there is the book by F. Berezin [55].

B.1 Grassmann algebra

The Grassmann algebra Λ_N is an associative algebra generated by N linearly independent anticommutating elements ξ^i

$$\left\{\xi^i, \xi^j\right\} = 0 \qquad i, j = 1, ..., N\,. \tag{B.1}$$

For infinitely many ξ $(N \to \infty)$ we get the Grassmann algebra Λ_∞. The elements of Λ_∞ are called supernumbers. They can be written as

$$\begin{aligned} z &= z_B + z_S \qquad \text{with} \\ z_S &= \sum_{k=1}^{\infty} \frac{1}{k!} C_{i_1 i_2 \cdots i_k}\, \xi^{i_1} \xi^{i_2} \cdots \xi^{i_k}\,, \\ & \quad z_B,\, C_{i_1 i_2 \cdots i_k} \in \mathbb{C}\,. \end{aligned} \tag{B.2}$$

z_B is called the body, z_S the soul of the supernumber z. A bodyless supernumber is called a "pure" supernumber. Every supernumber can be decomposed into an "even" part z_c and an "odd" part z_a:

$$z = z_c + z_a\,,$$

$$
\begin{aligned}
z_c &= z_B + \sum_{k=1}^{\infty} \frac{1}{(2k)!} C_{i_1 i_2 \cdots i_{2k}} \xi^{i_1} \xi^{i_2} \cdots \xi^{i_{2k}} \,, \\
z_a &= \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} C_{i_1 i_2 \cdots i_{2k+1}} \xi^{i_1} \xi^{i_2} \cdots \xi^{i_{2k+1}} \,. \qquad \text{(B.3)}
\end{aligned}
$$

A supernumber that has z_c or z_a equal zero is called an "a-" or "c-number", respectively. The two sets containing all the a- or c-numbers are denoted as $\mathbb{C}_a$ ($\mathbb{R}_a$) and $\mathbb{C}_c$ ($\mathbb{R}_c$) respectively. Obviously $\mathbb{C}_c$ ($\mathbb{R}_c$) is a commutative subalgebra of Λ_∞, whereas $\mathbb{C}_a$ ($\mathbb{R}_a$) is not closed since the product of two a-numbers gives a bodyless c-number.

It is possible to define the operation of complex conjugation on Λ_∞ by

$$
z^* = z^* + \sum_{k=1}^{\infty} \frac{1}{k!} C^*_{i_1 i_2 \ldots i_k} \xi^{i_k} \ldots \xi^{i_2} \xi^{i_1} . \qquad \text{(B.4)}
$$

Hence we can talk about a real ($z^* = z$) or imaginary ($z^* = -z$) supernumber.

B.2 Supervector spaces

A supervector space is defined as a set $\mathscr{V}$ of elements called "supervectors" provided with a binary operation, a left and right multiplication by a supernumber and eventually a complex conjugation. The following axioms must be satisfied.

(i) $\vec{X} + \vec{Y} = \vec{Y} + \vec{X} \qquad \forall \vec{X}, \vec{Y} \in \mathscr{V}$

(ii) $(\vec{X} + \vec{Y}) + \vec{Z} = \vec{X} + (\vec{Y} + \vec{Z}) \qquad \forall \vec{X}, \vec{Y}, \vec{Z} \in \mathscr{V}$

(iii) $\exists\, \vec{0} \in \mathscr{V}$ such that $\vec{X} + \vec{0} = \vec{X} \qquad \forall \vec{X} \in \mathscr{V}$

(iv) $\forall \vec{X} \in \mathscr{V} \quad \exists\, \vec{Y} \in \mathscr{V}$ such that $\vec{X} + \vec{Y} = \vec{0}$

(v) $\forall z_1 z_2 \in \Lambda_\infty$ and $\forall \vec{X}, \vec{Y} \in \mathscr{V}$ we have

$$
\begin{gathered}
(z_1 + z_2)\vec{X} = z_1\vec{X} + z_2\vec{X} \qquad \vec{X}(z_1 + z_2) = \vec{X}z_1 + \vec{X}z_2 \\
z_1(\vec{X} + \vec{Y}) = z_1\vec{X} + z_1\vec{Y} \qquad (\vec{X} + \vec{Y})z_1 = \vec{X}z_1 + \vec{Y}z_1 \\
(z_1 z_2)\vec{X} = z_1(z_2\vec{X}) \qquad \vec{X}(z_1 z_2) = (\vec{X}z_1)z_2 \\
1\vec{X} = \vec{X} \qquad \vec{X}1 = \vec{X} \qquad 0\vec{X} = \vec{X}0 = \vec{0} \qquad z_1\vec{0} = \vec{0}z_1 = \vec{0} \\
\vec{X} + (-1)\vec{X} = \vec{X} + \vec{X}(-1) = \vec{0}
\end{gathered}
$$

(vi) The left and right multiplication must satisfy

- $(z_1\vec{X})z_2 = z_1(\vec{X}z_2) \quad \forall\, z_1,\, z_2 \in \Lambda_\infty \quad \forall \vec{X} \in \mathscr{V}$
- $z\vec{X} = \vec{X}z \quad \forall\, z \in \mathbb{C}_c \quad \forall\, \vec{X} \in \mathscr{V}$
- $\forall\, \vec{X} \in \mathscr{V} \quad \exists! \quad \vec{X}_c,\, \vec{X}_a \in \mathscr{V}$ such that

$$\vec{X} = \vec{X}_c + \vec{X}_a \qquad z\vec{X}_c = \vec{X}_c z \qquad z\vec{X}_a = -\vec{X}_a z \qquad \forall\, z \in \mathbb{C}_a$$

(vii) For arbitrary $z \in \Lambda_\infty$ and $\vec{X},\, \vec{Y} \in \mathscr{V}$ the complex conjugation is constructed by

$$\vec{X}^{**} = \vec{X} \qquad (\vec{X} + \vec{Y})^* = \vec{X}^* + \vec{Y}^* \qquad (z\vec{X})^* = \vec{X}^* z^* \qquad (\vec{X}z)^* = z^* \vec{X}^*$$

In analogy to the case of supernumbers, the supervectors $\vec{X}_c$ and $\vec{X}_a$ are called the "even" and "odd" part of $\vec{X}$. Let me define now the Grassmann parity ε for supernumbers z and supervectors $\vec{X}$:

$$\varepsilon(z) = \begin{cases} 0 & \text{if } z \text{ even} \\ 1 & \text{if } z \text{ odd} \end{cases} \qquad \varepsilon(\vec{X}) = \begin{cases} 0 & \text{if } \vec{X} \text{ even} \\ 1 & \text{if } \vec{X} \text{ odd} \end{cases} \tag{B.5}$$

For a pure supernumber z and an arbitrary pure supervector $\vec{X}$ we then have the identity

$$z\vec{X} = (-1)^{\varepsilon(z)\varepsilon(\vec{X})}\vec{X}z\,. \tag{B.6}$$

In complete analogy to the case of an ordinary vector space, a set $\{\vec{e}_M\} \subseteq \mathscr{V}$ is said to be linearly independent if for $z^M_{(\pm)} \in \Lambda_\infty$

$$\vec{e}_M z^M_{(+)} = z^M_{(-)}\vec{e}_M = 0 \quad \Rightarrow \quad z^M_{(\pm)} = 0 \;\; \forall\, M = 1, 2, \ldots, d \tag{B.7}$$

$\mathscr{V}$ is finite dimensional if every supervector $\vec{X} \in \mathscr{V}$ can be written as $\vec{X} = \vec{e}_M z^M_{(+)} = z^M_{(-)}\vec{e}_M$ with a finite number of vectors $\vec{e}_M$, i.e $d < \infty$. Obviously $\{\vec{e}_M\}$ is a basis for $\mathscr{V}$. It is possible to prove that there always exists a pure basis $\vec{E}_M$. That is a set with p even and q odd linearly independent supervectors ($p + q = \dim(\mathscr{V})$). A pure basis will be denoted as $\vec{E}_M = (\vec{E}_\mu, \vec{E}_\alpha)$ with $\mu = 1, 2, \ldots, p$ and $\alpha = 1, 2, \ldots, q$. I can now write every supervector $\vec{X} \in \mathscr{V}_c$ (c-type or even supervector) as

$$\vec{X} = y^\mu \vec{E}_\mu + \theta^\alpha \vec{E}_\alpha \qquad y^\mu \in \mathbb{C}_c \qquad \theta^\alpha \in \mathbb{C}_a\,. \tag{B.8}$$

Thus I have found a one-to-one correspondence between $\mathscr{V}_c$ and the space $\mathbb{C}^{p|q} \doteq \mathbb{C}^p_c \times \mathbb{C}^q_a$. More precisely

$$\mathbb{C}^{p|q} = \{(y^1,\, y^2, \cdots, y^p, \theta^1,\, \theta^2, \cdots, \theta^q);\; y^\mu \in \mathbb{C}_c,\, \theta^\alpha \in \mathbb{C}_a\}\,. \tag{B.9}$$

If y^μ and θ^α are real supernumbers, I get the space $\mathbb{R}^{p|q}$. The construction for $\mathscr{V}_a$ (a-type or odd supervectors) is analogous to the one made above for $\mathscr{V}_c$.

Supermatrices

A supermatrix is a matrix whose elements are supernumbers. Since I can decompose every supernumber in its body and soul this is also possible for supermatrices. A supermatrix F is invertible if its body F_B is invertible ($\det(F_B) \neq 0$).

Proof. If $\det(F_B) \neq 0$, F can be written as $F = F_B(\mathbb{1} + F_B^{-1}F_S)$. $F_B^{-1}F_S$ has no body and therefore there exists some integer n with $(F_B^{-1}F_S)^n = 0$ (since it is made of anticommutating elements). The inverse of F is then

$$F^{-1} = F_B^{-1} + \sum_{k=1}^{n}(-1)^k(F_B^{-1}F_S)^k F_B^{-1}\,. \tag{B.10}$$

$$\begin{aligned}
FF^{-1} &= (F_B + F_S)\left(F_B^{-1} + \sum_{k=1}^{n}(-1)^k(F_B^{-1}F_S)^k F_B^{-1}\right) \\
&= \mathbb{1} + \sum_{k=1}^{n}(-1)^k F_B(F_B^{-1}F_S)^k F_B^{-1} + F_S F_B^{-1} + \sum_{k=1}^{n}(-1)^k F_S(F_B^{-1}F_S)^k F_B^{-1} \\
&= \mathbb{1} - F_S F_B^{-1} + \sum_{k=2}^{n}(-1)^k F_B(F_B^{-1}F_S)^k F_B^{-1} + F_S F_B^{-1} + \\
&\quad \sum_{k=1}^{n}(-1)^k F_S(F_B^{-1}F_S)^k F_B^{-1} \\
&= \mathbb{1} + \sum_{k=2}^{n}(-1)^k F_B(F_B^{-1}F_S)^k F_B^{-1} + (-1)\sum_{k=1}^{n}(-1)^{k+1} F_B(F_B^{-1}F_S)^{k+1} F_B^{-1} \\
&= \mathbb{1} + \sum_{k=1}^{n}(-1)^k F_B(F_B^{-1}F_S)^k F_B^{-1} + (-1)\sum_{k=2}^{n}(-1)^k F_B(F_B^{-1}F_S)^k F_B^{-1} \\
&= \mathbb{1} \qquad \text{analogously for } F^{-1}F \;=\; \mathbb{1}
\end{aligned}$$

□

Every invertible supermatrix G can be used to perform a change of basis in a supervector space:

$$\vec{e}\,'_M = G_M{}^N \vec{e}_N \tag{B.11}$$

If I have two pure bases $\vec{E}_M$ and $\vec{E}'_M$ related to each other by the supermatrix G, I can write G as

$$G_M{}^N = \begin{pmatrix} A_\mu{}^\nu & B_\mu{}^\beta \\ C_\alpha{}^\nu & D_\alpha{}^\beta \end{pmatrix} \qquad \text{with} \qquad A_\mu{}^\nu,\, D_\alpha{}^\beta \in \mathbb{C}_c \quad B_\mu{}^\beta,\, C_\alpha{}^\nu \in \mathbb{C}_a\,. \tag{B.12}$$

Left and right linear operator $\mathscr{F}_l$ and $\mathscr{F}_r$ on $\mathscr{V}$ are defined as

- $\mathscr{F}_l : \mathscr{V} \to \mathscr{V}$ with
  $\mathscr{F}_l(\vec{X}+\vec{Y}) = \mathscr{F}_l(\vec{X}) + \mathscr{F}_l(\vec{Y}) \qquad \forall \vec{X}, \vec{Y} \in \mathscr{V}$
  $\mathscr{F}_l(\vec{X}z) = \mathscr{F}_l(\vec{X})z \qquad \forall z \in \Lambda_\infty \quad \vec{X} \in \mathscr{V}$

- $\mathscr{F}_r : \mathscr{V} \to \mathscr{V}$ with
  $(\vec{X}+\vec{Y})\mathscr{F}_r = (\vec{X})\mathscr{F}_r + (\vec{Y})\mathscr{F}_r \qquad \forall \vec{X}, \vec{Y} \in \mathscr{V}$
  $(z\vec{X})\mathscr{F}_r = z(\vec{X})\mathscr{F}_r \qquad \forall z \in \Lambda_\infty \quad \vec{X} \in \mathscr{V}$

To every left (analogously for a right) linear operator $\mathscr{F}_l$ on $\mathscr{V}$ I can associate a supermatrix as follows ($\vec{e}_M$ is a basis of $\mathscr{V}$):

$$\begin{aligned} \mathscr{F}_l(\vec{e}_N) &= \vec{e}_M F^M{}_N \quad (\neq F^M{}_N \vec{e}_M\,!) \\ \Rightarrow \mathscr{F}_l(\vec{X}) &= \vec{X}' = \vec{e}_M z'^M_{(+)} \quad z'^M_{(+)} = F^M{}_N z^N_{(+)}\,. \end{aligned} \tag{B.13}$$

Denote the left (right) dual supervector space to $\mathscr{V}$ as ${}^*\mathscr{V}$ ($\mathscr{V}^*$). Elements of ${}^*\mathscr{V}$ ($\mathscr{V}^*$) are called "left (right) super 1-forms".

The supertranspose of a left (right) linear operator is a one-to-one mapping of the space of left (right) linear operators on $\mathscr{V}$ to the space of linear operators on ${}^*\mathscr{V}$ ($\mathscr{V}^*$). For a pure basis $\{\vec{E}_M\}$ of $\mathscr{V}$ its dual basis is $\{\mathbf{E}^M\}$ satisfying $\vec{E}_M \cdot \mathbf{E}^N = \delta^N_M$. The supertranspose is

$$\begin{aligned} \mathscr{F}_l(\vec{E}_M) &= \vec{E}_N F^N{}_M \\ \mathscr{F}_l^{sT}(\mathbf{E}^M) &= \mathbf{E}^N (F^{sT})_N{}^M\,, \end{aligned} \tag{B.14}$$

where

$$(F^{sT})_M{}^N = (-1)^{\varepsilon(\mathscr{F})(\varepsilon_M+\varepsilon_N)+\varepsilon_M+\varepsilon_M\varepsilon_N} F^N{}_M\,, \tag{B.15}$$

$$\varepsilon_M = \begin{cases} 0 & \text{if } M=\mu \\ 1 & \text{if } M=\alpha \end{cases} \qquad \varepsilon(\mathscr{F}) = \begin{cases} 0 & \text{if } \mathscr{F}: \mathscr{V}_c \to \mathscr{V}_c \text{ or } \mathscr{V}_a \to \mathscr{V}_a \\ 1 & \text{if } \mathscr{F}: \mathscr{V}_c \to \mathscr{V}_a \text{ or } \mathscr{V}_a \to \mathscr{V}_c \end{cases}\,. \tag{B.16}$$

The properties of the supertransposition are

$$\begin{aligned} (\mathscr{F}_1 + \mathscr{F}_2)^{sT} &= \mathscr{F}_1^{sT} + \mathscr{F}_2^{sT}\,, \\ (\mathscr{F}_1 \cdot \mathscr{F}_2)^{sT} &= (-1)^{\varepsilon(\mathscr{F}_1)\varepsilon(\mathscr{F}_2)} \mathscr{F}_1^{sT} \mathscr{F}_2^{sT}\,, \\ (\mathscr{F}^{sT})^{sT} &= \mathscr{F}\,. \end{aligned} \tag{B.17}$$

In order to define the superdeterminant (also called "Berezinian" after F. Berezin) let us write a invertible supermatrix (suppressing the indices) as

$$F = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\,. \tag{B.18}$$

The superdeterminant is then defined as

$$\mathrm{sdet}(F) \equiv \mathrm{Ber}(F) = \det(A - BD^{-1}C)\det^{-1}(D) = \det(A)\det^{-1}(D - CA^{-1}B)\,. \quad \text{(B.19)}$$

"det" is defined as the usual determinant for complex n×n matrices with the complex numbers replaced by pure supernumbers. This rather special definition is needed in order to have the right transformation properties for integrals (Jacobian) and the property

$$\mathrm{Ber}(F^{sT}) = \mathrm{Ber}(F)\,. \quad \text{(B.20)}$$

B.3 Superanalysis

B.3.1 Superfunctions

The well known definition of "analytic" functions on $\mathbb{R}^n$ (or $\mathbb{C}^n$) is to be generalized to supernumber-valued functions. The case of complex supernumbers is not treated here since it is analogous to the real one. The function $f : \mathbb{R}^{p|q} \to \Lambda_\infty$ is called "superanalytic" if it can be expanded in a Taylor series

$$\begin{aligned} f(z) &= f(x^1, \ldots, x^p, \theta^1, \ldots, \theta^q) \\ &= \sum_{k=0}^{\infty} f_{M_1 \ldots M_k} z^{M_1} \cdots z^{M_k}\,, \qquad f_{M_1,\ldots,M_k} \in \Lambda_\infty\,. \end{aligned} \quad \text{(B.21)}$$

This rather complicated definition has some very nice features for special cases. The most simple case is $p = 0,\ q = 1$ because the expansion is just

$$f(\theta) = \varphi + \psi\theta\,, \qquad \varphi,\, \psi \in \Lambda_\infty\,. \quad \text{(B.22)}$$

For $p = 0$ but $q \neq 0$ I have

$$f(\theta^1, \ldots, \theta^q) = f_0 + \sum_{k=1}^{q} \frac{1}{k!} f_{[\alpha_1 \ldots \alpha_k]} \theta^{\alpha_1} \cdots \theta^{\alpha_k}\,, \qquad f_0,\, f_{[\alpha_1 \ldots \alpha_k]} \in \Lambda_\infty\,. \quad \text{(B.23)}$$

where $f_{[\alpha_1 \ldots \alpha_k]}$ is totally antisymmetric in its indices in order not to make any summand vanish. For $p \neq 0$ the whole thing becomes somewhat more complicated. Have a look at the function $\tilde{f} : \mathbb{R} \to \mathbb{R}$ of ordinary real variables. If I split the variable $z \in \mathbb{R}_c$, i.e. $p = 1$ and $q = 0$, into its body and soul I can write

$$\tilde{f}(z_B) = \sum_{k=0}^{\infty} \tilde{f}_k \cdot (z_B)^k\,, \qquad \tilde{f}_k \in \mathbb{R}\,,$$

$$\check{f}(z = z_B + z_S) \;=\; \tilde{f}(z_B) + \sum_{k=0}^{m} \frac{1}{k!} \tilde{f}^{(k)}(z_B) \cdot (z_S)^k \,, \tag{B.24}$$

where $\tilde{f}^{(k)}$ is the k^{th} derivative of $\tilde{f}$ and m is the number for which $(z_s)^{m+1} = 0$. $\check{f}$ is now said to be the super extension of $\tilde{f}$ and it is analytic on $\mathbb{R}_c$. Every superanalytic function on $\mathbb{R}_c$ has now the general form

$$f(z) = \sum_{k=0}^{\infty} \check{f}_{i_1 \ldots i_k}(z)\, \xi^{i_1} \ldots \xi^{i_k} \,, \tag{B.25}$$

with $\check{f}_{i_1 \ldots i_k}$ being the super extensions of some (ordinary) functions $\tilde{f}_{i_1 \ldots i_k}$. It is now straight forward to extend this concept to $\mathbb{R}_c^p$. Finally, every superanalytic function $f : \mathbb{R}^{p|q} \to \Lambda_\infty$ ($p \neq 0$, $q \neq 0$) can be written as

$$f(x^1, \ldots, x^p, \theta^1, \ldots, \theta^q) = f_0(x^1, \ldots, x^p) + \sum_{k=1}^{q} \frac{1}{k!} f_{[\alpha_1 \ldots \alpha_k]}(x^1, \ldots, x^p)\, \theta^{\alpha_1} \cdots \theta^{\alpha_k} \,, \tag{B.26}$$

with $f_{[\alpha_1 \ldots \alpha_k]}(x^1, \ldots, x^p)$ being superanalytic functions on $\mathbb{R}_c^p$ with totally antisymmetric index structure.

B.3.2 Derivatives with respect to supernumbers

With the definitions of superanalytic functions given in the previous section I am now able to define derivatives of super functions. For the function $\check{f}(z)$ defined in equation (B.24) the n^{th} derivative with respect to $z \in \mathbb{R}_c$ is

$$\check{f}^{(\text{n})}(z) = \tilde{f}^{(\text{n})}(z_B) + \sum_{k=1}^{m} \frac{1}{k!} \tilde{f}^{(\text{n+k})}(z_B)(z_S)^k \,. \tag{B.27}$$

The partial derivatives of the superanalytic function $f(x^1, \ldots, x^p, \theta^1, \ldots, \theta^q)$ of equation (B.26) are then given by

$$\begin{aligned}
\frac{\partial}{\partial x^\mu} f(x^1, \ldots, x^p, \theta^1, \ldots, \theta^q) \;&=\; \frac{\partial}{\partial x^\mu} f_0(x^1, \ldots, x^p) \\
&\quad + \sum_{k=1}^{q} \frac{1}{k!} \left(\frac{\partial}{\partial x^\mu} f_{[\alpha_1 \ldots \alpha_k]}(x^1, \ldots, x^p) \right) \theta^{\alpha_1} \cdots \theta^{\alpha_k} \,, \\
\frac{\partial}{\partial \theta^\beta} f(x^1, \ldots, x^p, \theta^1, \ldots, \theta^q) \;&=\; (-1)^{\varepsilon(f)} \sum_{k=1}^{q} (-1)^k \frac{1}{(k-1)!} f_{[\beta \alpha_1 \ldots \alpha_{k-1}]}(x^1, \ldots, x^p) \cdot \\
&\quad \theta^{\alpha_1} \cdots \theta^{\alpha_{k-1}} \,, \\
\text{with} \qquad \varepsilon(f) \;&=\; \begin{cases} 0 & \text{if } f : \mathbb{R}^{p|q} \to \mathbb{C}_c \text{ (bosonic)} \\ 1 & \text{if } f : \mathbb{R}^{p|q} \to \mathbb{C}_a \text{ (fermionic)} \end{cases} \,.
\end{aligned} \tag{B.28}$$

B.3.3 Integration of supernumbers

In this section I will only look at the integration over $\mathbb{R}^{p|q}$. For details about complex superspace $\mathbb{C}^{p|q}$ see [55]. As in equation (B.24) $\check{f}$ is defined as function over $\mathbb{R}_c$ ($\check{f} : \mathbb{R}_c \rightarrow \Lambda_\infty$). I define the integral $\int_{z_1}^{z_2} dz \check{f}(z)$ with z_1, $z_2 \in \mathbb{R}_c$ trough the extension (B.24).

$$\tilde{F}(z_B) = \int_0^{z_B} dz'_B\, \tilde{f}(z'_B)$$

$$\begin{aligned} \check{F}(z) &= \tilde{F}(z_B) + \sum_{k=1}^{\infty} \frac{1}{k!} \tilde{F}^{(k)}(z_B) \cdot (z_S)^k \\ &= \tilde{F}(z_B) + \sum_{k=1}^{\infty} \frac{1}{k!} \tilde{f}^{(k-1)}(z_B) \cdot (z_S)^k \end{aligned}$$

$$\Rightarrow \qquad \check{F}'(z) = \check{f}(z)$$

$$\Rightarrow \qquad \int_{z_1}^{z_2} dz\, \check{f}(z) = \check{F}(z_2) - \check{F}(z_1)\,. \tag{B.29}$$

As for the integration over $\mathbb{R}$, integrals over $\mathbb{R}_c^p$ are sequences of integrals over $\mathbb{R}_c$ defined in equation (B.29).

The integration over $\mathbb{R}_a$ is somewhat simpler and has some unexpected properties. In this case I cannot "construct" the integral $\int d\theta f(\theta)$ like I did before. I can just stipulate some basic properties i.e. the following axioms:

$$\int d\theta\, [f(\theta) + h(\theta)] = \int d\theta f(\theta) + \int d\theta h(\theta)\,, \tag{B.30}$$

$$\int d\theta\, z f(\theta) = (-1)^{\varepsilon(z)} z \int d\theta f(\theta)\,, \tag{B.31}$$

$$\int d\theta\, \frac{d}{d\theta} f(\theta) = 0\,, \tag{B.32}$$

$$\int d\theta\, \theta = 1\,. \tag{B.33}$$

The first two axioms are established in analogy to the linearity of ordinary integrals. (B.32) takes into account that $\mathbb{R}_a$ is boundless and (B.33) plays the role of a normalization (which is absolutely arbitrary). The requirement of the invariance under translations on $\mathbb{R}_a$ causes the "volume" of $\mathbb{R}_a$ to be zero : $\int d\theta = 0$.

$$\int d\theta f(\theta + \eta) = \int d\theta\, [A + B(\theta + \eta)] = \int d\theta\, [A + B\eta] + \int d\theta B\theta = \int d\theta f(\theta) \tag{B.34}$$

$$\int d\theta f(\theta + \eta) \doteq \int d\theta f(\theta) = \int d\theta\, [A + B\theta] \tag{B.35}$$

$$\Rightarrow \int d\theta \, [A + B\eta] = \int d\theta A = 0 \tag{B.36}$$

These axioms imply also the following properties:

$$\int d\theta f(\theta) = \frac{d}{d\theta} f(\theta) \tag{B.37}$$

$$\int d\theta \, \theta f(\theta) = f(0) \qquad \Rightarrow \qquad \delta(\theta) \doteq \theta \tag{B.38}$$

Integrating over $\mathbb{R}_a^q$ is again understood as sequence of integrations over $\mathbb{R}_a$ defined in this section.

As shown in [20] the Jacobian for superspace integrals is the Berezinian defined in (B.19).

Appendix C

Feynman Rules

C.1 N=1 super Yang-Mills theory

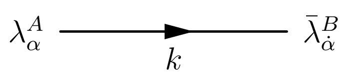
The Feynman rules for the super Yang-Mills theory (2.69) are presented in what follows. How to derive them from the Lagrangian of the theory can be found in textbooks on quantum field theories such as [56]. The Lagrangian of equation (2.69) reads

$$\mathcal{L}_{SYM} = \mathrm{Tr}\left[-\frac{1}{4}G^{\mu\nu}G_{\mu\nu} - i\lambda^{\alpha}\mathrm{D}_{\alpha\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}} + 2D^2\right] . \tag{C.1}$$

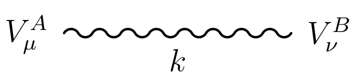
As D is an auxiliary field, it does neither have a propagator nor is it interacting with the gauge boson or gaugino. Thus there are no Feynman rules for D.

C.1.1 Propagators

I use the Feynman gauge for the propagator of the gauge boson.



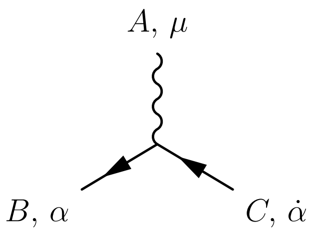
$$\equiv \delta^{AB}\,\frac{i(\sigma^{\mu})_{\alpha\dot{\alpha}}\,k_{\mu}}{k^2 + i\epsilon} \tag{C.2}$$



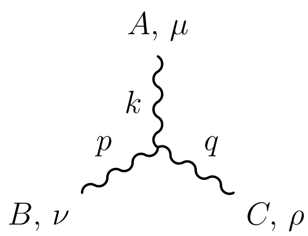
$$\equiv \delta^{AB}\,\frac{g_{\mu\nu}}{k^2 + i\epsilon} \tag{C.3}$$

C.1.2 Vertices

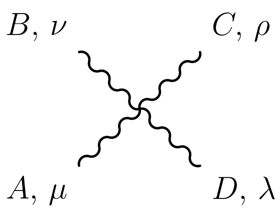
As for the propagators, I do only present the momentum space Feynman rules since it is more convenient to work therein.



$$\equiv \quad -g(\sigma^{\mu})_{\alpha\dot\alpha} f^{ABC} \tag{C.4}$$



$$\begin{aligned} \equiv \quad g f^{ABC}\Big[& g^{\mu\nu}(p-k)^{\rho} + g^{\nu\rho}(q-p)^{\mu} \\ & + g^{\mu\rho}(k-q)^{\nu}\Big] \end{aligned} \tag{C.5}$$



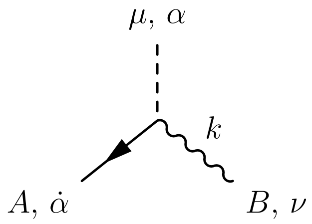
$$\begin{aligned} \equiv \quad -ig^2\Big[& f^{ADE} f^{BCE}(g^{\mu\nu}g^{\rho\lambda} - g^{\mu\rho}g^{\nu\lambda} \\ & + f^{ABE} f^{CDE}(g^{\mu\rho}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\rho}) \\ & + f^{ACE} f^{BDE}(g^{\mu\nu}g^{\rho\lambda} - g^{\mu\lambda}g^{\nu\rho})\Big] \end{aligned} \tag{C.6}$$

C.2 Supercurrent

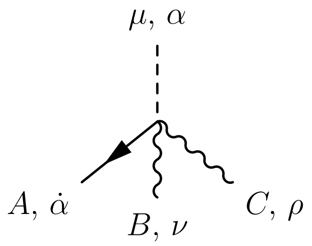
The supercurrent of equation (2.78)

$$S^{\mu}{}_{\alpha} \;=\; -2(\sigma_{\nu})_{\alpha\dot\alpha} \mathrm{Tr}\left[G^{\mu\nu}\bar\lambda^{\dot\alpha}\right] \tag{C.7}$$

is not found directly in the Lagrangian. Therefore it does only have the following two vertices.



$$\equiv \quad 2i\,(\sigma^{\rho})_{\alpha\dot{\alpha}}\,\delta^{AB}\left(k_{\mu}\,\delta_{\nu\rho} - k_{\rho}\,\delta_{\mu\nu}\right) \tag{C.8}$$



$$\equiv \quad -2g\,(\sigma^{\lambda})_{\alpha\dot{\alpha}}\,f^{ABC}\,\delta_{\mu\nu}\,\delta_{\rho\lambda} \tag{C.9}$$

Appendix D

Quantum Effective Action

In this appendix I ignore the serious problems one encounters in defining a path integral. A more detailed introduction to the quantum effective action can be found in [56, 57] and especially for supersymmetric Theories in [20].

D.1 Definition

Let us take a quantum field theory defined by the Lagrangian $\mathcal{L}[\vec{\varphi}]$ where the components of $\vec{\varphi}$ denote all fields of the theory. The φ^i's can be bosonic or fermionic and are defined on Minkowski space.

The generating functional of the theory in the presence of an external source $\vec{J}$ is given by

$$\begin{aligned} Z[\vec{J}] &= \int \mathscr{D}\vec{\varphi}\, \exp\left[i \int d^4x \left(\mathcal{L}[\vec{\varphi}] + \vec{J}\vec{\varphi}\right)\right] \\ &\doteq \exp\left[iW[\vec{J}]\right] . \end{aligned} \tag{D.1}$$

This definition has a close analogy to statistical thermodynamics e.g. to a ferromagnet described by the Heisenberg or Ising model. For a detailed discussion of this analogy see [57]. From the definition of the functional integral I know that the functional derivative of $W[\vec{J}]$ is the vacuum expectation value of the fields φ^i:

$$\begin{aligned} \frac{\delta}{\delta J^i(x)} W[\vec{J}] &= -i \frac{\delta}{\delta J^i(x)} \log Z[\vec{J}] \\ &= \frac{\int \mathscr{D}\vec{\varphi}\, \mathrm{e}^{i\int(\mathcal{L}+\vec{J}\vec{\phi})}\, \phi^i(x)}{\int \mathscr{D}\vec{\varphi}\, \mathrm{e}^{i\int(\mathcal{L}+\vec{J}\vec{\phi})}} \\ &= \langle 0|\, \varphi^i(x)\, |0\rangle_{\vec{J}} \\ &\doteq \Phi^i(x) . \end{aligned} \tag{D.2}$$

The quantum effective action is now defined as the Legendre transformation of $W[\vec{J}]$:

$$\Gamma[\vec{\Phi}] \quad \doteq \quad -W[\vec{J}] + \int d^4y \; \vec{J}(y)\vec{\Phi}(y) \, . \tag{D.3}$$

This is the same construction as for the Gibbs fee energy. As very nicely shown in [58], $\Gamma[\vec{\Phi}]$ is the sum of all one-particle irreducible graphs. The quantum effective action contains all the quantum correction to the classical theory. This is what makes it so important, since knowing $\Gamma[\vec{\Phi}]$ allows one to make non-perturbative statements about the theory.

Unfortunately, one can anticipate from the definition of $W[\vec{J}]$ that in general it is not possible to calculate $\Gamma[\vec{\Phi}]$ explicitly. What one usually does is to expand $\Gamma[\vec{\Phi}]$ around the classical action and to do a straightforward perturbative calculation.

The above construction of $\Gamma[\vec{\Phi}]$ can be done also with an operator $\mathbb{O}(\vec{\varphi})$ instead of the fundamental fields $\vec{\varphi}$. $\mathbb{O}(\vec{\varphi})$ is a composite operator of $\vec{\varphi}$ which couples to an external source J. Instead of $\langle 0| \, \varphi^i(x) \, |0\rangle_{\vec{J}}$ I then get the vacuum expectation value of the operator $\langle 0| \, \mathbb{O}(\vec{\varphi}) \, |0\rangle_{\vec{J}}$. Of course the "perturbative" meaning of this new effective action changes, e.g. for $\mathbb{O}(\vec{\varphi}) = \varphi^2$, Γ is the sum of all two-particle irreducible graphs. For the rest of this appendix I shall only look at the case where $\mathbb{O}(\vec{\varphi}) = \vec{\varphi}$.

D.2 Properties of $\Gamma[\vec{\Phi}]$

For a magnetic system (ferromagnet) in an external magnetic field H the derivative of the Gibbs free energy with respect to the average magnetization returns H. What is the analogon for $\Gamma[\vec{\Phi}]$?

$$\begin{aligned}
\frac{\delta}{\delta\Phi^i(x)}\Gamma[\vec{\Phi}] &= -\frac{\delta}{\delta\Phi^i(x)}W[\vec{J}] + \int d^4y \, \left(\frac{\delta \, J^k(y)}{\delta\Phi^i(x)} \cdot \Phi^k(y) + J^k(y) \, \delta^{ki} \, \delta^4(x-y) \right) \\
&= -\int d^4y \, \frac{\delta \, J^k(y)}{\delta\Phi^i(x)} \frac{\delta W[\vec{J}]}{\delta J^k(y)} + \int d^4y \, \frac{\delta \, J^k(y)}{\delta\Phi^i(x)} \cdot \Phi^k(y) + J^i(x) \\
&= -\int d^4y \, \frac{\delta \, J^k(y)}{\delta\Phi^i(x)} \Phi^k(y) + \int d^4y \, \frac{\delta \, J^k(y)}{\delta\Phi^i(x)} \cdot \Phi^k(y) + J^i(x) \\
&= J^i(x) \, .
\end{aligned} \tag{D.4}$$

As one could expect, the result is the external source. Taking the limit where the source goes to zero I get the equation

$$\frac{\delta}{\delta\Phi_i(x)}\Gamma[\vec{\Phi}] \quad = \quad 0 \, , \tag{D.5}$$

whose solutions are the vacuum expectation values of $\varphi^i(x)$ in the stable quantum states (all quantum corrections are already included!). This is why $\Gamma[\vec{\Phi}]$ is so important for the question of symmetry breaking. It is in principle possible to calculate condensates without fearing that quantum corrections could break the symmetry at some higher order in perturbation theory.

As already mentioned, $\Gamma[\vec{\Phi}]$ is not so easy to calculate explicitly. Instead of doing perturbative calculation I can use another crucial property of $\Gamma[\vec{\Phi}]$. The effective action inherits the symmetries of the classical action. I can therefore establish Ward identities for $\Gamma[\vec{\Phi}]$ and try to solve the resultant differential equations. It is also possible to make an ansatz (hopefully the most general) and draw some conclusions from it.

D.3 Effective Potential

Supposing that $\vec{\Phi}$ is slowly varying, I can expand $\Gamma[\vec{\Phi}]$ in a power series of derivatives of $\vec{\Phi}$:

$$\Gamma[\vec{\Phi}] \;=\; \int d^4x \left(-V_{eff} - \frac{1}{2} Z(\vec{\Phi})\, \partial_\mu \vec{\Phi} \partial^\mu \vec{\Phi} + \;\ldots\right) . \tag{D.6}$$

V_{eff} is called the effective potential. Its importance lies in the following statement: If $\Phi_i^0 = const.$ is a solution of equation (D.5) then this is equivalent to

$$\frac{\partial V_{eff}}{\partial \Phi_i} \;=\; 0\,, \tag{D.7}$$

which provides us with the constant values of $\vec{\Phi}$ for which the effective action takes a minimum. Obviously, if there are such non-zero values that solve equation (D.7), we have a broken symmetry somewhere in our theory.

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