Peter Minkowski Bern, 18. March 2006

**On a theory of isotopic spin**

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Publication of the section of Milano of the INFN
Casa editrice Pleion, Milano

[1]

The arguments exposed here were subject of a series of seminars held at the Institute for theoretical Physics of the University of Milano. This subject constitutes a study effectuated towards the end of 1953 at the E.T.H. in Zurich and it enters the context of a theory developed by Prof. W. Pauli.
Since the author (W. Pauli, note added by PM) has not published in any journal the mentioned theory, the author of this work (P. Gulmanelli, note added by PM) has considered it opportune to give also thereof an ample exposition in appendix II, up to paragraph g) .

# 1 Isotopic spin and quantization of mass

It is known that if one wants an irreducible wave equation describing simultaneously proton and neutron, i.e. if one wants to introduce in an irreducible way (as an irreducible representation, PM) a vector of matrices $\vec{\tau}$ as providing a complete specification of the particles with spin 1/2 (the two nucleons, PM) it is necessary to amplify the space of ordinary spin components ( and to extend it to isospin-space, PM) .
PAIS ( Physica, 1953, 19, 869 ) has proposed to treat this isospin space as a manifold ( $\omega$ ) invariant with respect to a three dimensional, real orthogonal transformation group ( SO3, PM ) , i.e. for example the surface of a three dimensional sphere ( $S_2$ , PM ) .
Results :

1) The ( e.m. - ) charge independence of nuclear forces follows directly from the invariance of the interaction term under rotations of the space $\omega$ .

2) The iterated wave equation satisfied by the 8 component spinor leads to a mass spectrum, the different levels of which depend on the new

[1] Translation from italian by Peter Minkowski . Thanks to Lorenzo Mercolli for his proof reading .

quantum numbers ( in particular those related to the space $\omega$ , PM ) of the particle . The lowest level corresponds to the nucleon (nucleons, PM ) .

PAULI has remarked that in the scheme of PAIS :

1) the tensorial (i.e. pseudoscalar ) nature of the meson field ( coupled to the nucleons, PM ) does not follow from the theory for an intrinsic reason, but rather is postulated ;

2) the conservation of ( e.m. , PM ) charge ( the operators corresponding to observable quantities are obviously expressed by integrals over the variables ( coordinates , PM ) of the space $\omega$ and as a consequence do not explicitely depend on the latter ( space , PM ) ) – is connected with a phase transformation , with constant phase ( over $x$ , $\omega$ , PM ) . It would be interesting , instead , to be able to derive it ( charge conservation , PM ) from something more general, which could be the analog of ( local , PM ) gauge transformations of the electromagnetic potentials . The criterion , which inspired PAULI in the formulation of his theory , was the one of ( characterizing , PM ) unified theories , i.e. to attribute to ( the full space $x$ , $\omega$ , PM ) space a structure from which as possible results it would prove feasible to derive the existence of forces amenable to physical interpretation. In our case the forces are the nuclear forces and the potentials we are looking for are those of the mesonic field .

## 2 The group of transformations of the ( internal , PM ) space $\omega$ and the metric field

In every point of space-time within general relativity there is postulated the existence of a ( fibre space, PM ) space $\omega$ , of dimension n .
The connection between the two spaces ( totally separated in the theory of PAIS ) is introduced by postulating that rotations in $\omega$ ( with coordinates $\omega : \Omega^A ; A = 1, \cdots, n$ , PM ) shall depend on the point ( x in space-time , PM ) .
The group of homogeneous transformations considered is the following

$$
\begin{aligned}
&x'^{\,i} = x^{i} && , \; i = 1,2,3,4\,; \\
&\Omega'^{\,A} = a^{A}_{B}(x)\,\Omega^{B} && , \; A\,,\,B = 1,2\,\cdots,\,n\,; \\
&\Omega^{B} = \overline{a}^{B}_{A}(x')\,\Omega'^{\,B} && , \\
&a^{A}_{B}(x)\,\overline{a}^{B}_{C}(x') = \delta^{A}_{B}\,. &&
\end{aligned}
\qquad (1)
$$

The coefficients $a^{A}_{B}$ shall depend on x but not on $\Omega$ .

## a) Transformation of tensors with respect to the group 1 .

A vector F with contravariant components $F^{\varrho}$ , ( $\varrho = 1,2,3,4\,;1,2\cdots,n$ ) and a vector G with covariant components $G_{\varrho}$ transform as follows

$$
F'^{\,i} = F^{i} \quad , \quad F'^{\,A} = \begin{array}{c} a^{A}_{B}(x)\,F^{B} + \\ + (\partial_{x^k}\, a^{A}_{B})(x)\,\Omega^{B}\,F^{k} \end{array} \qquad (2)
$$

$$
G'_{i} = \begin{array}{c} G_{i} + (\partial_{x'^{i}}\,\overline{a}^{A}_{B})(x') \times \\ \times\, \Omega'^{\,B}\,G_{A} \end{array} \quad , \quad G'_{A} = \overline{a}^{B}_{A}(x')\,G_{B}\,. \qquad (3)
$$

( It suffices to remember , e.g.

$$
F'^{\,i} = (\partial_{x^k}\, x'^{\,i})\,F^{k} + (\partial_{\Omega^A}\, x'^{\,i})\,F^{A}\,. \quad )
$$

One demonstrates immediately that

$$
F^{i}\,G_{i} + F^{A}\,G_{A} = \text{invariant} \qquad (4)
$$

( It suffices to remember the relation existing between the coefficients $a$ and $\overline{a}$ :

$$
(\partial_{x^i}\,\overline{a}^{A}_{B})\,a^{B}_{C} = -\,(\partial_{x^i}\,a^{B}_{C})\,\overline{a}^{A}_{B}\,. \quad )
$$

## b) The metric tensor $g_{\varrho\sigma}(x, \Omega)$ .

One introduces a symmetric metric tensor, the components $g_{ik}$ , $g_{iA}$ , $g_{AB}$ of which transform in the following way :

$$
\begin{aligned}
g'_{ik} &= \begin{array}{c} g_{ik} + (\partial_{x^i} \overline{a}^A_B)\, \Omega'^B g_{Ak} + \\ + (\partial_{x^k} \overline{a}^A_B)\, \Omega'^B g_{Ai} + \\ + (\partial_{x^i} \overline{a}^A_C)(\partial_{x^k} \overline{a}^B_D)\, \Omega'^C \Omega'^D g_{AB} \end{array} , \\
g'_{iA} &= \begin{array}{c} \overline{a}^B_A g'_{iB} + \\ + (\partial_{x^i} \overline{a}^B_C)\, \Omega'^C \overline{a}^D_A g_{BD} \end{array} , \\
g'_{AB} &= \overline{a}^C_A \overline{a}^D_B g_{CD}
\end{aligned} \tag{5}
$$

It may prove convenient to introduce also a condition of homogeneity, e.g. of the type :

$$g_{AB}\, \Omega^A \Omega^B = 1 . \tag{6}$$

( In the ordinary space this would correspond to consider only spherical surfaces of rays with unit length . )

## c) Normal form of the line element .

From general relativity we know that, given a generic point, it is always possible to reduce there the line element to its normal form. In our case this corresponds to have :

$$g_{iA} = 0 \, ; \; g_{AB} = \delta^B_A \, ; \; g_{ik} = \overset{o}{g}_{ik} \, ; \overset{o}{g}_{ik} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} \tag{7}$$

It is easy to convince oneself that this implies :

( 8*a* ) $g_{AB}$ independent of $\Omega$ ;

( 8*b* ) $g_{Ai} = f_{AB,i}(x)\, \Omega^B$ , i.e. linearly dependent on $\Omega$ .

Note primarily that these properties are invariant with respect to the group transformations defined in 1 .
( Making such a transformation, PM) In the new reference frame we have: $g'_{A\,B} = \delta^{B}_{A}$ . By the third relation in eq. 5 it follows hence:
$\overline{a}^{C}_{A}\,\overline{a}^{D}_{B}\,g_{C\,D} = \delta^{B}_{A}$ . Since the $\overline{a}$ depend only on x this is possible only if $g_{C\,D}$ does not depend on $\Omega$ .
Reasoning in an analogous way one finds :

$$\overline{a}^{B}_{A}\,g_{i\,B} + (\partial_{x^i}\,\overline{a}^{B}_{C})\,\Omega'^{\,C}\,\overline{a}^{D}_{A}\,g_{B\,D} = 0 \,.$$

The second term is linear, homogeneous in $\Omega$ and this property must thus hold also for the first.

## d) The operation of underlining the indices .

The subsequent equations define new tensors with contravariant indices $^{A}$ and covariant indices $_{i}$ underlined ( the reasons which suggest the introduction of this notation will become clear in a moment ) :

$$\begin{cases} g^{i\,l}\,g_{\underline{k}\,\underline{l}} = \delta^{i}_{k}\,, \\ g^{\underline{A}\,\underline{B}}\,g_{B\,C} = \delta^{A}_{C}\,, \\ F^{\underline{A}} = F^{A} + g^{\underline{A}\,\underline{B}}\,g_{B\,k}\,F^{k}\,, \\ G_{\underline{k}} = G_{k} - g^{\underline{A}\,\underline{B}}\,g_{B\,k}\,G_{A}\,. \end{cases} \tag{9}$$

We introduce the abbreviated notation:

$$g^{\underline{A}}_{K} \equiv g^{\underline{A}\,\underline{B}}\,g_{B\,K}\,. \tag{10}$$

It is worth while to observe that:

$$g^{\underline{A}}_{K} \neq \delta^{A}_{K}$$

If in the relation above the equal sign would apply, then $g_{A\,B}$ would become dependent on $\Omega$ .
It is obvious how one passes from the operation of underlining of the index of a vector to the one of underlining one or several of the indices of a tensor: it is enough to think of the tensor as transforming like a product of vectors .
One verifies easily among other relations that:

$$g_{A\underline{k}} = g^{\underline{A}k} = 0\,.$$

Since $g^{\underline{A}\,\underline{B}} g_{BC} = \delta^A_C$ , the transformation law of $g^{\underline{A}\,\underline{B}}$ results in beeing the reciprocal of the one for $g_{BC}$ :

$$g'^{\underline{A}\,\underline{B}} = a^A_F\, a^B_G\, g^{\underline{F}\,\underline{G}}\,.$$

It follows then:

$$g'^{\underline{A}}_k = g'^{\underline{A}\,\underline{B}} g'_{Bk} = a^A_F\, g^{\underline{F}\,\underline{G}} g_{Gk} - (\partial_{x^k} a^A_I)\,\Omega^I\,.$$

It is now easy to conclude that:

$$\begin{aligned}
&F'^{\underline{A}} = a^A_B\, F^{\underline{B}}\,;\\
&G'_{\underline{K}} = G_{\underline{K}}\,, \text{ (invariant)}\,;\\
&F^{\underline{A}} G_A + F^K G_{\underline{K}} = F^A G_A + F^K G_K\,, \text{ (invariant by eq. 4)}\,.
\end{aligned} \qquad (11)$$

The lowering and raising respectively of an index can be defined in the following way:

$$\begin{aligned}
g_{BA} F^A + g_{BK} F^K = F_B \;&;\; g_{iA} F^A + g_{iK} F^K = F_i\\
g^{BA} F_A + g^{BK} F_K = F^B \;&;\; g^{iA} F_A + g^{iK} F_K = F^i\,.
\end{aligned}$$

It is not difficult to verify that the first members (in the above pairs of formulae, PM) transform exactly as the indices appearing in the second members imply and that from the described definitions the following relations follow

$$\begin{aligned}
F_A = g_{AB} F^{\underline{B}} \;&,\; F^i = g_{iK} F_{\underline{K}}\\
F^{\underline{A}} = g^{\underline{A}\,\underline{B}} F_B \;&,\; F_{\underline{i}} = g^{\underline{i}\,\underline{K}} F^K\,.
\end{aligned}$$

These relations together with those in eq. 11 indicate clearly the reason for which the underlining operation was introduced.
Finally it is not difficult to convince oneself that the relations satisfied by $g$ with upper and lower indices, underlined or not underlined, are such that, once the quantities $g_{\underline{i}\,\underline{K}}$ , $g_{iA}$ , $g_{AB}$ are assigned, it is possible to recover all derivations. One shall observe that if at this point one would set $g_{\underline{i}\,\underline{K}} = \overset{o}{g}_{iK}$ (special relativity) , this choice would be invariant with respect to the transformation group defined in eq. 1 .

## e) <u>Covariant derivative in the space</u> $\omega$ .

The symbol

$$D_{\underline{k}} \equiv \partial_{x^k} - g_{k}^{A} \partial_{\Omega^A} \tag{12}$$

defines an operation invariant under the transformation group of the space $\omega$ , and yet the index $_k$ (in $D_{\underline{k}}$ , PM) has bin underlined . We remark that $G'_{\underline{k}} = G_{\underline{k}}$ and $G_{\underline{k}} = G_k - g_k^A G_A$ . It will be sufficient to show that $\partial_{x^k}$ transforms as $G_k$ and $\partial_{\Omega^A}$ like $G_A$ :

$$\begin{aligned}
\partial_{x'^k} &= (\partial_{x'^k} x^i)\, \partial_{x^i} + \partial_{x'^k} \Omega^A )\, \partial_{\Omega^A} = \\
&= \partial_{x^k} + (\partial_{x'^k} \overline{a}_C^A)\, \Omega'^C\, \partial_{\Omega^A} ; \\
\partial_{\Omega'^A} &= (\partial_{\Omega'^A} \Omega^B)\, \partial_{\Omega^B} + (\partial_{\Omega'^A} x^i)\, \partial_{x^i} = \\
&= \overline{a}_A^B\, \partial_{\Omega^B} , \text{ q.e.d.}
\end{aligned}$$

## f) <u>Tensor of forces</u> .

Operating with $D_{\underline{k}}$ on the quantities $g_k^A$ (i.e. in the end on the quantities $g_{A\,i}$, which among the metric tensor are those components which establish the connection between the (fibre- , PM) space $\omega$ and ordinary space) one can construct the tensor antisymmetric in $_i$ and $_k$ :

$$F^{\underline{A}}_{,\underline{i\,k}} = D_{\underline{k}}\, g_k^A - D_{\underline{i}}\, g_i^A \tag{13}$$

The indices $^A$ , $_i$ , $_k$ are underlined from which property the transformation law results to be the following:

$$F'^{\underline{A}}_{,\underline{i\,k}} = a_B^A\, F^{\underline{B}}_{,\underline{i\,k}} . \tag{14}$$

Eq. 14 expresses the fact that the quantity $F^{\underline{A}}_{,\underline{i\,k}}$ behaves like a vector with respect to the index $\underline{A}$ pertaining to the space $\omega$ and like an invariant with respect to the indices $_{i\,k}$ pertaining to ordinarey space (-time , PM) . This is a direct consequence of the transformation law of the quantities $g_k^{\underline{A}}$ :

$$g'^{\underline{A}}_k = a_B^A\, g_k^{\underline{B}} - (\partial_{x^k} a_C^A)\, \Omega^C . \tag{15}$$

The form and behaviour of the quantities (related to the field strengths, PM) $F^{\underline{A}}_{,\underline{i\,k}}$ with respect to the spatial axes justify the denomination of tensor of forces and eq. 15 expresses a kind of gauge transformation induced on its "potentials" $g^{A}_{k}$ ( or $g_{Ai}$ ) by the transformation(s) specified in eq. 1 .

Observe that if $\omega$ is supposed twodimensional and homogeneous ($S_1$ , PM) and $g_{AB} = \delta^{B}_{A}$ eq. 6 becomes :

$$(\Omega_1)^2 + (\Omega_2)^2 = 1 .$$

Setting in this case $\Omega_1 = \cos x_5$ and $\Omega_2 = \sin x_5$ , it is easy to see that the (gauge- , PM) group of KLEIN and KALUZA :

$$x'_5 = x_5 + f(x_1, \cdots, x_4)$$

is contained in the transformations defined in eq. 1 . In fact :

$$\cos x'_5 = \cos(x_5 + f) = \begin{array}{l} \cos f(x) \cos x_5 - \\ - \sin f(x) \sin x_5 \end{array}$$

$$\rightarrow \quad \Omega'_1 = a(x)\,\Omega_1 + b(x)\,\Omega_2$$

Finally we want to underline explicitely the strict analogy between the transformation law of the quantities $g_{Ai}$ and the gauge tranformations of the electromagnetic potentials , writing them (the former, PM) in the onedimensional case :

$$\begin{array}{l} g'_{Ai} = \partial_{\Omega^A} g_{Ai} + \\ + (\partial_{x^i} \overline{a}_A)\,\Omega'^{A}\,\overline{a}_A\, g_{AA} \quad \text{(without summations)} . \end{array}$$

It is easy to convince oneself, given that the components $g_{Ai}$ are linear and homogeneous in (the coordinates, PM) $\Omega$, that we can write:

$$F^{A}_{,\underline{ik}} = f^{A}_{B,ik}\,\Omega^{B} . \tag{16}$$

With a simple calculation one finds (for the 4-index quantities $f^{A}_{B,ik}$ defined in eq. 16 , PM) the following expression:

$$f^{A}_{B,\underline{ik}} = \partial_{x^i} f^{A}_{Bk} - \partial_{x^k} f^{A}_{Bi} - f^{A}_{C,k} f^{C}_{B,i} + f^{A}_{C,i} f^{C}_{B,k} , \tag{17}$$

with (the three-index quantities $f^{A}_{B\,k}$ , PM) (see eq. (8b) and eq. 10 defined by

$$g^{A}_{k} = f^{A}_{B\,k}\,\Omega^{B}\ ,\ f^{A}_{B\,k} = g^{\underline{A}\,\underline{B}} f_{C\,B\,,\,k}\ .$$

Besides the tensor (4-index quantity, PM) defined in eq. 17 it is possible to define an associated one with all 4 components lower case ones.
One notes the formal similarity of the (4- , PM) tensor $f^{A}_{B\,,\,\underline{i}\,\underline{k}}$ with the Riemann tensor. In other words it is possible to demonstrate that the former represents (or assigns , PM) an analogous role to the space $\omega$ as the latter (Riemann tensor, PM) plays for space-time . Indeed, it is a necessary and sufficient condition for $g_{A\,i} = 0$ , i.e. for the components $g_{A\,i}$ to vanish everywhere, that this is so for the quantities $f^{A}_{B\,,\,\underline{i}\,\underline{k}}$ i.e. that $f^{A}_{B\,,\,\underline{i}\,\underline{k}} = 0$ . It is obvious that this condition is necessary. And it is intuitive, in view of the given interpretation , that , if the forces are zero , the potential(s , PM) turn(s) out to be constant. We omit here to prove this.
Finally it will be appropriate to anticipate at this point that, in order to have $g_{A\,B} = \delta^{B}_{A}$ <u>everywhere</u> , it is necessary to satisfy another condition (see below) .

## g) <u>The scalar potential</u> .

We consider the expression

$$\psi_{\underline{i}\,,\,\{A\,B\}} = \tfrac{1}{2}\,(f_{i\,,\,A\,B} + f_{i\,,\,B\,A} - \partial_{x^{i}}\,g_{A\,B})$$

constructed with the coefficients of the potentials ( see eq. (8b) ) . This expression is symmetric in $A$ , $B$ and the index $i$ is underlined (in the definition of $\psi$ above, PM) , because, as one easily verifies, $\psi$ transforms with respect to the tranformation group defined in eq. 1 as a tensor.
We define the antisymmetric quantity:

$$\psi_{\underline{i}\,,\,[A\,B]} = \tfrac{1}{2}\,(f_{i\,,\,A\,B} - f_{i\,,\,B\,A})\ .$$

Then we will have:

$$f_{i\,,\,A\,B} = \tfrac{1}{2}\,\partial_{x^{i}}\,g_{A\,B} + \psi_{\underline{i}\,,\,\{A\,B\}} + \psi_{\underline{i}\,,\,[A\,B]}\ . \qquad (18)$$

We have in this way decomposed the coefficients $f_{i\,,\,A\,B}$ into a symmetric and an antisymmetric part: one sees easily that this decomposition is invariant with respect to the transformation group defined in eq. 1 .
We envisage now two possibilities

1) if one sets $\psi_{\underline{i}, \{AB\}} = 0$ (this condition is invariant) the forces

$f_{AB, \underline{i}\,\underline{k}}$ will result to be antisymmetric with respect to $A$ , $B$ in addition to beeing antisymmetric in $\underline{i}\,\underline{k}$

( $f^{\underline{A}'}_{B, \underline{i}\,\underline{k}} = g^{\underline{A}'\underline{A}} f_{AB, \underline{i}\,\underline{k}}$ , PM)

2) If we set more generally $\psi$ equal to the covariant derivative ($D_{\underline{i}}$ above, PM) of a function $\Phi_{\{AB\}}$ , symmetric in $A$ , $B$

$\psi_{\underline{i}}, \{AB\} = D_{\underline{i}} \Phi_{\{AB\}}$ ,

one is led to introduce a field $\Phi_{\{AB\}}$, a scalar with respect to the Lorentz group and having isotopic spin 2, besides the quantities $f_{i, AB}$, which transform as a vector.

The above hypotheis 1) is not necessary, but it has the advantage to simplify significantly the metric. Indeed, when hypothesis 1) is valid and if the tensor in eq. 17 vanishes, eq. 18 reduces to $\partial_{x^i} g_{AB} = 0$, i.e. it is possible to have $g_{AB} = \delta^B_A$ not only in one point, but in the entire space.

# 3 The Dirac equation in space-time

[2]

# 4 Concluding observations

The theory exposed in the previous sections accomplishes, as we have seen, its goal to deduce the form of the coupling of baryon fields with the mesonic field(s , PM) from the hypothesis of invariance with respect to a certain transformation group, avoiding thereby to be forced to directly postulate it ( ad hoc, PM ) as done by PAIS .
The meson field(s , PM ) resulting in this theory is (are , PM) however a vector-field(s , PM) with respect to ordinary space(-time , PM) . The considerations made in II-g) show that one is able to introduce also a scalar (or if needed pseudoscalar) field, but at the price of hypotheses, which appear to be somewhat artificial, and which do not suffice to exclude the simultaneous

[2] This chapter is not essential to the construction of *the bosonic part* of nonabelian gauge modes and is left out .

presence of the vector field(s , PM) . In addition the mass of the (vector-, PM)meson cannot be introduced in the Lagrangian and thus must result from the interaction with the other fields.
Finally we want to note, that the elctromagnetic field can not be deduced in this theory as the mesonic field but can only be introduced (in addition and , PM ) in the ordinary way .