On the class of chiral symmetry representations with scalar and pseudoscalar fields

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Abstract
In the following few pages an account is given of a theme, which I began in 1966 and continued to the present.
1 \[ \Sigma = \frac{1}{\sqrt{2}} \left( \sigma - i \pi \right) \] scalar - pseudoscalar fields and the class of their chiral symmetry representations

Let's denote by \( t, s, n \cdots \) quark flavor indices with
\[ t, s, n \cdots = 1, \cdots, N \equiv N_{fl} \] (1)
and by \( \chi^a \) the \( N^2 \) hermitian \( U_N \) matrices with the normalization
\[
\chi^a = \left( \chi^a \right)_{ts} ; \quad tr \chi^a \chi^b = \delta_{ab} \\
a = 0, 1, \cdots, N^2 - 1 ; \quad \chi^0 = \sqrt{N}^{-1/2} ( \Psi )_{N \times N} \\
tr \chi^a = 0 \text{ for } a > 0 ; \quad \chi^a |_{\text{conv.}} = \sqrt{2} \chi^a
\] (2)

In order to maintain clear quark field association we choose the convention and restriction projecting out color and spin degrees of freedom from the complete set of \( \overline{q} q \) bilinears
\[
\Sigma_{s\dot{i}} \sim \overline{q}^i \gamma^5 \left( 1 + \gamma 5 R \right) q^c \\
\gamma 5 R = \frac{1}{i} \gamma 0 \gamma 1 \gamma 2 \gamma 3 ; \quad c, \dot{c} = 1, 2, 3 \text{ color}
\] (3)
The logical structure of \( \Sigma \) - variables is different, when used to derive the dynamics of quarks, i.e. QCD, or before this, when used in their own right as by M. Gell-Mann and M. Lévy [1], or else associating chiral symmetry with superconductivity as by Y. Nambu and G. Jona-Lasinio [2].

Here the chiral \( UN_{flR} \times UN_{flL} \) transformations correspond to
\[
UN_{flR} : \frac{1}{2} \left( 1 + \gamma 5 R \right) q^c_s \rightarrow V_{ss'} \frac{1}{2} \left( 1 + \gamma 5 R \right) q^c_{s'} \\
UN_{flL} : \frac{1}{2} \left( 1 - \gamma 5 R \right) q^c_s \rightarrow W_{ss'} \frac{1}{2} \left( 1 - \gamma 5 R \right) q^c_{s'} \\
\downarrow \\
\Sigma \rightarrow V \Sigma W^{-1}
\] (4)

The construction in eq. 4 can be interpreted as group-complexification, discussed below. The \( \Sigma \)-variables arise as classical field configurations, Legendre transforms of the QCD generating functional driven by general \( x \)-dependent complex color neutral mass terms.
The latter represent external sources with \( UN_{fl} R \times UN_{fl} L \) substitutions aligned with the \( \Sigma \) - variables

\[
- \mathcal{L}_m = m_{ts} (x) \left\{ \bar{q} \frac{1}{2} \left( 1 - \gamma_5 R \right) q \right\} + h.c.
\]

\[
\propto tr \left( m \Sigma \dagger + \Sigma m \dagger \right)
\]

\[
m \rightarrow V m W^{-1} \quad \Sigma \rightarrow V \Sigma W^{-1}
\]

The so defined (classical) target space variables \(^1\) form

- upon the exclusion of values for which \( Det \Sigma = 0 \) -

the group

\[
GL \left( N , C \right) = \left\{ \Sigma \mid Det \Sigma \neq 0 \right\}
\]

the general linear group over the complex numbers in \( N \) dimensional target-space.

We proceed to define the hermitian chiral currents generating \( UN_{fl} R \times UN_{fl} L \) (global) pertaining to \( \Sigma \)

\[
j^a_{\mu R} = tr \Sigma \dagger \left( \frac{1}{2} \lambda^a \right) \bar{\partial}_\mu \Sigma \sim \bar{q} \gamma_\mu \frac{1}{2} \lambda^a P_{R, q}
\]

\[
j^a_{\mu L} = tr \Sigma \dagger i \bar{\partial}_\mu \Sigma \left( -\frac{1}{2} \lambda^a \right) \sim \bar{q} \gamma_\mu \frac{1}{2} \lambda^a P_{L, q}
\]

\[
A \bar{\partial}_\mu B = A \partial_\mu B - \left( \partial_\mu A \right) B ; \quad P_{R, L} = \frac{1}{2} \left( 1 \pm \gamma_5 R \right)
\]

We avoid here to couple external sources to all other \( \bar{q} q \) bilinears except the scalar - pseudoscalar ones as specified in eq. 5 for two reasons

1) – to retain a minimum set of external sources capable to reproduce spontaneous chiral symmetry breaking alone as a restricted but fully dynamical spontaneous phenomenon.

2) – in order to avoid a nonabelian anomaly structure. The latter would force either the consideration of leptons in addition to quarks, or the inclusion of nonabelian Wess-Zumino terms obtained from connections formed from the \( \Sigma \) fields \( \left[3\right] \).

\(^1\) The notion of target-space is used as defined in modern context of string theories.
For completeness we display the equal time current algebra relations inherited from \( \overline{q} q \)

\[
\begin{align*}
    \left[ j^a_0 R (t, \vec{x}), j^b_0 R (t, \vec{y}) \right] &= i f_{abn} j^n_0 R (t, \vec{x}) \delta^3 (\vec{x} - \vec{y}) \\
    \left[ j^a_0 L (t, \vec{x}), j^b_0 L (t, \vec{y}) \right] &= i f_{abn} j^n_0 L (t, \vec{x}) \delta^3 (\vec{x} - \vec{y}) \\
    \left[ j^a_0 R (t, \vec{x}), j^b_0 L (t, \vec{y}) \right] &= 0 \\
    \left[ \frac{1}{2} \lambda^a, \frac{1}{2} \lambda^b \right] &= i f_{abn} \frac{1}{2} \lambda^n
\end{align*}
\]

The GL \((N, C)\) group structure defined in eq. 6 enables bilateral multiplication of the \(\Sigma\), \(\text{Det} \Sigma \neq 0\) elements, of which the left- and right-chiral currents defined in eq. 7 are naturally associated with the Lie-algebra of \(UN_{fl R} \times UN_{fl L}\) through the exponential mapping with subgroups of \(GL(N, C)_R \times GL(N, C)_L\). These (sub)groups act by multiplication of the base-group-manifold by respective multiplication from the left \(\leftrightarrow G_R\) and from the right \(\leftrightarrow G_L\). The reverse association – here – is accidental.

\[
\begin{align*}
    GL(N, C)_R (L) &\rightarrow G_R (L) = G \\
    \Sigma &\in G ; \ g \in G_R ; \ h \in G_L : \\
    G_R \bullet G &\leftrightarrow \Sigma \rightarrow g \Sigma \\
    G_L \bullet G &\leftrightarrow \Sigma \rightarrow \Sigma h^{-1} \\
    G_R \otimes G_L \bullet G &\leftrightarrow \Sigma \rightarrow g \Sigma h^{-1} \\
    \Sigma &= \Sigma (x) ; \ g, h : x\text{-independent or 'rigid'}
\end{align*}
\]

The exponential mapping and compactification(s) of \(G(\Sigma)\)

The condition \(\text{Det} \Sigma \neq 0\) in the restriction to \(GL(N, C)\) (eq. 6) is very special and surprising in conjunction with the field variable definition.

In fact such a condition is completely untenable and shall be discussed below.

This was a stumbling block for a while.
This condition is equivalent to the relation with the Lie algebra of $GL(N, C)$ through the exponential mapping and its inverse (log):

$$\Sigma = \exp b ; \quad b = b^a \frac{1}{2} \lambda^a ; \quad \frac{1}{2} \lambda^0 = (2N)^{-1/2} (\mathbb{I})_{N \times N}$$

$$\text{Det} \Sigma = \exp (\text{tr} b) = \exp \beta ; \quad \beta = \sqrt{\frac{2}{N}} b^0$$

$$\text{Det} \Sigma = 0 \iff \Re \beta = -\infty ; \quad \beta \sim \beta + 2\pi i \nu ; \quad \nu \in \mathbb{Z}$$

Of course eliminating – from general dynamical $\Sigma$-variables – the subset with $\text{Det} \Sigma = 0$ affects only the non-solvable (and non-semi-simple$^2$) part of the associated group, whence the former are interpreted as a manifold, which simply is not a group. It may thus appear that the restriction in order to enforce a group structure is characterized by the notion of 'group-Plague', infecting the general structure at hand.

This said we continue to treat $\Sigma$-variables as if they were identifiable with $GL(N, C)$.

The next reductive step is to consider the solvable (simple) subgroup

$$SL(N, C) \subset GL(N, C) \subset \{ \Sigma \}$$

$$SL(N, C) = \left\{ \hat{\Sigma} \bigg| \text{Det} \hat{\Sigma} = 1 \right\}$$

$$\hat{\Sigma} \sim \Sigma / (\text{Det} \Sigma)^{1/N} ; \text{ allowing all N roots}$$

The advantage of the above reduction to $SL(N, C)$ is that it allows the exponential mapping to an irreducible (simple) Lie-algebra, refining eq. 10

$$\hat{\Sigma} = \exp \hat{b} ; \quad \hat{b} = \hat{b}^a \frac{1}{2} \lambda^a ; \quad a = 1, 2, \cdots, N^2 - 1$$

$$\hat{b}^0 = 0 ; \quad \text{tr} \lambda^a = 0$$

i.e. eliminating the unit matrix $\propto \lambda^0$ from the latter.

---

$^2$ The words testify to the fight for definite mathematical notions.
1.1 Relaxing the condition $\text{Det } \Sigma = 0$ and the unique association

$$\Sigma \xrightarrow{\text{Det } \Sigma \neq 0} \text{GL}(N, C)$$

We transform $\Sigma_{st}$ as defined or better associated in eq. 3 by means of the $N^2$ hermitian matrices $\overline{\lambda}^a$ in eq. 2:

$$\Sigma_{st} = \Sigma^{a} \left( \overline{\lambda}^{a} \right)_{st}$$

$$\Sigma^{a} = \text{tr} \overline{\lambda}^{a} \Sigma ; \ a = 0, 1, \cdots , N^2 - 1$$

(13)

The complex (field valued) quantities $\Sigma^a$ are components of a complex $N^2$-dimensional space $C_{N^2}$ and in one to one correspondence with the matrix elements $\Sigma_{st}$:

$$C_{N^2} = \left\{ \left( \Sigma^0, \Sigma^1, \cdots \Sigma^{N^2-1} \right) \right\}$$

(14)

This serves to become aware of the second algebraic relation ($\oplus$), beyond ($\otimes$), i.e. to add matrices and not to just multiply them.\(^3\)

The $\oplus$ operation is also encountered upon 'shifting' general (pseudo)scalar fields relative to a spontaneous vacuum expected value. This is relevant here for spontaneous breaking of chiral symmetry.

It arises independently for the $SU2_L$-doublet scalar (Higgs) fields.

Hence the idea that the combination of $\oplus$ and $\otimes$ – which form the full motion group (of matrices) – are related to 'fields' (‘Körper’ in german). Thus we are led to consider quaternion- and octonion-algebras in the next sections.

1.2 Octonions (or Cayleigh numbers) as pairs of quaternions

Let

$$q = q^0 i_0 + q^a i_a ; \ a = 1, 2, 3 ; \ (q^0, \overline{q}) \in R_4$$

$$i_0 = 1 ; \ i_a i_b = -\delta_{ab} i_0 + \varepsilon_{abc} i_c \ | \ \text{for } a, b, n = 1, 2, 3$$

$$\overline{q} = q^0 i_0 - q^a i_a$$

(15)

denote a quaternion over the real numbers.

Then a single octonion is represented (modulo external automorphisms\(^4\))

\(^3\) Elements of a $N \times N$-matrix can equivalently be arranged along a line.

\(^4\) These automorphisms form the exceptional group $G_2$.\(^5\)
by a pair of quaternions \((p, q)\) with the nonassociative multiplication rule

\[
o = (p, q) = p^0 j_0 + p^a j_a + q^0 j_4 + q^a j_4 + a
\]

\[
o^\alpha = (p^\alpha, q^\alpha) ; \alpha = 1, 2, \ldots
\]

\[
o^1 \odot o^2 = (p^1 p^2 - \bar{\eta}^2 q^1, q^2 p^1 + q^1 \bar{\eta}^2)
\]

\[
\overline{\sigma} = (\overline{\eta}, -q)
\]

\[
\rightarrow \text{ for } o^2 = \overline{\eta}^1 ; \quad o^2 = (\overline{p}^1, -q^1)
\]

\[
o^1 \odot (o^2 = \overline{\sigma}^1) = \left(p^1 \overline{p}^1 + \overline{\eta}^1 q^1, -q^1 p^1 + q^1 \overline{\eta}^1\right)
\]

\[
= \left\{ |p^1|^2 + |q^1|^2 \right\} j_0 + 0
\]

\[
j_0 = j_1, j_1, j_7 ; \quad j_{1,2,3} \simeq i_{1,2,3} \quad (16)
\]

In eq. 16 we used the involutory properties

\[
\overline{\eta} = q ; \quad \overline{\sigma} = o \quad (17)
\]

It follows that unitary quaternions \((q \overline{q} = \overline{q} q = \eta)\) are equivalent to \(S_3 \simeq SU2 \subset \mathbb{R}^4\), whereas unitary octonions \((o \overline{o} = \overline{o} o = \eta)\) are equivalent to \(S_7 \subset \mathbb{R}^8\).

This leads together with the complex numbers to the algebraic association of \(N = 1\) and \(N = 2 - \Sigma\) variables to the three inequivalent ‘field’-algebras

\[
1 \quad N = 1 \iff \mathbb{C} \simeq \mathbb{R}_2 \supset S_1
\]

\[
2 \quad N = 2 \iff \mathbb{Q} \simeq \mathbb{R}_4 \supset S_3
\]

\[
3 \quad N = 2 \iff \mathbb{O} \simeq \mathbb{R}_8 \supset S_7
\]

The group structures of cases 1 - 3 in eq. 18 correspond to

\[
1 : \quad S_1 \simeq U1 \iff U1_R \otimes U1_L
\]

\[
2 : \quad S_3 \simeq SU2 \iff SU2_R \otimes SU2_L
\]

\[
3 : \quad S_7 \iff U2_L \otimes U2_R
\]

While the model introduced by M. Gell-Mann and M. Lévy [1] corresponds to case 2 (eq. 18, 19), it is case 3 (also for \(N = 2\)) which is different and the only one extendable to \(N > 2\).

This shall be illustrated for \(N = 3\) and from there back to case 3 with \(N = 2\) in the next section.
1.3 \[ \Sigma = \frac{1}{\sqrt{2}} \left( \sigma - i \pi \right) \] for \( N = N_{fl} = 3 \) (\( m_u \sim m_d \sim m_s \))

For \( N = 3 \) the \( \Sigma \) - variables describe a \( U(3) \) - nonet of scalars and pseudoscalars (one each). I shall use the notation \( \Sigma \rightarrow \pi, K, \eta, \eta' \) labelled by the names of pseudoscalars, yet denoting associated pairs scalars \( \leftrightarrow \) pseudoscalars

\[
\Sigma = \begin{pmatrix}
\Sigma_{11} & \Sigma_{\pi} - \Sigma_{K} \\
\Sigma_{\pi} + \Sigma_{22} & \Sigma_{K^0} \\
\Sigma_{K} + \Sigma_{K^0} & \Sigma_{33}
\end{pmatrix}
\]

(20)

\[
\Sigma_{11} = \frac{1}{\sqrt{3}} \Sigma_{\eta'}, \quad \Sigma_{\pi} + \frac{1}{\sqrt{2}} \Sigma_{\pi^0} + \frac{1}{\sqrt{6}} \Sigma_{\eta}
\]

\[
\Sigma_{22} = \frac{1}{\sqrt{3}} \Sigma_{\eta'}, - \frac{1}{\sqrt{2}} \Sigma_{\pi^0} + \frac{1}{\sqrt{6}} \Sigma_{\eta}
\]

In the chiral limit \( m_{u,d,s} \rightarrow 0 \) – 8 pseudoscalar Goldstone modes become massless: \( \pi, (3); K, \bar{K}, (4); \eta, (1) \), whereas \( \eta' \) and all 9 scalars remain massive.

\( \pi_0 \leftrightarrow \eta \leftrightarrow \eta' \) – mixing – eventually different for scalars relative to pseudoscalars – is not discussed here [4].

Projecting back on case 3 and \( N = 2 \) in the limit \( m_s \rightarrow \infty \) an \( SU(2)_{fl} \) - singlet pair – denoted \( \Sigma_{\eta(2)} \) – forms as (singlet) combinations of \( \Sigma_{\eta}, \Sigma_{\eta'} \), and a corresponding isotriplet pair \( \Sigma_{\pi} \rightarrow \bar{\Sigma}_{\pi} \).

Instead of the \( 2 \times 2 \) matrix form pertinent to case 3 and \( N = 2 \) we can equivalently display the double quaternion basis from the octonion -structure (eq. 16)

\[
p \leftrightarrow \left( \sigma_{\eta(2)}, \bar{\pi} \right) \rightarrow [1]
\]

\[
q \leftrightarrow \left( \eta_{(2)}, \bar{\sigma}_{\pi} \right)
\]

(21)
2 From \( \langle \Sigma \rangle \) as spontaneous real parameter to \( f_\pi \)

As shown in section 1, the \( \Sigma \) - variables are chosen such, that the spontaneous breaking of \( \text{just} \) chiral symmetry can be explicitly realized. For \( N \) equal (positive) quark masses it follows

\[
\langle \Sigma \rangle = S \frac{1}{\sqrt{2}} N \times N
\]

\[
S = \frac{1}{\sqrt{2}} \langle \sigma^0 \rangle \quad ; \quad \Sigma = \frac{1}{\sqrt{2}} (\sigma - i \pi)_{N \times N}
\]

\[
j^a_{\mu R} = i S tr \frac{1}{2} \lambda^a \partial_{\mu} (\Sigma - \Sigma^\dagger) + \cdots
\]

\[
= S \partial_{\mu} \pi^a + \cdots
\]

\[
\Sigma - \Sigma^\dagger = -i \pi^b \lambda^b
\]

\[
\langle \Omega | j^a_{\mu R} | \pi^b, p \rangle = i \frac{1}{2} f_\pi p_{\mu} \delta^{ab} \text{ for } a, b > 0
\]

\[
-S = \frac{1}{2} f_\pi \leftrightarrow -\langle \sigma^0 \rangle = \left( \frac{N}{2} \right)^{1/2} f_\pi \quad ; \quad f_\pi \sim 92.4 \text{ MeV for } \vec{\pi}
\]

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