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On the class of chiral symmetry representations with scalar and pseudoscalar fields

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Abstract

In the following few pages an account is given of a theme , which I began in 1966 and continued to the present.

1 $\Sigma = \frac{1}{\sqrt{2}} (\sigma - i \pi)$ scalar - pseudoscalar fields and the class of their chiral symmetry representations

Lets denote by $t , s , n \cdots$ quark flavor indices with

$$t , s , n \cdots = 1 , \cdots , N \equiv N_{fl} \tag{1}$$

and by $\overline{\lambda}^{a}$ the N^{2} hermitian U_{N} matrices with the normalization

$$\begin{array}{l} \overline{\lambda}^{a} = \left(\overline{\lambda}^{a} \right)_{ts} \; ; \; tr \, \overline{\lambda}^{a} \overline{\lambda}^{b} = \delta_{ab} \\ a = 0 , 1 , \cdots , N^{2} - 1 \; ; \; \overline{\lambda}^{0} = \sqrt{N}^{\, - 1/2} \; (\P)_{N \times N} \\ tr \, \overline{\lambda}^{a} = 0 \text{ for } a > 0 \; ; \; \lambda^{a} |_{conv.} = \sqrt{2} \, \overline{\lambda}^{a} \end{array} \tag{2}$$

In order to maintain clear quark field association we choose the convention *and restriction* projecting out color and spin degrees of freedom from the complete set of $\overline{q} \, q$ bilinears

$$\begin{array}{l} \Sigma_{s \, \dot{t}} \sim \overline{q}^{\, \dot{c}}_{\, t} \, \frac{1}{2} \, (1 + \gamma_{5 \, R}) \, q^{\, c}_{\, s} \\ \gamma_{5 \, R} = \frac{1}{i} \, \gamma_{0} \, \gamma_{1} \, \gamma_{2} \, \gamma_{3} \; ; \; c, \dot{c} = 1,2,3 \text{ color} \end{array} \tag{3}$$

The *logical* structure of Σ - variables is different, when used to *derive* the dynamics of quarks, i.e. QCD, or before this, when used in their own right as by M. Gell-Mann and M. Lévy [1] , or else associating chiral symmetry with superconductivity as by Y. Nambu and G. Jona-Lasinio [2].
Here the chiral $UN_{fl \, R} \times UN_{fl \, L}$ transformations correspond to

$$\begin{array}{c} UN_{fl \, R} \, : \, \frac{1}{2} \, (1 + \gamma_{5 \, R}) \, q^{\, c}_{\, s} \rightarrow V_{ss'} \, \frac{1}{2} \, (1 + \gamma_{5 \, R}) \, q^{\, c}_{\, s'} \\ UN_{fl \, L} \, : \, \frac{1}{2} \, (1 - \gamma_{5 \, R}) \, q^{\, c}_{\, s} \rightarrow W_{ss'} \, \frac{1}{2} \, (1 - \gamma_{5 \, R}) \, q^{\, c}_{\, s'} \\ \updownarrow \\ \Sigma \rightarrow V \, \Sigma \, W^{-1} \end{array} \tag{4}$$

The construction in eq. 4 can be interpreted as group-complexification , discussed below. The Σ-variables arise as classical field configurations , Legendre transforms of the QCD generating functional driven by general x-dependent complex *color neutral* mass terms.

The latter represent external sources with $UN_{fl\,R} \times UN_{fl\,L}$ substitutions aligned with the Σ - variables

$$\begin{array}{rcl} - \mathcal{L}_{m} & = & m_{ts}(x) \left\{ \overline{q}\,^{\dot{c}}_{s} \frac{1}{2} (1 - \gamma_{5\,R}) \, q^{\,c}_{t} \right\} + h.c. \\ & \propto & tr \left(m \, \Sigma^{\dagger} + \Sigma \, m^{\dagger} \right) \\ m \rightarrow V \, m \, W^{-1} & \longleftrightarrow & \Sigma \rightarrow V \, \Sigma \, W^{-1} \end{array} \tag{5}$$

The so defined (classical) target space variables [1] form

– *upon the exclusion of values for which* $Det\, \Sigma = 0$ –

the group

$$GL(N, C) = \{ \Sigma \mid Det\, \Sigma \neq 0 \} \tag{6}$$

the general linear group over the complex numbers in N dimensional target-space .
We proceed to define the hermitian chiral currents generating
$UN_{fl\,R} \times UN_{fl\,L}$ (global) pertaining to Σ

$$\begin{array}{l} j^{\,a}_{\mu\,R} = tr\, \Sigma^{\dagger} \left(\frac{1}{2} \lambda^{a} \, i \, \overleftrightarrow{\partial}_{\mu} \right) \Sigma \quad \sim \quad \overline{q} \, \gamma_{\mu} \frac{1}{2} \lambda^{a} P_{R} \, q \\ j^{\,a}_{\mu\,L} = tr\, \Sigma^{\dagger} \, i \, \overleftrightarrow{\partial}_{\mu} \, \Sigma \left(- \frac{1}{2} \lambda^{a} \right) \quad \sim \quad \overline{q} \, \gamma_{\mu} \frac{1}{2} \lambda^{a} P_{L} \, q \\ A \overleftrightarrow{\partial}_{\mu} B = A \, \partial_{\mu} B - (\partial_{\mu} A) B \; ; \; P_{R\,(L)} = \frac{1}{2} (1 \pm \gamma_{5\,R}) \end{array} \tag{7}$$

We avoid here to couple external sources to all other $\overline{q}\, q$ bilinears except the scalar - pseudoscalar ones as specified in eq. 5 for two reasons

1) – to retain a minimum set of external sources capable to reproduce spontaneous *chiral* symmetry breaking alone as a restricted but fully dynamical spontaneous phenomenon.

2) – in order to avoid a nonabelian anomaly structure . The latter would force either the consideration of leptons in a ddition to quarks , or the inclusion of nonabelian Wess-Zumino terms obtained from connections formed from the Σ fields [3] .

[1] The notion of target-space is used as defined in modern context of string theories .

For completeness we display the equal time current algebra relations inherited from $\overline{q}\, q$

$$
\begin{aligned}
&\left[\, j^{\,a}_{\,0\,R} \left(\, t \, , \, \vec{x} \, \right) \, , \, j^{\,b}_{\,0\,R} \left(\, t \, , \, \vec{y} \, \right) \, \right] \; = \; i \, f_{\,abn} \, j^{\,n}_{\,0\,R} \left(\, t \, , \, \vec{x} \, \right) \delta^{\,3} \left(\, \vec{x} \, - \, \vec{y} \, \right) \\
&\left[\, j^{\,a}_{\,0\,L} \left(\, t \, , \, \vec{x} \, \right) \, , \, j^{\,b}_{\,0\,L} \left(\, t \, , \, \vec{y} \, \right) \, \right] \; = \; i \, f_{\,abn} \, j^{\,n}_{\,0\,L} \left(\, t \, , \, \vec{x} \, \right) \delta^{\,3} \left(\, \vec{x} \, - \, \vec{y} \, \right) \\
&\left[\, j^{\,a}_{\,0\,R} \left(\, t \, , \, \vec{x} \, \right) \, , \, j^{\,b}_{\,0\,L} \left(\, t \, , \, \vec{y} \, \right) \, \right] \; = \; 0 \\
&\left[\, \tfrac{1}{2} \, \lambda^{\,a} \, , \, \tfrac{1}{2} \, \lambda^{\,b} \, \right] \; = \; i \, f_{\,abn} \, \tfrac{1}{2} \, \lambda^{\,n}
\end{aligned}
\tag{8}
$$

The $GL \left(\, N \, , \, C \, \right)$ group structure defined in eq. 6 enables bilateral multiplication of the Σ , $Det \, \Sigma \, \neq \, 0$ elements , of which the left- and right-chiral currents defined in eq. 7 are *naturally* associated with the Lie-algebra of $UN_{\,fl\,R} \, \times \, UN_{\,fl\,L}$ through the exponential mapping with subgroups of $GL \left(\, N \, , \, C \, \right)_{\,R} \, \times \, GL \left(\, N \, , \, C \, \right)_{\,L}$. These (sub)groups act by multiplication of the base-group-manifold by respective multiplication from the left $\leftrightarrow \, G_{\,R}$ and from the right $\leftrightarrow \, G_{\,L}$. The reverse association – here – is *accidental*

$$
\begin{aligned}
& GL \left(\, N \, , \, C \, \right)_{\,R\,(L)} \; \rightarrow \; G_{\,R\,(L)} \; = \; G \\
& \Sigma \, \in \, G \; ; \; g \, \in \, G_{\,R} \; ; \; h \, \in \, G_{\,L} \; : \\
& \qquad G_{\,R} \, \bullet \, G \; \leftrightarrow \; \Sigma \; \rightarrow \; g \, \Sigma \\
& \qquad G_{\,L} \, \bullet \, G \; \leftrightarrow \; \Sigma \; \rightarrow \; \Sigma \, h^{\,-1} \\
& \quad G_{\,R} \, \otimes \, G_{\,L} \, \bullet \, G \; \leftrightarrow \; \Sigma \; \rightarrow \; g \, \Sigma \, h^{\,-1} \\
& \Sigma \; = \; \Sigma \left(\, x \, \right) \; ; \; g \, , \, h \; : \; \text{x-independent or 'rigid'}
\end{aligned}
\tag{9}
$$

The exponential mapping and compactification(s) of $G \left(\, \Sigma \, \right)$

The condition $Det \, \Sigma \, \neq \, 0$ in the restriction to $GL \, (N \, , C \,)$ (eq. 6) is very special and surprising in conjunction with the field variable definition.

In fact such a condition is completely untenable and shall be discussed below. This was a stumbling block for a while .

This condition is equivalent to the relation with the Lie algebra of $GL\ (\ N\ ,\ C\ )$ through the exponential mapping and its inverse (log)

$$
\begin{array}{l}
\Sigma \ = \ \exp \ b \ ; \ b \ = \ b^{\ a} \ \tfrac{1}{2} \ \lambda^{\ a} \ ; \ \tfrac{1}{2} \ \lambda^{\ 0} \ = \ (\ 2 \ N \ )^{\ -1/2} \ (\ \P \ )_{\ N \times N} \\
Det \ \Sigma \ = \ \exp \ (\ tr \ b \ ) \ = \ \exp \ \beta \ ; \ \beta \ = \ \sqrt{\tfrac{2}{N}} \ b^{\ 0} \\
Det \ \Sigma \ = 0 \ \leftrightarrow \ \Re \, \beta \ = \ - \ \infty \ ; \ \beta \ \sim \ \beta \ + \ 2 \ \pi \ i \ \nu \ ; \ \nu \ \in \ \mathbb{Z}
\end{array}
\tag{10}
$$

Of course eliminating – from general *dynamical* Σ-variables – the subset with $Det\ \Sigma\ =\ 0$ affects only the non-solvable (and non-semi-simple [2]) part of the associated group, whence the former are interpreted as a manifold, which simply is *not* a group . It may thus appear that the restriction in order to enforce a group structure is characterized by the notion of 'group-Plague', infecting the general structure at hand .
This said we continue to treat Σ-variables as if they were identifiable with $GL\ (\ N\ ,\ C\ )$.
The next reductive step is to consider the solvable (simple) subgroup

$$
\begin{array}{l}
SL \ (\ N \ , \ C \ ) \ \subset \ GL \ (\ N \ , \ C \ ) \ \subset \ \{ \ \Sigma \ \} \\
SL \ (\ N \ , \ C \ ) \ = \ \left\{ \ \widehat{\Sigma} \ \middle| \ Det \ \widehat{\Sigma} \ = \ 1 \ \right\} \\
\widehat{\Sigma} \ \sim \ \Sigma \ / \ (\ Det \ \Sigma \ )^{\ 1/N} \ \ ; \ \text{allowing } \textit{all} \text{ N roots}
\end{array}
\tag{11}
$$

The advantage of the above reduction to $SL\ (\ N\ ,\ C\ )$ is that it allows the exponential mapping to an irreducible (simple) Lie-algebra ,
refining eq. 10

$$
\begin{array}{l}
\widehat{\Sigma} \ = \ \exp \ \widehat{b} \ ; \ \widehat{b} \ = \ \widehat{b}^{\ a} \ \tfrac{1}{2} \ \lambda^{\ a} \ ; \ a \ = \ 1 \ , \ 2 \ , \ \cdots \ , \ N^{\ 2} \ - \ 1 \\
\widehat{b}^{\ 0} \ = \ 0 \ ; \ tr \ \lambda^{\ a} \ = \ 0
\end{array}
\tag{12}
$$

i.e. eliminating the unit matrix $\propto\ \lambda^0$ from the latter .

[2] The words testify to the fight for definite mathematical notions .

1.1 Relaxing the condition $Det \, \Sigma \, = \, 0$ and the unique association

$$\Sigma \underset{Det \, \Sigma \neq 0}{\longrightarrow} GL \, (\, N \, , C \,)$$

We transform $\Sigma_{\, st}$ as defined or better associated in eq. 3 by means of the $N^{\, 2}$ *hermitian* matrices $\overline{\lambda}^{\, a}$ in eq. 2 .

$$\begin{aligned} \Sigma_{\, st} \, &= \, \Sigma^{\, a} \left(\overline{\lambda}^{\, a} \right)_{st} \\ \Sigma^{\, a} \, &= \, tr \, \overline{\lambda}^{\, a} \, \Sigma \, ; \, a \, = \, 0 \, , \, 1 \, , \, \cdots \, , \, N^{\, 2} \, - \, 1 \end{aligned} \tag{13}$$

The complex (field valued) quantities $\Sigma^{\, a}$ are components of a complex $N^{\, 2}$-dimensional space $C_{\, N^{\, 2}}$ and in one to one correspondence with the matrix elements $\Sigma_{\, st}$

$$C_{\, N^{\, 2}} \, = \, \left\{ \left(\Sigma^{\, 0} \, , \, \Sigma^{\, 1} \, , \, \cdots \, \Sigma^{\, N^{\, 2} - 1} \right) \right\} \tag{14}$$

This serves to become aware of the *second* algebraic relation ($\oplus$) , beyond ($\otimes$) , i.e. to add matrices and not to just multiply them . [3]

The $\oplus$ operation is *also* encountered upon 'shifting' general (pseudo)scalar fields relative to a spontaneous vacuum expected value . This is relevant *here* for spontaneous breaking of chiral symmetry .

It arises independently for the $SU2_{\, L}$-doublet scalar (Higgs) fields .

Hence the idea that the combination of $\oplus$ and $\otimes$ – which form the full motion group (of matrices) – are related to 'fields' ('Körper' in german). Thus we are led to consider quaternion- and octonion-algebras in the next sections .

1.2 Octonions (or Cayleigh numbers) as pairs of quaternions

Let

$$\begin{aligned} & q \, = q^{\, 0} \, i_{\, 0} \, + \, q^{\, a} \, i_{\, a} \; ; \; a \, = \, 1,2,3 \; ; \; \left(q^{\, 0} \, , \, \vec{q} \right) \, \in \, R_{\, 4} \\ & i_{\, 0} \, = \, ¶ \; ; \; i_{\, a} \, i_{\, b} \, = \, - \, \delta_{\, ab} \, i_{\, 0} \, + \, \varepsilon_{\, abn} \, i_{\, n} \quad | \text{ for } a \, , \, b \, , \, n \, = \, 1,2,3 \\ & \overline{q} \, = \, q^{\, 0} \, i_{\, 0} \, - \, q^{\, a} \, i_{\, a} \end{aligned} \tag{15}$$

denote a quaternion over the real numbers .

Then a single octonion is represented (modulo external automorphisms [4])

[3] Elements of a $N \times N$-matrix can equivalently be arranged along a line .

[4] Thes automorphisms form the exceptional group $G_{\, 2}$.

by a pair of quaternions (p , q) with the nonassociative multiplication rule

$$
\begin{array}{l}
o \; = \; (\; p \; , \; q \;) \; = \; p^{\,0} \, j_{\,0} \; + \; p^{\,a} \, j_{\,a} \; + \; q^{\,0} \, j_{\,4} \; + \; q^{\,a} \, j_{\,4 \, + \, a} \\
o^{\,\alpha} \; = \; (\; p^{\,\alpha} \; , \; q^{\,\alpha} \;) \; ; \; \alpha \; = \; 1, 2, \cdots \\
o^{\,1} \; \odot \; o^{\,2} \; = \big(\; p^{\,1} \, p^{\,2} \; - \; \overline{q}^{\,2} \, q^{\,1} \; , \; q^{\,2} \, p^{\,1} \; + \; q^{\,1} \, \overline{p}^{\,2} \; \big) \\
\overline{o} \; = \; (\; \overline{p} \; , \; - \, q \;) \\
\rightarrow \text{ for } o^{\,2} \; = \; \overline{o}^{\,1} \; ; \; o^{\,2} \; = \; (\; \overline{p}^{\,1} \; , \; - \, q^{\,1} \;) \\
\quad o^{\,1} \; \odot \; \big(\; o^{\,2} \; = \; \overline{o}^{\,1} \; \big) \; = \; \Big(\; p^{\,1} \, \overline{p}^{\,1} \; + \; \overline{q}^{\,1} \, q^{\,1} \; , \; - \, q^{\,1} \, p^{\,1} \; + \; q^{\,1} \, \overline{\overline{p}}^{\,1} \; \Big) \\
\qquad\qquad\qquad\qquad\qquad = \; \Big\{ \; | \, p^{\,1} \, |^{\,2} \; + \; | \, q^{\,1} \, |^{\,2} \; \Big\} \; j_{\,0} \; + \; 0 \\
\hline
j_{\,0} \; = \; \P \; , \; j_{\,1} \; , \; j_{\,7} \; ; \; \; j_{\,1,2,3} \; \simeq \; i_{\,1,2,3}
\end{array}
\tag{16}
$$

In eq. 16 we used the involutory properties

$$\overline{\overline{q}} \; = \; q \; ; \; \overline{\overline{o}} \; = \; o \tag{17}$$

It follows that unitary quaternions ($q \, \overline{q} \; = \; \overline{q} \, q \; = \; \P$) are equivalent to $S_{\,3} \; \simeq \; SU2 \; \subset \; R_{\,4}$, whereas unitary octonions ($o \, \overline{o} \; = \; \overline{o} \, o \; = \; \P$) are equivalent to $S_{\,7} \; \subset \; R_{\,8}$.

This leads together with the complex numbers to the algebraic association of $N \; = 1$ and $N \; = 2 - \;\; \Sigma$ variables to the *three* inequivalent 'field'-algebras

$$
\begin{array}{l}
1 \quad N \; = \; 1 \quad \leftrightarrow \quad \mathbb{C} \; \simeq \; R_{\,2} \; \supset \; S_{\,1} \\
2 \quad N \; = \; 2 \quad \leftrightarrow \quad \mathbb{Q} \; \simeq \; R_{\,4} \; \supset \; S_{\,3} \\
3 \quad N \; = \; 2 \quad \leftrightarrow \quad \mathbb{O} \; \simeq \; R_{\,8} \; \supset \; S_{\,7}
\end{array}
\tag{18}
$$

The group structures of cases 1 - 3 in eq. 18 correspond to

$$
\begin{array}{lllll}
1 & : & S_{\,1} & \simeq \; U1 & \leftrightarrow \quad U1_{\,R} \; \otimes \; U1_{\,L} \\
2 & : & S_{\,3} & \simeq \; SU2 & \leftrightarrow \quad SU2_{\,R} \; \otimes \; SU2_{\,L} \\
3 & : & S_{\,7} & & \leftrightarrow \quad U2_{\,L} \; \otimes \; U2_{\,R}
\end{array}
\tag{19}
$$

While the model introduced by M. Gell-Mann and M. Lévy [1] corresponds to case 2 (eq. 18 , 19) , it is case 3 (also for $N \; = \; 2$) which is *different* and the *only* one extendable to $N \; > \; 2$.

This shall be illustrated for $N \; = \; 3$ and from there back to case 3 with $N \; = \; 2$ in the next section.

1.3 $\Sigma = \frac{1}{\sqrt{2}} (\sigma - i \pi)$ for $N = N_{fl} = 3 (m_u \sim m_d \sim m_s)$

For $N = 3$ the $\Sigma -$ variables describe a $U3_{fl} -$ nonet of *scalars and pseudoscalars* (one each) . I shall use the notation $\Sigma_{\rightarrow \pi , K , \eta , \eta'}$ labelled by the names of pseudoscalars , yet denoting associated pairs
scalars $\leftrightarrow$ *pseudoscalars*

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{\pi^-} & \Sigma_{K^-} \\ \\ \Sigma_{\pi^+} & \Sigma_{22} & \Sigma_{\overline{K}^0} \\ \\ \Sigma_{K^+} & \Sigma_{K^0} & \Sigma_{33} \end{pmatrix}$$

$$\begin{array}{l} \Sigma_{11} = \frac{1}{\sqrt{3}} \Sigma_{\eta'} + \frac{1}{\sqrt{2}} \Sigma_{\pi^0} + \frac{1}{\sqrt{6}} \Sigma_{\eta} \\ \\ \Sigma_{22} = \frac{1}{\sqrt{3}} \Sigma_{\eta'} - \frac{1}{\sqrt{2}} \Sigma_{\pi^0} + \frac{1}{\sqrt{6}} \Sigma_{\eta} \\ \\ \Sigma_{22} = \frac{1}{\sqrt{3}} \Sigma_{\eta'} \qquad\qquad\qquad - \frac{2}{\sqrt{6}} \Sigma_{\eta} \end{array} \tag{20}$$

In the chiral limit $m_{u,d,s} \rightarrow 0$ – 8 pseudoscalar Goldstone modes become massless : π , (3) ; K , $\overline{K}$, (4) ; η , (1) , whereas η' and all 9 scalars remain massive.
$\pi_0 \leftrightarrow \eta \leftrightarrow \eta'$ – mixing – eventually different for scalars relative to pseudoscalars – is not discussed here [4] .

Projecting back on case 3 and $N = 2$ in the limit $m_s \rightarrow \infty$ an $SU2_{fl} -$ singlet *pair* – denoted $\Sigma_{\eta\,(2)}$ – forms as (singlet) combinations of Σ_{η} , $\Sigma_{\eta'}$ and a corresponding isotriplet *pair* $\Sigma_{\pi} \rightarrow \vec{\Sigma}_{\pi}$.
Instead of the 2×2 matrix form pertinent to case 3 and $N = 2$ we can equivalently display the *double quaternion* basis from the octonion - structure (eq. 16)

$$\begin{array}{lll} p \leftrightarrow & (\sigma_{\eta\,(2)} \; , \; \vec{\pi} \;) & \rightarrow [1] \\ q \leftrightarrow & (\eta_{(2)} \; , \; \vec{\sigma}_{\pi}) & \end{array} \tag{21}$$

2 From $\langle\ \Sigma\ \rangle$ as spontaneous real parameter to $f_{\ \pi}$

As shown in section 1 , the $\Sigma -$ variables are chosen such , that the spontaneous breaking of *just* chiral symmetry can be explicitely realized .
For N equal (positive) quark masses it folows

$$\langle\ \Sigma\ \rangle\ =\ S\ \P_{\ N\times\ N}$$

$$S\ =\ \frac{1}{\sqrt{2\ N}}\ \left\langle\ \sigma^{\ 0}\ \right\rangle \quad ; \quad \Sigma\ =\ \frac{1}{\sqrt{2}}\ (\ \sigma\ -\ i\ \pi\ )_{\ N\times N}$$

$$j^{\ a}_{\ \mu\ R}\ =\ i\ S\ tr\ \tfrac{1}{2}\ \lambda^{\ a}\ \partial_{\ \mu}\ \left(\ \Sigma\ -\ \Sigma^{\ \dagger}\ \right)\ +\ \cdots$$

$$=\ S\ \partial_{\ \mu}\ \pi^{\ a}\ +\ \cdots$$

$$\Sigma\ -\ \Sigma^{\ \dagger}\ =\ -\ i\ \pi^{\ b}\ \lambda^{\ b}$$

$$\langle\ \Omega\ |\ j^{\ a}_{\ \mu\ R}\ \left|\ \pi^{\ b}\ ,\ p\ \right\rangle\ =\ i\ \tfrac{1}{2}\ f_{\ \pi}\ p_{\ \mu}\ \delta^{\ ab}\ \text{ for }\ a,b>0$$

$$-\ S\ =\ \tfrac{1}{2}\ f_{\ \pi}\ \leftrightarrow\ -\ \left\langle\ \sigma^{\ 0}\ \right\rangle\ =\ \left(\ \tfrac{N}{2}\ \right)^{\ 1/2}\ f_{\ \pi}\ ;\ f_{\ \pi}\ \sim\ 92.4\ \ \text{MeV} \quad \text{for } \vec{\pi} \tag{22}$$

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