

Oscillatory modes of quarks in baryons for 3 quark flavors u , d , s

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Abstract

The present notes are meant to illustrate the 'oscillatory modes of $N_{fl} = 3$ light quarks', – u , d , s –, using the $SU(2N_{fl} = 6) \times SO3(\vec{L})$ broken symmetry classification .

$\vec{L} = \sum_{n=1}^{N_{fl}} \vec{L}_n$ stands for the space rotation group generated by the sum of the 3 individual angular momenta of quarks in their c.m. system . The motivation arises from modeling the yields of hadrons in heavy ion collisions at RHIC and LHC , necessitating at the respective highest c.m. energies per nucleon pairs an increase of heavy hadron resonances relative to e.g. SPS energies , whence the included hadrons are treated as a noninteracting gas .

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1 - Introduction

The notes developed here have their root in my discussion of the oscillatory modes of quarks in baryons in ref. [1-1980] , as well as the presentation of two hadron resonance collections used in ref. [2-2010] , worked out subsequently in a notefile similar to this one in ref. [3-2011] .

In comparing the two collections used in ref. [2-2010] , denoted $N_{\text{type}}=65 \supset N_{\text{type}}=26$, turn out both to be too small to account for the hadron abundances , as measured at RHIC and LHC , comparing with the Hadron Resonance Gas (HRG) approach (see e.g. ref. [4-2010]) , with noninteracting hadrons . The detailed description of the collection of hadron resonances used in thermal fits to all , including RHIC and LHC hadron abundances is explicitly presented in ref. [5-2009] . The selection consists of including all hadron resonances in the PDG tables of 2008 [6-2008] .

Here we propose to consider the three light flavors of quark u , d , s , extending the modes discussed in ref. [1-1980] as a first step , in the next section .

2 - Modes of quarks in baryons according to the $\text{SU} (2 N_{fl} = 6) \times \text{SO3} (\vec{L})$ broken symmetry classification

The strategy to select hadron resonance followed in 2009 in ref. [5-2009] , as described in section 1 , and strategies grouping together , e.g. baryon states , between 1964-1979 in refs. [7-1964-1974] , [8-1977] , [9-1979] for 3 flavors of quarks , using the $\text{SU} (2 N_{fl} = 6) \times \text{SO3} (\vec{L})$ broken symmetry classification , forming the title of section 2 , reveal a break of tradition . →

2-1

Here we propose to illustrate with new data with respect to 1980 the reliability of the PDG with respect to missing as well as incorrectly included hadron resonances , as compared with spectroscopic theoretical valence quark models with extended broken symmetries .

2-1 - The reduction of an SUN group R-fold product representation through the symmetric group S_R

\longleftrightarrow Young tableaux Y_{R}^N

Alfred Young + 16 April 1873 in Widnes , Lancashire , England , cited from ref. [12-2012] .
† 15 December 1940 in Birdbrook , Essex , England

The three irreducible Young tableaux in figures 1 - 3 below , arise as tensors of rank 3 within a symmetry group of $SU (6 = SU2_{spin} \times SUN_{fl} = 3)$. Its broken character is discussed later .

Rank three corresponds to the wave function of a baryon formed from three valence quarks, confined with respect to color, exclusively . In the construction of this wave function the width of the clearly involved resonance shall be set approximately to zero .

We turn towards the functions associated with these Young tableaux , depending on 6 integer arguments

$$D_3^6 (m_6 , m_5 , \dots , m_1) = D (m_6 , m_5 , \dots , m_1)$$

$$(1) \quad m_6 > m_5 > \dots > m_1 > 0 : \text{integers}$$

$$D (m_6 , m_5 , \dots , m_1) = \prod_{j=2}^6 \prod_{k=1}^{j-1} (m_j - m_{j-k})$$

The D - functions for symmetric , mixed and antisymmetric representations of SU6 will be discussed after the 3 figures below . \longrightarrow

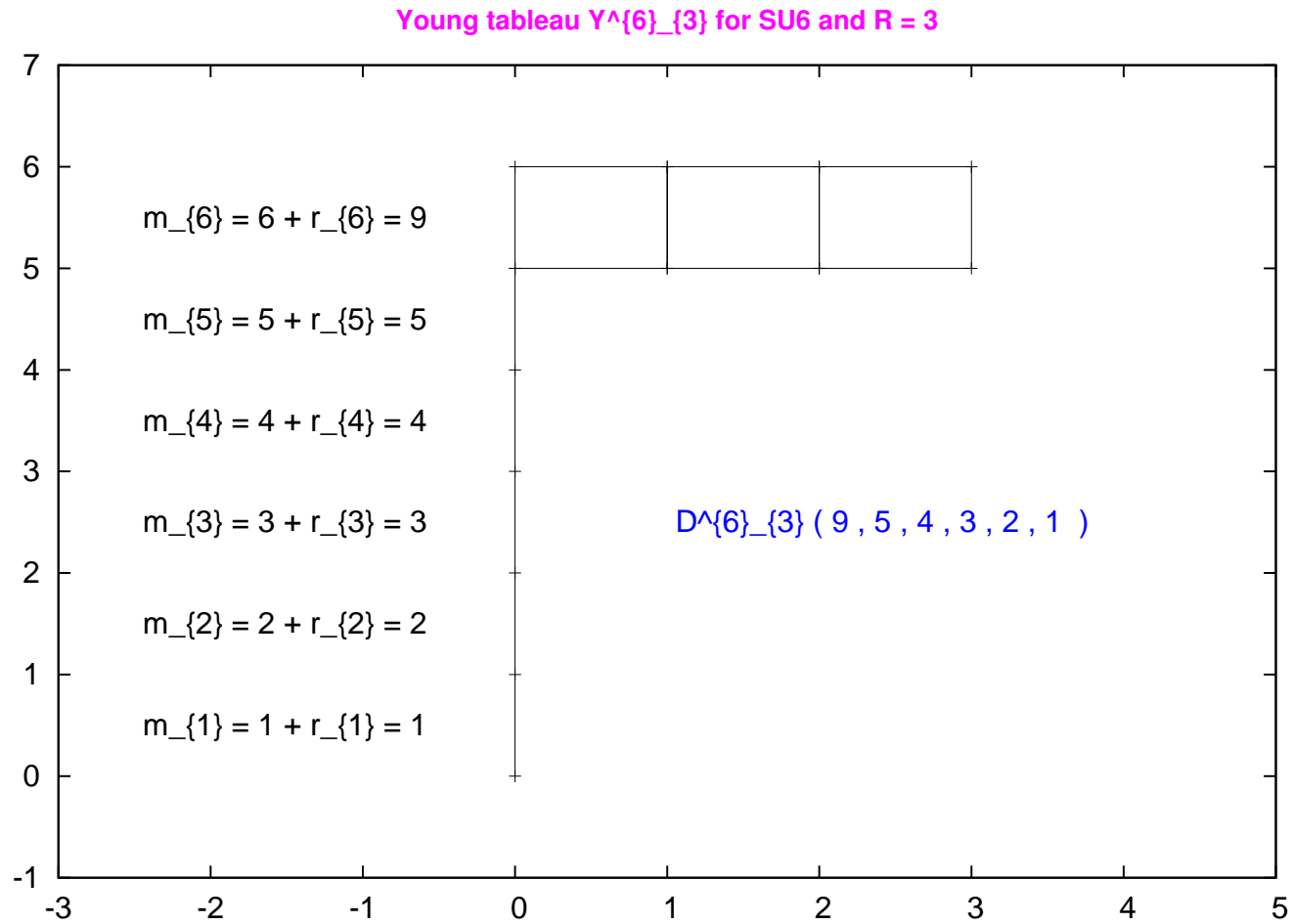


Fig 1 : The symmetric Young tableau for SU6 and rank $R = 3 \longleftrightarrow$

2-3

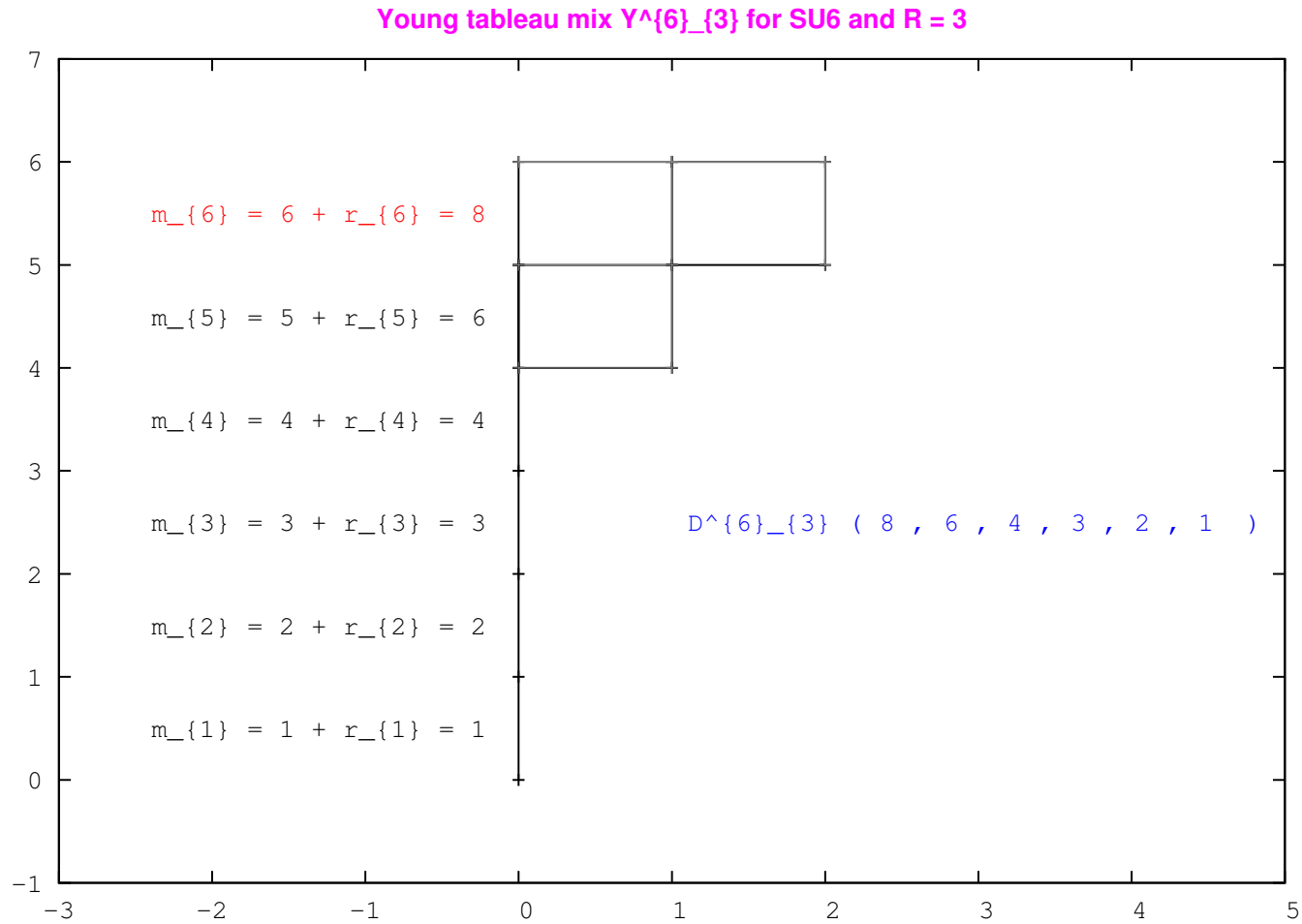


Fig 2 : The mixed Young tableau for SU_6 and rank $R = 3$ \longleftrightarrow

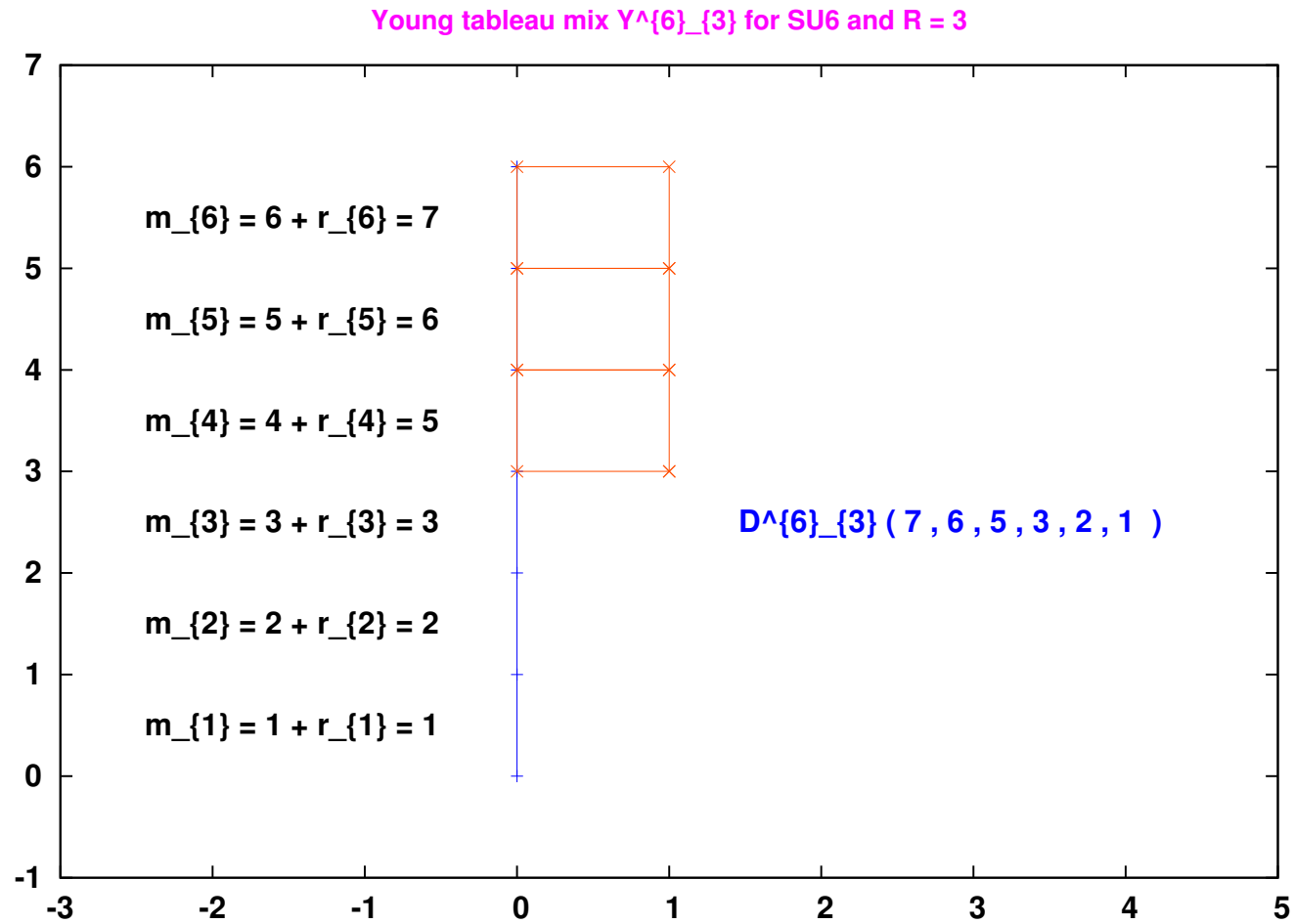


Fig 3 : The antisymmetric Young tableau for SU6 and rank $R = 3$ \longleftrightarrow

2-6

The dimensions of irreducible SUN representations belonging to a specific Young tableau are given by

$$(3) \quad \dim(\mathbf{Y-t}) = \frac{D(m_R, m_{R-1}, \dots, m_1)}{D(R, R-1, \dots, 1)}$$

Substituting eq. 2 in eq. 3 it follows , always for SU6

$$\dim \left(\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} \right) = \frac{4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}{5!} = 56$$

$$(4) \quad \dim \left(\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array} \right) = \frac{2 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{4!5!} = 70$$

$$\dim \left(\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \right) = \frac{2 \cdot 4 \cdot 5 \cdot 6 \cdot 3 \cdot 4 \cdot 5 \cdot 2 \cdot 3 \cdot 4}{3!4!5!} = 20$$



2-2 - The main $\underline{N}^P = 56^+$ baryon valence quark configuration multiplet – the entry point

The most stable baryon multiplet, restricted to the light three flavors of u, d, s quarks, shall be labelled by the total number of states including spin and flavor, with multiplicity and parity denoted \underline{N} and by the superfix P respectively.

The landmark pertaining to the $J^P = \frac{3}{2}$ states within this multiplet is the discovery picture of the Ω^- in 1964, shown in figure 4 below, as administered electronically today by Wikipedia, the free encyclopedia, ref. [13-1962] for the authors of the discovery of ref. [15-1964]. J denotes the total angular momentum formed from spins and orbital angular momenta. The strangeness 3 baryon was predicted in 1962 together with its mass region for a decuplet with $J = \frac{3}{2}$ in refs. [14-1962].

The configuration space wave functions of the 56^+ combined $SU_6 \times SU_2 \left(\vec{L} \right)$ - ground state multiplet are (assumed) independent of angular relative momenta of the 3 valence quarks and all equal as a consequence, and thus form an octet of $J = \frac{1}{2}$ and a decuplet of $J = \frac{3}{2}$ i.e. a multiplet of overall 56 states of equal parity, positive by convention.

This concludes the entry point presentation of Young tableaux. The nontrivial higher configurations shall be discussed in subsequent sections. →

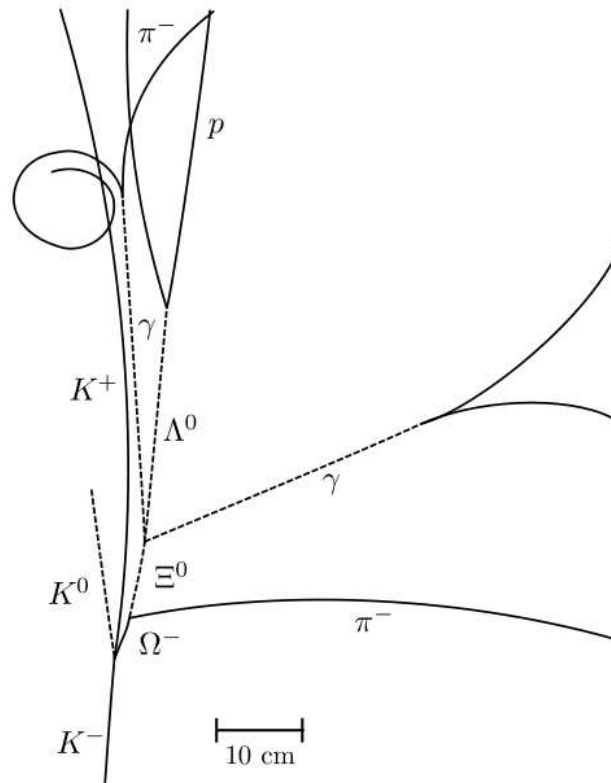


Fig 4 : The bubble chamber picture of the Ω^- in 1964 , see ref. [15-1964] \longleftrightarrow

2-3 - The first orbitally excited , negative parity , $N^P = 70^-$ based , $\underline{N}^- = 210^-$ baryon valence quark configuration multiplet – the confirmation from experimental spectroscopy ?

The $\underline{N}^- = 210^-$ negative parity u , d , s multiplet is well described in the current PDG review 'Quark Model' by C. Amsler , T. De Grand and B. Krusche in ref. [17-2011] . The mixed $SU6_{spin \times fl}$ Young tableau yielding the 70 representation is combined with an $SU2 \left(\vec{L} \right)$; $L = 1$ orbital angular momentum wave function in a way compatible with – including color – overall fermion permutation antisymmetry of three valence quarks u, d , s .

$$(5) \quad \vec{L} = \vec{L}_1 + \vec{L}_2 + \vec{L}_3 ; L = L_{123} = 1 ; 2L + 1 = 3$$

$$\underline{N} = (2L + 1) N = 210$$

The $N = 70$ representation of $SU6_{spin \times fl}$ decomposes with respect to total quark spin multiplicity and (times) $SU3_{fl}$ irreducible representations as

$$(6) \quad 70 = 2 \times 10 + 2 \times 8 + 4 \times 8 + 2 \times 1$$

This in turn combines the respective total spins $\frac{1}{2}$ and $\frac{3}{2}$ to total angular momenta J^P

$$(7) \quad \begin{array}{l} \text{spin} = \frac{1}{2} : J^P = \frac{3}{2} \oplus \frac{1}{2} \\ \text{spin} = \frac{3}{2} : J^P = \frac{5}{2} \oplus \frac{3}{2} \oplus \frac{1}{2} \end{array}$$



2-10

Finally also combining total orbital angular momentum and total spin the $\underline{N}^- = 210^-$ multiplet decomposes according to

$$(8) \quad \underline{N}^- = 210^- = \left\{ \begin{array}{ll} 10, \left(\frac{3}{2}\right)^- \oplus \left(\frac{1}{2}\right)^- + & \# \quad 60 \\ 8, \left(\frac{3}{2}\right)^- \oplus \left(\frac{1}{2}\right)^- + & \# \quad 48 \\ 8, \left(\frac{5}{2}\right)^- \oplus \left(\frac{3}{2}\right)^- \oplus \left(\frac{1}{2}\right)^- + & \# \quad 96 \\ 1, \left(\frac{3}{2}\right)^- \oplus \left(\frac{1}{2}\right)^- & \# \quad 6 \end{array} \right.$$

210



2-11

$S = 0$ $I = \frac{1}{2}$	$S = 0$ $I = \frac{3}{2}$	$S = -1$ $I = 0$	$S = -1$ $I = 1$	$S = -2$ $I = \frac{1}{2}$	$S = -3$ $I = 0$
$J^P = (\frac{3}{2})^-$ $M = 1520$	$J^P = (\frac{3}{2})^-$ $M = 1700$	$J^P = (\frac{3}{2})^-$ $M = 1690$	$J^P = (\frac{3}{2})^-$ $M = 1670$		
$J^P = (\frac{1}{2})^-$ $M = 1535$	$J^P = (\frac{1}{2})^-$ $M = 1620$	$J^P = (\frac{1}{2})^-$ $M = 1670$			
$J^P = (\frac{1}{2})^-$ $M = 1650$		$J^P = (\frac{1}{2})^-$ $M = 1405$	$J^P = (\frac{1}{2})^-$ $M = 1750$		
$J^P = (\frac{3}{2})^-$ $M = 1700$		$J^P = (\frac{3}{2})^-$ $M = 1520$	$J^P = (\frac{3}{2})^-$ $M = 1940 ?$	$J^P = (\frac{3}{2})^-$ $M = 1820$	
$J^P = (\frac{5}{2})^-$ $M = 1675$		$J^P = (\frac{5}{2})^-$ $M = 1830$	$J^P = (\frac{5}{2})^-$ $M = 1775$		

**Table 1 : Candidate states belonging to the mixed $N = 70^-$ negative parity Young tableau
M in MeV**

2-12

The lowest in mass negative parity states , without heavy flavor content , according to the current PDG tables [16-2012] , are listed in Table 1 , eq. 9 .

We list the numbers for SU_3 fl octets , decuplets and singlets in the $N = 70^-$; $\underline{N} = 210^-$ multiplet of states

	octets	$J^P = \left(\frac{1}{2}\right)^-$:	2	#	32
	octets	$J^P = \left(\frac{3}{2}\right)^-$:	2	#	64
	octets	$J^P = \left(\frac{5}{2}\right)^-$:	1	#	48
	decuplets	$J^P = \left(\frac{1}{2}\right)^-$:	1	#	20
	decuplets	$J^P = \left(\frac{3}{2}\right)^-$:	1	#	40
	singlets	$J^P = \left(\frac{1}{2}\right)^-$:	1	#	2
	singlets	$J^P = \left(\frac{3}{2}\right)^-$:	1	#	<u>4</u>
					#	210

(10)



2-13

Accepting all states listed in Table 1 and assigning $\Xi \left(\frac{3}{2}\right)^{-}; M = 1820$ to an octet, $\Sigma \left(\frac{3}{2}\right)^{-}; M = 1949$ to a decuplet and $\Lambda \left(\frac{3}{2}\right)^{-}; M = 1520$ and $\Lambda \left(\frac{1}{2}\right)^{-}; M = 1405$ to a singlet, the following states are either missing or corresponding quantum numbers cannot be assigned

(11)	$\Omega \left(\frac{3}{2}\right)^{-}$ decuplet	#	4	}	# missing states : 76 out of 210
	$\Omega \left(\frac{1}{2}\right)^{-}$ decuplet	#	2		
	$\Xi \left(\frac{3}{2}\right)^{-}$ decuplet	#	8		
	$\Xi \left(\frac{1}{2}\right)^{-}$ decuplet	#	4		
	$\Xi \left(\frac{5}{2}\right)^{-}$ octet	#	12		
	$\Xi \left(\frac{3}{2}\right)^{-}$ octet	#	8		
	$\Xi \left(\frac{1}{2}\right)^{-}$ octet	#	8		
	$\Sigma \left(\frac{1}{2}\right)^{-}$ decuplet	#	6		
	$\Sigma \left(\frac{3}{2}\right)^{-}$ octet	#	12		
	$\Sigma \left(\frac{1}{2}\right)^{-}$ octet	#	6		
	$\Lambda \left(\frac{3}{2}\right)^{-}$ octet	#	4		
	$\Lambda \left(\frac{1}{2}\right)^{-}$ octet	#	2		



Only states with nonvanishing strangeness are among the missing or non-assignable .

In order to put this number (76) in perspective we list among the $\underline{N}^- = 210^-$ states those with vanishing strangeness : a quartet per decuplet and a doublet per octet . Thus we have for the nonstrange number of states

$$(12) \quad \left. \begin{array}{ll} \Delta \left(\frac{3}{2} \right)^- \text{ decuplet} & \# \quad 16 \\ \Delta \left(\frac{1}{2} \right)^- \text{ decuplet} & \# \quad 8 \\ \mathcal{N} \left(\frac{5}{2} \right)^- \text{ octlet} & \# \quad 12 \\ \mathcal{N} \left(\frac{3}{2} \right)^- \text{ octlet} & \# \quad 16 \\ \mathcal{N} \left(\frac{1}{2} \right)^- \text{ octlet} & \# \quad 8 \end{array} \right\} \# \text{ nonstrange states : } 60$$

$$\longrightarrow \# \text{ strange states : } 150$$

Let me associate a quality factor relative to the spectroscopic recognition of the states with nonvanishing strangeness , pertaining to the negative parity $N = 70^-$; $\underline{N} = 210^-$ multiplet of states , abbreviated by $\{ 70^- \}$ →

restricted also to three u, d , s flavored valence quark configurations, lowest in mass

$$\begin{aligned}
 (13) \quad Qf(S < 0) &= 1 - \frac{\# \text{ correctly assigned strange states}}{\# \text{ all strange states } \{ 70^- \}} \quad \forall \text{ nonstrange states } \in \\
 &= \frac{74}{150} = 0.493 \sim 49.3 \%
 \end{aligned}$$

The result in eq. 13 prompts two remarks

- 1) The quality factor $Qf(S < 0) \sim 49.3 \%$, defined in eqs. 11 - 13 , is clearly insufficient .
- 2) This calls for explanations in the first instance by the experimenters having performed the pertinent experiments .

2-4 - The first orbitally excited , negative parity , $N^P = 20^-$ based , $\underline{N}^- = 20^-$ baryon valence quark configuration multiplet – about SU_3 fl singlet baryons

First we study the following symmetric (pseudo-) tensor structure obtained from a triplet of configuration space vectors $\vec{x}_{1,2,3}$ subject to the c.m. restriction

$$(14) \quad T_{L=2}^{mn} = \left\{ \begin{array}{l} + (x_1)^m (x_2 \wedge x_3)^n + (x_1)^n (x_2 \wedge x_3)^m \\ + (x_3)^m (x_1 \wedge x_2)^n + (x_3)^n (x_1 \wedge x_2)^m \\ + (x_2)^m (x_3 \wedge x_1)^n + (x_2)^n (x_3 \wedge x_1)^m \\ - 2 \delta^{mn} Det (x_1, x_2, x_3) \end{array} \right\}$$

$$\sum_{i=1}^3 \vec{x}_i = 0 \rightarrow Det (x_1, x_2, x_3) = 0$$

Under rotations of the configuration vectors $\vec{x}_i \rightarrow R \vec{x}_i ; i = 1, 2, 3$ the functions

$T_{L=2}^{mn} (x_1, x_2, x_3)$ form the 5 orbital angular momentum wave functions with $L = 2$.

The full wave functions , not restricted to oscillatory modes , as described in ref. [1-1980] , but with quantum numbers specified according to the $SU (2 N_{fl} = 6) \times SO_3 (\vec{L})$ broken symmetry classification , and associated Young tableau symmetry relations →

2-17

are then constructed as follows , where the orbital part – here $T_{L=2}^{mn}$ defined in eq. 14 – is a factor

$$\Psi^{\mu\nu}(x_1, x_2, x_3) = T_{L=2}^{mn}(x_1, x_2, x_3) \psi(x_1, x_2, x_3; L=2)$$

$$\psi(x_1, x_2, x_3; L=2) \longrightarrow \psi(x_1, x_2, x_3) \quad : \quad \text{to simplify notation}$$

$$\psi(x_1, x_2, x_3) = \psi(Rx_1, Rx_2, Rx_3) \quad \text{orbital rotation symmetry}$$

$$\psi(x_1, x_2, x_3) = \psi(x_{j_1}, x_{j_2}, x_{j_3}) \quad \forall \text{ permutations } \begin{pmatrix} 1 & 2 & 3 \\ j_1 & j_2 & j_3 \end{pmatrix}$$

$$\sum_{i=1}^3 \vec{x}_i = 0 \quad \text{overall center of mass condition}$$

(15)

Whence the rotational , combinatorial and c.m. related conditions , displayed in the last 3 lines of eq. 15 , are satisfied

– they are broken by the asymmetries of the quark masses m_u, m_d, m_s and independently by the breaking of $SU6_{spin \times fl}$ –

the remaining dynamical structure of QCD resides in the specific form of the residual wave function

$\psi(x_1, x_2, x_3)$. **It is a Gaussian function for oscillatory modes [1-1980]** . →

2-4-1 - The unitary scalar product adapted to rotational , combinatorial and c.m. related conditions of valence quark u,d,s modes in baryons

The present subsection fits nicely in the context of this chapter , even if it is a digression , preparing discussion of positive parity $SU (2 N_{fl} = 6) \times SO3 (\vec{L})$ broken symmetry baryon multiplets . To this end we replace the specific wave function $\Psi^{\mu\nu} (x_1 , , x_2 , x_3)$ defined in eq. 15 by a collection of generic ones

$$(16) \quad \Psi^{\mu\nu} (x_1 , , x_2 , x_3) \longrightarrow \bigcup_{(\alpha)} \Psi^{(\alpha)} (x_1 , , x_2 , x_3)$$

The unique unitary scalar product , compatible with all rotational , combinatorial and c.m. related conditions is then of the form

$$(17) \quad \langle \Psi^{(2)} | \Psi^{(1)} \rangle = (3)^{3/2} \int \prod_{i=1}^3 d^3 x_i \delta^3 (x_1 + x_2 + x_3) \times \\ \times \Psi^{*(2)} (x_1 , , x_2 , x_3) \Psi^{(1)} (x_1 , , x_2 , x_3)$$

We calculate the the square norm of a Gaussian , using the barycentric coordinates for three valence quarks in the c.m. system

$$(18) \quad z_1 = \frac{1}{\sqrt{2}} (x_1 - x_2) , \quad z_2 = \frac{1}{\sqrt{6}} (x_1 + x_2 - 2 x_3) \\ z_3 = \frac{1}{\sqrt{3}} (x_1 + x_2 + x_3) = \sqrt{3} X_{c.m.} \rightarrow 0$$



2-19

It is good to remember from ref. [1-1980] that only 2 out of the three barycentric three-vector variables as well as their conjugate momenta , i.e.

$$(19) \quad z_1, z_2 \longleftrightarrow \pi_1, \pi_2$$

exhibit oscillatory (confined) motion , wheres the c.m. related position and conjugate momentum are left in free motion . Thus the universal , i.e. N_c independent oscillatory frequency appears in the form of the induced \mathcal{M}^2 operator in the N_c (= 3 here) dependent combination

$$\mathcal{M}^2 = \sum_{\nu=1}^{N_c-1} \left[K_{N_c} \pi_{\nu}^2 + (K_{N_c})^{-1} \Lambda^2 z_{\nu}^2 \right]$$

$$(20) \quad K_{N_c} = N_c / (N_c - 1) \quad \left(= \frac{3}{2} \text{ here } \right)$$

$$\pi_{\mu} = \frac{1}{i} \partial_{z_{\mu}} ; \quad \mu = 1, 2$$

Finally in the identification of oscillatory variables we reduce to the actual oscillator ones through a canonical transformation

$$(21) \quad z_{\mu} = \lambda \bar{z}_{\mu} \longleftrightarrow \pi_{\mu} = \lambda^{-1} \bar{\pi}_{\mu} ; \quad \mu = 1, 2$$

under which



\mathcal{M}^2 in eq. 20 becomes

$$(22) \quad \mathcal{M}^2 = \sum_{\nu=1}^{N_c-1} \left[\frac{K_{N_c}}{\lambda^2} \bar{\pi}_\nu^2 + \frac{\lambda^2 \Lambda^2}{K_{N_c}} \bar{z}_\nu^2 \right]$$

The quantity λ of dimension length, introduced in eqs. 21 and 22, is to be chosen such that

$$(23) \quad \frac{K_{N_c}}{\lambda^2} = \frac{\lambda^2 \Lambda^2}{K_{N_c}} \rightarrow \lambda^4 = \left(\frac{K_{N_c}}{\Lambda} \right)^2 \rightarrow \lambda^2 = \frac{K_{N_c}}{\Lambda}$$

whereupon \mathcal{M}^2 in eqs. 20, 22 assumes the reduced universal form

$$\mathcal{M}^2 = \Lambda \sum_{\nu=1}^{N_c-1} [\bar{\pi}_\nu^2 + \bar{z}_\nu^2]$$

$$(24) \quad z_\mu = \lambda \bar{z}_\mu \longleftrightarrow \pi_\mu = \lambda^{-1} \bar{\pi}_\mu ; \mu = 1, 2 ; \lambda^2 = \frac{K_{N_c}}{\Lambda}$$

$$i \bar{\pi}_\mu = \partial_{\bar{z}_\mu} \longleftrightarrow \bar{z}_\mu ; \mu = 1, 2$$



The associated (universal) oscillator vector-variables derive from the relation on the last line in eq. 24

$$(25) \quad a_{\mu k} = \frac{1}{\sqrt{2}} \left(i \bar{\pi}_{\mu k} + \bar{z}_{\mu k} \right) ; \quad a_{\mu k}^{\dagger} = \frac{1}{\sqrt{2}} \left(-i \bar{\pi}_{\mu k} + \bar{z}_{\mu k} \right)$$

$$\mu = 1, 2 ; \quad k = 1, 2, 3$$

which in turn yields the relativistic structure of the oscillatory \mathcal{M}^2 operator , substituting in eqs. 20 , 22 and 24

$$(26) \quad \mathcal{M}^2 = (2 \Lambda) \sum_{\nu=1, k=1}^{\nu=2, k=3} \left[\bar{a}_{\nu k}^{\dagger} \bar{a}_{\nu k} + \frac{1}{2} \right] ; \quad 2 \Lambda = 1 / \alpha'$$

The contribution of the zero mode oscillations , $\frac{1}{2}$ for each dimension of configuration space (= 3) is inherent to the classical limiting form of oscillatory motion , and is accompanied in the sense of a long range approximation (especially given finite quark masses) by a constant correction .The latter remains non zero also in the limit of vanishing quark masses , as discussed in ref. [1-1980] .

In eq. 26 $1 / \alpha'$ denotes the inverse of the Regge slope. →

If we determine it from the positive parity Λ trajectory from the present PDG tables [16-2012]

$$\begin{array}{rcl}
 \Lambda, J^P & : & \frac{1}{2}^+ \quad \frac{5}{2}^+ \quad \frac{9}{2}^+ \\
 M_j & : & 1.115683 \quad 1.820 \quad 2.350 \\
 M_j^2 & : & 1.2447485 \quad 3.3124 \quad 5.5225 \\
 \frac{1}{2} \Delta M^2 & : & \quad \quad 1.034 \quad 1.105
 \end{array}$$

(27)


and average the two half mass square difference entries in the last line of eq. 27 with weights two to one we obtain

$$(28) \quad 1 / \alpha' = \frac{1}{3} (M_2^2 - M_1^2) + \frac{1}{6} (M_3^2 - M_2^2) \sim 1.06 \text{ GeV}^2$$

I remark that in ref. [16-2012] $\Lambda \frac{9}{2}^+$ has only three stars, and furthermore the trajectory contains only three entries, whereas I think to remember that it contained four sometimes back ^a.

$\Lambda \frac{13}{2}^+$ would extrapolate to 2.755 GeV using eq. 28. →

^a "Tempora mutantur nos et mutamur in illis."

We will come back to eigenvalues , number partitions and counting of 3 flavor- , 2 spin- and 6 orbital oscillator-states , deriving from the mass square operator as characterized in eq. 26 in a subsequent subsection and return to the scalar product of the ground state oscillator wave function with itself . The latter is constructed from the orbital oscillator operators specified in eq. 25 , neglecting here flavor and spin quantum numbers within the approximations discussed in this subsection . 

3-1

3 - From oscillatory modes to counting of states

The form of the mass square operator , as displayed in eqs. 20 - 26 is – as a long distance approximation – not specified , in particular with respect to the oscillatory zero modes , as well as other constant contributions in the configuration space distances at large .

The next step can be inferred from the way the universal inverse Regge slope is determined in eqs. 27 - 28 and amounts to the parametrization starting with eq. 26 repeated below

$$(29) \quad \mathcal{M}^2 = (2\Lambda) \sum_{\nu=1, k=1}^{\nu=2, k=3} \left[\bar{a}_{\nu k}^\dagger \bar{a}_{\nu k} + \frac{1}{2} \right] ; \quad 2\Lambda = 1/\alpha'$$

From eqs. 26 , 29 we introduce the decomposition

$$(30) \quad \begin{aligned} \mathcal{M}^2 &= \Delta \mathcal{M}^2 + \mathcal{M}_{(0)}^2 ; \quad \mathcal{M}_{(0)}^2 = (2\Lambda) C_{(0)} \\ \Delta \mathcal{M}^2 &= (2\Lambda) \sum_{\nu=1, k=1}^{\nu=2, k=3} \left[\bar{a}_{\nu k}^\dagger \bar{a}_{\nu k} \right] \end{aligned} \quad \left| \text{extended to spin and flavor} \right.$$

The extension to include spin and flavor degrees of freedom in the universal part $\Delta \mathcal{M}^2$ as displayed in eq. 30 corresponds to the definition of a direct product label \mathcal{A}

$$(31) \quad \mathcal{A} = \left(\begin{array}{cccccc} \mu & \otimes & dim & \otimes & spin & \otimes & flavor \\ 2 & \times & 3 & \times & 2 & \times & 3 \end{array} \right) = \{ 1, 2, \dots, 36 \}$$



3-2

The universal part $\Delta \mathcal{M}^2$ defined in eq. 30 – for valence quarks u, d, s in baryons – thus becomes

$$(32) \quad \Delta \mathcal{M}^2 = (2\Lambda) \sum_{\mathcal{A}=1}^R \bar{a}_{\mathcal{A}}^\dagger \bar{a}_{\mathcal{A}} ; \quad R = 36$$

This prompts the reparametrization of \mathcal{M}^2 and subsequent *approximate* universality extension of the operator $C_{(0)}$ and its eigenvalues, both defined in eq. 30

$$(33) \quad \begin{aligned} \overline{\mathcal{M}}^2 &= (2\Lambda)^{-1} \mathcal{M}^2 = \sum_{\mathcal{A}=1}^R \bar{a}_{\mathcal{A}}^\dagger \bar{a}_{\mathcal{A}} + C_{(0)} \equiv C + C_{(0)} \\ C &= \sum_{\mathcal{A}=1}^R \bar{a}_{\mathcal{A}}^\dagger \bar{a}_{\mathcal{A}} ; \quad R = 36 \end{aligned}$$

The *approximate* universality extension of the eigenvalues of the operator $C_{(0)}$, mentioned above, implies for the eigenvalues the Ansatz

$$(34) \quad \begin{aligned} E \left(\overline{\mathcal{M}}^2 \right) &= N + \langle C_{(0)} \rangle \\ N &= \sum_{\mathcal{A}=1}^R n_{\mathcal{A}} ; \quad n_{\mathcal{B}} = 0, 1, \dots ; \quad \mathcal{B} = 1, 2, \dots, R \\ \langle C_{(0)} \rangle &\text{ approximately independent of } \{ n_1, n_2, \dots, n_R \} \end{aligned}$$

The approximation defined in eqs. 29 - 34 reduces the counting of baryon states to the counting of points on a unit grid in $R = 36$ dimensions . →

3-3

3-1 - Counting the number of points with nonnegative coordinates on a unit grid in $R = 36$ dimensions

Incorrect procedure according to subsection 3-1(-1) – superseded by subsection 3-1-3d

The sought approximate counting we are envisaging is the number of points $\wp (N , R)$ with coordinates

$$(35) \quad (n_1 , n_2 , \dots , n_R) ; n_B = 0 , 1 \dots ; B = 1 . 2 . \dots , R$$

and the condition for a given positive integer N

$$(36) \quad S = \sum_{\mathcal{A}} n_{\mathcal{A}} \leq N$$

By reflection symmetry on any one of the R coordinates (quadrant symmetry for $R = 2$) we can extend the coordinates in eq. 35 to negative integer values so that the condition in eq. 36 becomes, but only *approximately* . reflection symmetry is reduced for all points with some $n_B \neq 0$.

$$(37) \quad S_{\pm} = \sum_{\mathcal{A}} | n_{\mathcal{A}} | \leq N ; n_B = 0 , \pm 1 \dots$$

It follows

$$(38) \quad \wp (N , R) \sim 2^{-R} \wp_{\pm} (N , R)$$

Since here N may be smaller than $\sqrt{R} = 6$ care must be taken especially when trying to circumscribe a R -sphere to the R -cube of length $2N$ to majorize $\wp_{\pm} (N , R)$. The R -volume of the R -cube is

$$(39) \quad V_{R-cube} = (2N)^R \quad \rightarrow$$

3-1-1 - Exact recursive combinatorics

The power of the set $\wp_{\pm}(N, R)$ defined in eq. 38 as the number of signed integers $\{n_1, n_2, \dots, n_R\}$; $n_{\mathcal{B}} = 0, \pm 1 \dots$; $\mathcal{B} = 1, 2, \dots, R$ which satisfy the condition given in eq. 37 repeated below

$$(40) \quad S_{\pm} = \sum_{\mathcal{A}} |n_{\mathcal{A}}| \leq N ; n_{\mathcal{B}} = 0, \pm 1 \dots ; \mathcal{B} = 1.2 \dots, R$$

satisfies successive recursion relations on N and R

$$\begin{aligned}
 \wp_{\pm}(N, R) &= \sum_{N_1=0}^N (2N - 2N_1 + 1) \wp_{\pm}(N_1, R-1) \\
 &= \sum_{N_1=0}^N (2N - 2N_1 + 1) \times \\
 &\quad \times \sum_{N_2=0}^{N_1} (2N_1 - 2N_2 + 1) \wp_{\pm}(N_2, R-2) \\
 &= \dots \\
 (41) \quad &= \sum_{N_1=0}^N (2N - 2N_1 + 1) \times \\
 &\quad \times \sum_{N_2=0}^{N_1} (2N_1 - 2N_2 + 1) \\
 &\quad \times \dots \\
 &\quad \times \sum_{N_{R-1}=0}^{N_{R-2}} (2N_{R-2} - 2N_{R-1} + 1) \wp_{\pm}(N_{R-1}, 1)
 \end{aligned}$$



3-5

From eq. 41 we can anchor the recursion at $R = 1$

$$(42) \quad \wp_{\pm} (N_{R-1}, 1) = 2 N_{R-1} + 1$$

Introducing the fixed endpoints $N = N_0, N_R = 0$ we obtain the base form of $\wp_{\pm} (N, R)$

$$\wp_{\pm} (N, R) = \left\{ \sum_{N \geq N_1 \geq \dots \geq N_{R-1} \geq 0} \right\} \left[\begin{aligned} & (2 N - 2 N_1 + 1) (2 N_1 - 2 N_2 + 1) \times \dots \\ & \times (2 N_{R-2} - 2 N_{R-1} + 1) \times \\ & \times (2 N_{R-1} - 2 N_R + 1) \end{aligned} \right]$$

$$(43) \quad N_0 = N, N_R = 0 \text{ fixed end points ; summation variables : } N_1, \dots, N_{R-1}$$

Since eq. 43 is exact and exact recursion relations for both $\wp_{\pm} (N, R)$ (eq. 41) and $\wp (N, R)$ exist , contrary to eq. 38 , which is only approximate is I proceed to derive the corresponding relations for $\wp (N, R) \sim 2^{-R}$ below . →

3-6

The recursion relations for $\wp (N , R)$ is of the form

$$\begin{aligned}
 \wp (N , R) &= \sum_{N_1=0}^N (N - N_1 + 1) \wp_{\pm} (N_1 , R - 1) \\
 &= \sum_{N_1=0}^N (N - N_1 + 1) \times \\
 &\quad \times \sum_{N_2=0}^{N_1} (N_1 - N_2 + 1) \wp_{\pm} (N_2 , R - 2) \\
 &= \dots\dots\dots \\
 (44) \quad &= \sum_{N_1=0}^N (N - N_1 + 1) \times \\
 &\quad \times \sum_{N_2=0}^{N_1} (N_1 - N_2 + 1) \\
 &\quad \times \dots \\
 &\quad \times \sum_{N_{R-1}=0}^{N_{R-2}} (N_{R-2} - N_{R-1} + 1) \wp (N_{R-1} , 1)
 \end{aligned}$$

and arises from eq. 41 by the substitution

$$2 (N_{\kappa-1} - N_{\kappa}) \rightarrow N_{\kappa-1} - N_{\kappa} ; \text{ for } \kappa = 0, 1, \dots, R.$$

The algebraic expression analogous to eq. 43 but for $\wp (N , R)$ arises through the same substitution and takes the form →

$$\varphi (N , R) = \sum_{\left\{ \begin{array}{l} N \geq N_1 \geq \dots \geq N_{R-1} \geq 0 \end{array} \right\}} \left[\begin{array}{l} (N - N_1 + 1) (N_1 - N_2 + 1) \times \dots \\ \times (N_{R-2} - N_{R-1} + 1) \times \\ \times (N_{R-1} - N_R + 1) \end{array} \right]$$

$N_0 = N, N_R = 0$ fixed end points ; summation variables : N_1, \dots, N_{R-1}

3-1-1a - The power of the set $\varphi (N , R)$ is summable , i.e. calculable in terms of the sum of integer powers p over the integers $0, 1, \dots, K$ denoted $I (K , p)$ below

The quantities

$$(46) \quad I (K , p) = \sum_{r=0}^K r^p ; \quad K , p , r : \text{integers , with } K , p \geq 1$$

to which we proceed to reduce the counting of baryon states with masses up to the limit

$$(47) \quad \left(1 / \alpha' \right) M_{limit}^2 - \langle C_{(0)} \rangle = N$$

α' , $\langle C_{(0)} \rangle$ in eq. 47 being defined in eqs. 28 and 33 respectively , and which is given by the integer quantity $\varphi (N , R)$ defined in eq. 45 , are classical ones in number theory . For a modern presentation see e. g. ref. [19-2013] .



3-1-1b - The binomial reduction formula $I (K , p + 1) \rightarrow I (K , p)$

We postpone to the next subsection the calculational reduction of $\varphi (N , R)$ in eq. 45 , in order to focus on the systematic reduction with respect to the power p of the quantities $I (K , p)$, the sum of powers of integers as given in eq. 46 .

$$\begin{aligned}
 (48) \quad I (K + 1 , p + 1) &= I (K , p + 1) + (K + 1)^{p+1} \\
 &= \sum_{r=r'+1=1}^{K+1} (r' + 1)^{p+1} \\
 &= \sum_{r'=0}^K (r' + 1)^{p+1}
 \end{aligned}$$

Next we express the summands in the last line of eq. 48 $(r' + 1)^{p+1}$ through their binomial expansion

$$(49) \quad (r' + 1)^{p+1} = (r')^{p+1} + \sum_{q=0}^p \binom{p+1}{q} (r')^q$$

Subtracting $I (K , p + 1)$ from both sides of eq. 48 we obtain

$$(50) \quad \sum_{q=0}^p \binom{p+1}{q} I (K , q) = (K + 1)^{p+1}$$



3-9 add

In eq. 50 repeated below for clarity the value $I (K , q = 0)$ needs special care , relative to all positive values $q > 0$.

$$(51) \quad \sum_{q=0}^p \binom{p+1}{q} I (K , q) = (K + 1)^{p+1}$$

This can be seen decomposing eq. 49

$$(52) \quad (r' + 1)^{p+1} = (r')^{p+1} + \sum_{q=1}^p \binom{p+1}{q} (r')^q + 1$$

Inserting the expression on the right hand side of eq. 52 into the last line on the right hand side of eq. 48 repeated below

$$(53) \quad \begin{aligned} I (K + 1 , p + 1) &= I (K , p + 1) + (K + 1)^{p+1} \\ &= \sum_{r=r'+1=1}^{K+1} (r' + 1)^{p+1} \\ &= \sum_{r'=0}^K (r' + 1)^{p+1} \end{aligned}$$

it follows



3-9 add1

$$(54) \quad I(K+1, p+1) = \left[\begin{array}{l} I(K, p+1) + \\ + \sum_{q=1}^p \binom{p+1}{q} I(K, q) + \sum_{r'=0}^K (1) \end{array} \right]$$

The last term on the right hand side of eq. 54 serves to define the limiting values for $q \rightarrow 0$

$$(55) \quad \sum_{r'=0}^K (1) = K+1 \longrightarrow \\ I(K, 0) = K+1 ; \quad \binom{p+1}{0} = 1$$

It appears surprising that singling out the last term in the sum on the left hand side of eq. 50 the recursion index has dropped from $p+1$ to p

$$(56) \quad (p+1) I(K, p) = (K+1)^{p+1} - \sum_{q=0}^{p-1} \binom{p+1}{q} I(K, q)$$



3-9

We substitute the summation variable $q' = p - q$ and rewrite eq. 56

$$(57) \quad (p + 1) I(K, p) = (K + 1)^{p+1} - \sum_{q'=1}^p \binom{p+1}{p-q'} I(K, p - q')$$

Finally we substitute $p \rightarrow p + 1$ in eq. 57 to promote the recursion

$I(K, p + 1) \rightarrow I(K, p' \leq p)$, dropping the prime on the dummy summation variable $q' \rightarrow q$

$$(58) \quad \begin{aligned} I(K, p + 1) &= \\ &= \left[\begin{aligned} &(p + 2)^{-1} (K + 1)^{p+2} - \\ &-(p + 2)^{-1} \sum_{q=1}^{p+1} \binom{p+2}{p+1-q} I(K, p + 1 - q) \end{aligned} \right] \end{aligned}$$

$$I(K, 0) = K + 1 ; \quad \binom{p+1}{0} = 1 \text{ for } p \geq 0$$

This ends the subsection on recursions of power sums on the power .



3-10

3-1-2 - Preparing the summation variables and tools to perform the nested

$N_0 = N \geq N_1 \geq N_2 \geq \dots \geq N_{R-1} \geq N_R = 0$ summation of $\wp (N , R)$ in eq. 45

First I repeat eq. 45 , the defining equation for the nested summation to be carried out below

$$(59) \quad \wp (N , R) = \left\{ \sum_{N \geq N_1 \geq \dots \geq N_{R-1} \geq 0} \right\} \left[\begin{aligned} & (N - N_1 + 1) (N_1 - N_2 + 1) \times \dots \\ & \times (N_{R-2} - N_{R-1} + 1) \times \\ & \times (N_{R-1} - N_R + 1) \end{aligned} \right]$$

$N_0 = N , N_R = 0$ fixed end points ; summation variables : N_1 , \dots , N_{R-1}

Next we want to distinguish fixed nonnegative integers : $N = N_0 ; N_R = 0$ from variables . also nonnegative integers : N_1 , \dots , N_{R-1} as defined and restricted in eqs. 45 = 59 .

To this end we make the notational identifications

$$(60) \quad (N_1 , \dots , N_{R-1}) \equiv (V_1 , \dots , V_{R-1})$$

nonnegative summation variables $\longrightarrow (V_1 , \dots , V_{R-1})$

and rewrite the quantity $\wp (N , R)$ in eq. 59 using the variables substituted in eq. 60 →

3-10-1



3-1-3 - Continuum integral approximations

First we repeat eq. 45 (= 59) below

$$(61) \quad \varphi (N , R) = \left\{ \sum_{N \geq N_1 \geq \dots \geq N_{R-1} \geq 0} \right\} \left[\begin{aligned} & (N - N_1 + 1) (N_1 - N_2 + 1) \times \dots \\ & \times (N_{R-2} - N_{R-1} + 1) \times \\ & \times (N_{R-1} - N_R + 1) \end{aligned} \right]$$

$N_0 = N , N_R = 0$ fixed end points ; summation variables : N_1 , \dots , N_{R-1}

In order to check the exact result for the quantity $\varphi (N , R)$ – the power of the set of states as oscillatory modes with main quantum number

$$(62) \quad N^* = \sum_{\kappa=1}^{35} N_{\kappa} \leq N$$

as worked out in detail in Appendix 2 (eq. 205) we consider an interpolation of the integer valued summands in eq. 61 and replace the integer logic , nested step-function sum , by a 35-fold nested integral , an approximation to the exact result . →

3-11-1

3-1-3a - The barycentric 6 spatial oscillatory variables and the set $\varphi (N , R)$

It is preferable to work out first the barycentric oscillatory variables as conditioned by the quark flavor
 Young tableau reduced flavor \times spin multiplets as derived in eq. 4

$$\begin{aligned}
 (63) \quad \dim \left(\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} \right) &= \frac{4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}{5!} = 56 \\
 \dim \left(\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array} \right) &= \frac{2 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{4! 5!} = 70 \\
 \dim \left(\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \right) &= \frac{2 \cdot 4 \cdot 5 \cdot 6 \cdot 3 \cdot 4 \cdot 5 \cdot 2 \cdot 3 \cdot 4}{3! 4! 5!} = 20
 \end{aligned}$$

We recall the definition of the barycentric variables in eqs. 19 - 25 and 28



3-11-2

$$\mathcal{M}^2 = \Lambda \sum_{\nu=1}^{N_c-1} [\bar{\pi}_\nu^2 + \bar{z}_\nu^2]$$

$$(64) \quad z_\mu = \lambda \bar{z}_\mu \longleftrightarrow \pi_\mu = \lambda^{-1} \bar{\pi}_\mu ; \mu = 1, 2 ; \lambda^2 = \frac{K_{N_c}}{\Lambda}$$

$$i \bar{\pi}_\mu = \partial_{\bar{z}_\mu} \longleftrightarrow \bar{z}_\mu ; \mu = 1, 2$$

$$K_{N_c} = N_c / (N_c - 1) \left(= \frac{3}{2} \text{ here} \right) ; 2 \Lambda = 1 / \alpha' \sim 1.06 \text{ GeV}^2$$

It is worthwhile to perform the calculation of the universal scale factors (λ, λ^{-1}) step by step, using the GeV fm conversion factor

$$(65) \quad \hbar c = 0.1973269718(44) \text{ GeV fm}$$



3-11-3

We add at this point the Regge slope determination from the mesonic ϱ trajectory , analogous to the Λ baryonic one in eq. 27

$$\begin{array}{rcl}
 \varrho, J^P & : & \begin{array}{ccc} 1^- & 3^- & 5^- \end{array} \\
 M_j & : & 0.77549 \pm 0.00034 \quad 1.6888 \pm 0.0021 \quad 2.330 \pm 0.035 \\
 M_j^2 & : & 0.6013847401 \quad 2.85204544 \quad 5.42890000 \\
 \pm & : & 0.0005274488 \quad 0.00709737 \quad 0.16432500 \\
 \frac{1}{2} \Delta M^2 & : & 1.12533035 \quad 1.28842728 \\
 \pm & : & 0.00355847 \quad 0.08223910
 \end{array}$$

(66)

For the inverse error-square weighted average of the root mean square of the two $\frac{1}{2} \Delta M^2$ determinations in eq. 66 we obtain

$$\begin{aligned}
 \frac{1}{2} \overline{\Delta M^2} &= (1.125656787 \pm 0.007090759292) \text{ GeV}^2 \sim 1 / \alpha'_\varrho \\
 &\rightarrow (1.126 \pm 0.007) \text{ GeV}^2
 \end{aligned}$$

(67)

The error in eq, 67 represents the statistical error only , it does not account for the systematic error caused by the widths of the resonances involved . The value of $1 / \alpha'_\varrho$ can be compared wit the one obtained for the Λ trajectory in eq. 28 . →

3-11-4

If we use the value for $1 / \alpha'_e$ also for the pion trajectory , we expect what in todays nomenclature is called $\pi(4)$ at a mass

$$(68) \quad m_{\pi(4)} \sim \sqrt{m_{\pi}^2 + 4 \times 1.126} = 2.13 \text{ GeV}$$

yet no entry exists in the present PDG tables in ref. [16-2012] , while a $J^{PC} = 4^{-+}$, $I = 1$ resonane was listed near this mass in earlier PDG tables . Here I must admit that I have never checked the quality of this $\pi(4)$ resonance in the mass range derived in eq. 68 .

After this digression we return to eqs. 64 and 65 , in order to complete the scale relation of configuration space variables to the dimensionless barycentric variables , defined in eq. 18 repeated below

$$(69) \quad \begin{aligned} z_1 &= \frac{1}{\sqrt{2}} (x_1 - x_2) \quad , \quad z_2 = \frac{1}{\sqrt{6}} (x_1 + x_2 - 2x_3) \\ z_3 &= \frac{1}{\sqrt{3}} (x_1 + x_2 + x_3) = \sqrt{3} X_{c.m.} \rightarrow 0 \end{aligned}$$

Using rational units for which $\hbar = c = 1$ and choosing as energy and momentum units

$$(70) \quad [E] = [p] = 1 \text{ GeV}$$

the unit of length follows from eq. 65

$$(71) \quad [L] = 1 \text{ GeV}^{-1} = 0.1973269718(44) \text{ fm} \sim \frac{1}{5} \text{ fm}$$



3-11-5

From eq. 64 we obtain

$$z_{\mu} = \lambda \bar{z}_{\mu} \longleftrightarrow \pi_{\mu} = \lambda^{-1} \bar{\pi}_{\mu} ; \mu = 1, 2$$

$$(72) \quad \lambda = \sqrt{2 K_{N_c} \alpha'} = 1.682316462 \text{ GeV}^{-1} = 0.3319664131 \text{ fm}$$
$$\lambda^{-1} = 0.5944184834 \text{ GeV}$$

$$\text{for : } (\alpha')^{-1} = 1.06 \text{ GeV}^2$$

3-1-3b - The barycentric 6 spatial oscillatory variables and their symmetries with respect to the 3 quark positions

We extend the barycentric dimensionless coordinates to include a general c.m. position

$$(73) \quad \left(\begin{array}{l} x_{1k} = \lambda \bar{x}_{1k} \\ x_{2k} = \lambda \bar{x}_{2k} \\ x_{3k} = \lambda \bar{x}_{3k} \end{array} \right) \longrightarrow X_k = \lambda \bar{X}_k = \frac{1}{3} (x_{1k} + x_{2k} + x_{3k})$$

$k = 1, 2, 3$: configuration space coordinate labels →

3-11-6

In the following we will suppress the configuration space coordinate labels $k = 1, 2, 3$ displayed in eq. 73 and restrict configuration space three vectors to their dimensionless representatives $\bar{x}_{1, 2, 3}$ and functions thereof.

The first thing to do is to transform to universal dimensionless configuration space variables the scalar product defined in eqs. 17 , 18 repeated below

$$\langle \Psi^{(2)} | \Psi^{(1)} \rangle = (3)^{3/2} \int \prod_{i=1}^3 d^3 x_i \delta^3 (x_1 + x_2 + x_3) \times \\ \times \Psi^{*(2)} (x_1, , x_2, x_3) \Psi^{(1)} (x_1, , x_2, x_3)$$

(74)

$$z_1 = \frac{1}{\sqrt{2}} (x_1 - x_2) , \quad z_2 = \frac{1}{\sqrt{6}} (x_1 + x_2 - 2x_3)$$

$$z_3 = \frac{1}{\sqrt{3}} (x_1 + x_2 + x_3) = \sqrt{3} X_{c.m.} \rightarrow 0$$

$$z_j = \lambda \bar{z}_j , \quad x_j = \lambda \bar{x}_j ; \quad j = 1, 2, 3$$

The normalizing factor $(3)^{3/2}$ in the integral in eq. 17 and 74 , which were missing in earlier versions, are now included in order to generate a conventionally normalized 6 dimensional L_2 space , which we do next step by step . →

3-11-7

The scalar product in eq. 74 becomes

$$\langle \Psi^{(2)} | \Psi^{(1)} \rangle = \lambda^6 \int \prod_{i=1}^3 d^3 \bar{\xi}_i \delta^3 (\bar{\xi}_3) \times \\ \times \Psi^{*(2)} (\bar{x}_1, \bar{x}_2, \bar{x}_3) \Psi^{(1)} (\bar{x}_1, \bar{x}_2, \bar{x}_3)$$

$$(75) \quad \bar{\xi}_1 = \frac{1}{\sqrt{2}} (\bar{x}_1 - \bar{x}_2) \quad , \quad \bar{\xi}_2 = \frac{1}{\sqrt{6}} (\bar{x}_1 + \bar{x}_2 - 2\bar{x}_3)$$

$$\bar{\xi}_3 = \frac{1}{\sqrt{3}} (\bar{x}_1 + \bar{x}_2 + \bar{x}_3) = \sqrt{3} \bar{X}_{c.m.} \rightarrow 0$$

$$z_j = \lambda \bar{z}_j \quad , \quad x_j = \lambda \bar{x}_j \quad ; \quad j = 1, 2, 3 \quad ; \quad X_{c.m.} = \lambda \bar{X}_{c.m.} = \frac{1}{3} \sum_{i=1}^3 \bar{x}_i$$

In order to eliminate the scale factor λ we redefine the wave functions $\Psi^{(1)}, (2)$ in eq. 75

$$(76) \quad \psi^{(1), (2)} = \lambda^3 \Psi^{(1), (2)}$$

Using the dimensionless wave functions

$$(77) \quad \psi^{(1), (2)} (\bar{x}_1, \bar{x}_2, \bar{x}_3)$$

defined in eq. 76 the scalar product (eq. 75) becomes →

3-11-8

$$(78) \quad \langle \psi^{(2)} | \psi^{(1)} \rangle = \int \prod_{i=1}^3 d^3 \bar{\xi}_i \delta^3 \left(\bar{\xi}_3 \right) \times \\ \times \psi^{*(2)} \left(\bar{x}_1, \bar{x}_2, \bar{x}_3 \right) \psi^{(1)} \left(\bar{x}_1, \bar{x}_2, \bar{x}_3 \right)$$

3-1-3c - The barycentric 6 spatial oscillatory variables and their symmetries with respect to the 3 quark positions in dimensionless universal variables

The central properties under the 3 quark position permutation group S_3 can perfectly be discussed according to the dimensionless variables $\bar{x}_i; i = 1, 2, 3$

$$(79) \quad \pi \begin{pmatrix} 1 & 2 & 3 \\ i_1 & i_2 & i_3 \end{pmatrix} \longrightarrow \begin{pmatrix} \bar{x}_1 & \bar{x}_2 & \bar{x}_3 \\ \bar{x}_{i_1} & \bar{x}_{i_2} & \bar{x}_{i_3} \end{pmatrix}$$

To this end we invert the linear relations in eq. 75

$$(80) \quad \begin{aligned} \bar{x}_1 &= \frac{1}{\sqrt{2}} \bar{\xi}_1 + \frac{1}{\sqrt{6}} \bar{\xi}_2 \\ \bar{x}_2 &= -\frac{1}{\sqrt{2}} \bar{\xi}_1 + \frac{1}{\sqrt{6}} \bar{\xi}_2 \\ \bar{x}_3 &= -\frac{2}{\sqrt{6}} \bar{\xi}_2 = \frac{1}{3} (2\bar{x}_3 - \bar{x}_1 - \bar{x}_2) \quad (\checkmark) \end{aligned}$$

The relation on the last line of eq. 80 takes into account the vanishing of $\sum_{i=1}^3 \bar{x}_i$. →

3-11-9

For completeness we give both barycentric variable transformations alongside (eqs. 75 and 80)

$$\bar{\xi}_1 = \frac{1}{\sqrt{2}} (\bar{x}_1 - \bar{x}_2) \quad , \quad \bar{\xi}_2 = \frac{1}{\sqrt{6}} (\bar{x}_1 + \bar{x}_2 - 2\bar{x}_3)$$

$$\bar{\xi}_3 = \frac{1}{\sqrt{3}} (\bar{x}_1 + \bar{x}_2 + \bar{x}_3) = \sqrt{3} \bar{X}_{c.m.} \rightarrow 0$$

$$z_j = \lambda \bar{z}_j \quad , \quad x_j = \lambda \bar{x}_j \quad ; \quad j = 1, 2, 3 \quad ; \quad X_{c.m.} = \lambda \bar{X}_{c.m.} = \frac{1}{3} \sum_{i=1}^3 \bar{x}_i$$

(81)

$$\bar{x}_1 = \frac{1}{\sqrt{2}} \bar{\xi}_1 + \frac{1}{\sqrt{6}} \bar{\xi}_2$$

$$\bar{x}_2 = -\frac{1}{\sqrt{2}} \bar{\xi}_1 + \frac{1}{\sqrt{6}} \bar{\xi}_2$$

$$\bar{x}_3 = -\frac{2}{\sqrt{6}} \bar{\xi}_2$$



3-11-10

3-1-3d - The barycentric coordinates dimension by dimension : cartesian and skew hexagonal coordinates as appropriate for 3 quark position variables

I shall sketch the hexagonal versus cartesian coordinate association simplifying first to one space dimension and two oscillator variables (ξ_2, ξ_1) in their two dimensional representation in figure 4 below



3-11-11

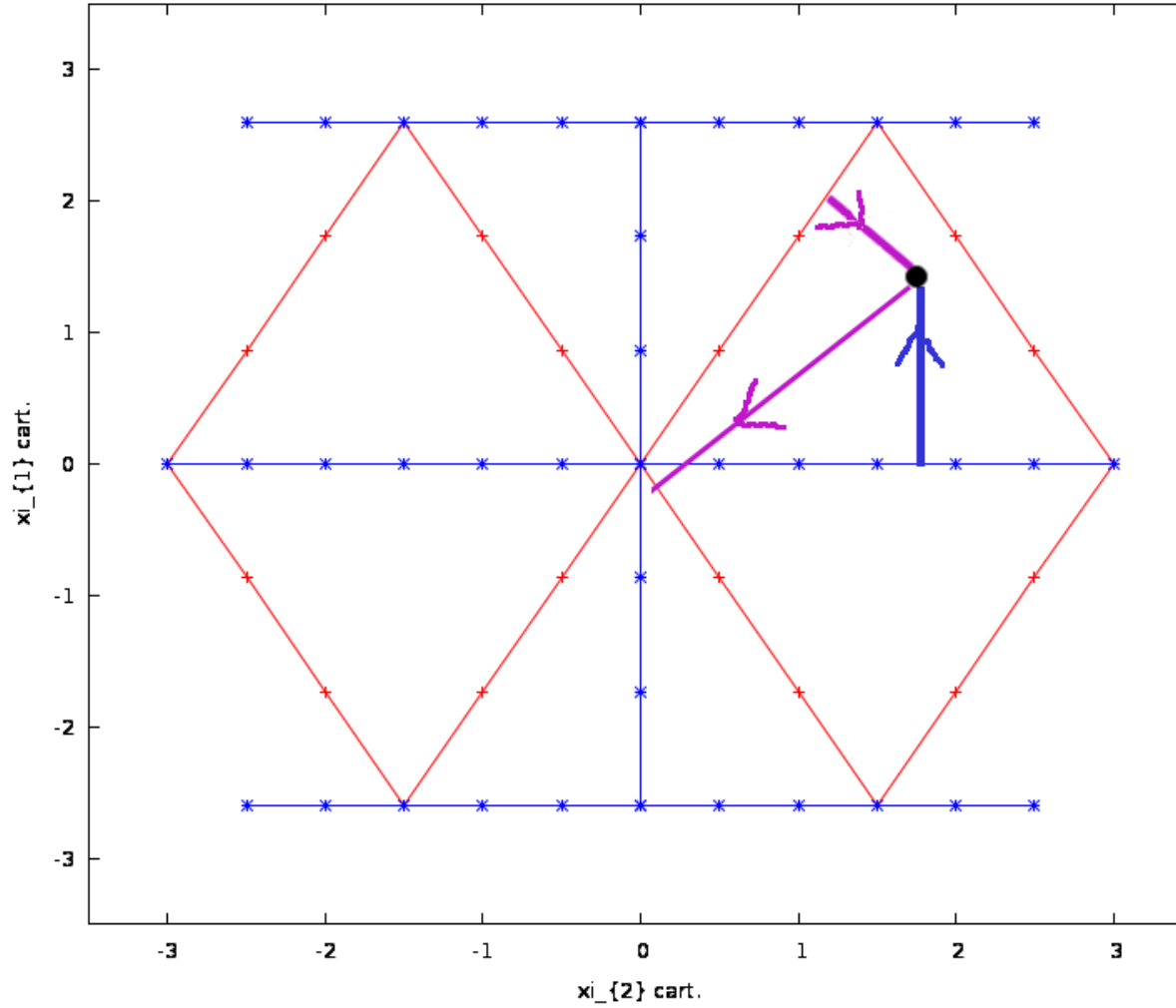


Fig 4 : The hexagonal logic in the (ξ_2, ξ_1) plane \longleftrightarrow

3-pmodes-1

Modes of a pair of onedimensional oscillators – pairmodes and the complex plane

The even dimension already for just 1 space dimension : 2 , of the barycentric *relative* coordinates with vanishing values for the c.m. coordinate(s) derive from the 3 base positions of the quarks bound in baryons , as discussed here in subsections 2-4 , and all subsections of section 3 .

The pairing mode allows to reveal explicitly the hidden SU2 symmetry

$$(82) \quad \begin{aligned} \zeta &= \frac{1}{\sqrt{2}} (x + i y) \quad ; \quad x = \xi_2 , y = \xi_1 \\ a &= \frac{1}{\sqrt{2}} \left(\partial_\zeta + \bar{\zeta} \right) \quad ; \quad b = \frac{1}{\sqrt{2}} \left(\partial_{\bar{\zeta}} + \zeta \right) \end{aligned}$$

$$[a , b] = 0$$

In eq. 82 a and b are **two** independent (commuting) , bosonic , absorption oscillator operators acting from the left on a wave function $\psi \left(\zeta , \bar{\zeta} \right)$. They can also be expressed in the real variables x and y as defined in eq. 82

$$(83) \quad \begin{aligned} x &= \frac{1}{\sqrt{2}} \left(\zeta + \bar{\zeta} \right) \quad ; \quad y = -i \frac{1}{\sqrt{2}} \left(\zeta - \bar{\zeta} \right) \\ \partial_\zeta &= \frac{1}{\sqrt{2}} \left(\partial_x - i \partial_y \right) \quad ; \quad \partial_{\bar{\zeta}} = \frac{1}{\sqrt{2}} \left(\partial_x + i \partial_y \right) \end{aligned}$$



3-pmodes-2

Thus we arrive at the (x, y) representation of the paired oscillators $(a, b) \equiv (a_1, a_2)$. We do this assembling the parts of $a_{1,2}$ as defined in eq. 82 in the table-equations below

$\frac{1}{\sqrt{2}} \zeta = \frac{1}{2} (x + i y)$	$\frac{1}{\sqrt{2}} \bar{\zeta} = \frac{1}{2} (x - i y)$
$\frac{1}{\sqrt{2}} \partial_{\bar{\zeta}} = \frac{1}{2} (\partial_x + i \partial_y)$	$\frac{1}{\sqrt{2}} \partial_{\zeta} = \frac{1}{2} (\partial_x - i \partial_y)$
$\frac{1}{\sqrt{2}} (\zeta + \partial_{\bar{\zeta}}) = \frac{1}{2} \begin{bmatrix} x + \partial_x + \\ + i (y + \partial_y) \end{bmatrix}$	$\frac{1}{\sqrt{2}} (\bar{\zeta} + \partial_{\zeta}) = \frac{1}{2} \begin{bmatrix} x + \partial_x - \\ - i (y + \partial_y) \end{bmatrix}$

(84)

Further it follows for the adjoint operators from eq. 84

$$(85) \quad a = \frac{1}{\sqrt{2}} (\zeta + \partial_{\bar{\zeta}}) ; \quad a^\dagger = \frac{1}{2} \begin{bmatrix} x - \partial_x - \\ - i (y - \partial_y) \end{bmatrix} = \frac{1}{\sqrt{2}} (\bar{\zeta} - \partial_{\zeta})$$

$$b = \frac{1}{\sqrt{2}} (\bar{\zeta} + \partial_{\zeta}) ; \quad b^\dagger = \frac{1}{2} \begin{bmatrix} x - \partial_x + \\ + i (y - \partial_y) \end{bmatrix} = \frac{1}{\sqrt{2}} (\zeta - \partial_{\bar{\zeta}}) \quad \rightarrow$$

3-pmodes-3

The polynomial basis of normalized wave function associated with the two paired absorption oscillators (a_1, a_2) follows from the associated construction of the creation oscillators $a_{1,2}^\dagger$ in eq. 85

$$\psi_{n_1, n_2}(\zeta, \bar{\zeta}) = \mathcal{N} 2^{-\frac{1}{2}(n_1 + n_2)} (\bar{\zeta} - \partial_\zeta)^{n_1} (\zeta - \partial_{\bar{\zeta}})^{n_2} \exp(-\zeta \bar{\zeta})$$

$$\zeta \bar{\zeta} = \frac{1}{2} (x^2 + y^2)$$

(86)

In eq. 86 \mathcal{N} denotes the normalization constant of the ground state with $N = 0$

$$\mathcal{N}^{-2} = (n_1!) (n_2!) \int \frac{1}{2} |d\zeta \wedge d\bar{\zeta}| \exp(-2\zeta \bar{\zeta})$$

$$\frac{1}{2} d\zeta \wedge d\bar{\zeta} = \frac{1}{4} (dx + i dy) \wedge (dx - i dy) = \frac{1}{2i} (dx \wedge dy)$$

$$\frac{1}{2} |d\zeta \wedge d\bar{\zeta}| = d\zeta d\bar{\zeta} = dx dy$$

(87)

$$\begin{aligned} \mathcal{N}^{-2} &= (n_1!) (n_2!) \int dx dy \exp(-x^2 - y^2) = \\ &= (n_1!) (n_2!) \pi \int_0^\infty d\rho e^{-\rho} = (n_1!) (n_2!) \pi \end{aligned}$$



3-pmodes-4

Finally we come to the paired oscillator mode orthogonal polynomials , being not Hermite polynomials , which prevail for unpaired modes, but simple monomials. This structure is derived from substituting the two expressions for the creation operators $a_{1,2}^\dagger : \left(\bar{\zeta} - \partial_\zeta \right)^{n_1}$ and $\left(\zeta - \partial_{\bar{\zeta}} \right)^{n_2}$ in eq. 86 , as combined operators inside the powers , acting on the left on the given paired mode ground state

$$(88) \quad \begin{aligned} \sqrt{2} a_1^\dagger &= \bar{\zeta} - \partial_\zeta = \exp \left(\zeta \bar{\zeta} \right) \left(-\partial_\zeta \right) \exp \left(-\zeta \bar{\zeta} \right) \\ \sqrt{2} a_2^\dagger &= \zeta - \partial_{\bar{\zeta}} = \exp \left(\zeta \bar{\zeta} \right) \left(-\partial_{\bar{\zeta}} \right) \exp \left(-\zeta \bar{\zeta} \right) \end{aligned}$$

The expression for the paired wave function $\psi_{n_1, n_2} \left(\zeta, \bar{\zeta} \right)$ in eq. 86 then takes the form

$$(89) \quad \begin{aligned} \psi_{n_1, n_2} \left(\zeta, \bar{\zeta} \right) &= \\ &= \mathcal{N} 2^{-\frac{1}{2} (n_1 + n_2)} \exp \left(\zeta \bar{\zeta} \right) \left(-\partial_\zeta \right)^{n_1} \left(-\partial_{\bar{\zeta}} \right)^{n_2} \exp \left(-2 \zeta \bar{\zeta} \right) \\ &= \mathcal{N} 2^{\frac{1}{2} (n_1 + n_2)} \bar{\zeta}^{n_1} \zeta^{n_2} \exp \left(-\zeta \bar{\zeta} \right) \end{aligned}$$

We use polar coordinates , as they are representing finite rotations of the complex ζ - plane →

3-pmodes-5

leading to the wave function representation

$$(90) \quad \psi_{n_1, n_2}(\zeta, \bar{\zeta}) = \left(\frac{2^{(n_1 + n_2)}}{\pi (n_1!) (n_2!)} \right)^{\frac{1}{2}} \exp(i(n_2 - n_1)\varphi) [\varrho^{(n_1 + n_2)} \exp(-\varrho^2)]$$

$$\varrho = |\zeta| \quad ; \quad \varphi = \arg(\zeta)$$

The functions ψ_{n_1, n_2} in eq. 90 form a complete basis in the space $L_2(\zeta_1, \zeta_2)$. They are combined with the restrictions from overall Fermi statistics – including an overall color antisymmetric selection rule in conjunction with the three Young tableaux as displayed in eq. 63.

Thus we study the action of the symmetric group S_3 on these base functions, as defined in eq. 79.

$$(91) \quad \left(U \left[\pi \begin{pmatrix} 1 & 2 & 3 \\ i_1 & i_2 & i_3 \end{pmatrix} \right] \psi_{n_1, n_2} \right) (\zeta, \bar{\zeta}) = \psi_{n_1, n_2} \left(\left[\pi \begin{pmatrix} 1 & 2 & 3 \\ i_1 & i_2 & i_3 \end{pmatrix} \right]^{-1} (\zeta, \bar{\zeta}) \right)$$



3-12

Eq. 91 needs to be elaborated, as follows

$$(92) \quad \left[\pi \begin{pmatrix} 1 & 2 & 3 \\ i_1 & i_2 & i_3 \end{pmatrix} \right]^{-1} = \pi \begin{pmatrix} i_1 & i_2 & i_3 \\ 1 & 2 & 3 \end{pmatrix}$$

Next we identify the subgroup of even permutations – $A_3 = Z_3$ – of S_3

$$(93) \quad \begin{aligned} Z &= \pi \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \quad \text{or} \quad \pi \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \\ Z^2 &= \pi \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad \text{or} \quad \pi \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} ; \quad Z^3 = \mathbb{1} \\ \rightarrow Z^{-1} &= Z^2 ; \quad Z^{-2} = Z \end{aligned}$$

Here is the place to emphasize that the discussion within subsection 3-1-3d is for the time being restricted to oscillatory modes in *one* space dimension, to be generalized to three subsequently, but after finishing the selection rules staying with 1 space dimension for the time being.

We proceed to identify the permutation Z as defined in eq. 93 with a rotation of the ζ plane by 120 degrees , completing the action of S_3 displayed in eq. 91



3-13

$$(94) \quad \left(U \left[\pi \begin{pmatrix} 1 & 2 & 3 \\ i_1 & i_2 & i_3 \end{pmatrix} \right] \psi_{n_1, n_2} \right) (\zeta, \bar{\zeta}) = \\ = D_{m_1 m_2 n_1 n_2} (\pi(\cdot)) \psi_{m_1, m_2} (\zeta, \bar{\zeta})$$

and for the abelian cyclic subgroup A_3 of even permutations eq. 94 becomes

$$(95) \quad \left(U \left[\pi \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \right] \psi_{n_1, n_2} \right) (\zeta, \bar{\zeta}) = \\ = D_{n_1 n_2}(Z) \psi_{n_1, n_2} (\zeta, \bar{\zeta}) = \psi_{n_1, n_2} (Z^{-1} \zeta, Z \bar{\zeta})$$

Eqs. 90 and 95 imply

$$(96) \quad D_{n_1 n_2}(Z) \psi_{n_1, n_2} (\zeta, \bar{\zeta}) = \\ = \exp(i(n_1 - n_2)(2\pi/3)) \psi_{n_1, n_2} (\zeta, \bar{\zeta}) = \\ = \psi_{n_1, n_2} (Z^{-1} \zeta, Z \bar{\zeta})$$

It follows from eq. 96

$$(97) \quad D_{n_1 n_2}(Z) = Z^{n_1 - n_2}, \quad Z = \exp(i(2\pi/3)) \quad \rightarrow$$

3-14

Next we decompose the action of Z on ζ into real and imaginary parts

$$\begin{aligned}
 \zeta &\longrightarrow Z \zeta = \zeta' : \\
 (98) \quad \zeta &= \begin{pmatrix} \bar{\xi}_2 \\ \bar{\xi}_1 \end{pmatrix}, \quad \zeta' = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \bar{\xi}_2 \\ \bar{\xi}_1 \end{pmatrix} \longrightarrow \\
 \bar{\xi}'_2 &= -\frac{1}{2} \bar{\xi}_2 - \frac{\sqrt{3}}{2} \bar{\xi}_1 \\
 \bar{\xi}'_1 &= \frac{\sqrt{3}}{2} \bar{\xi}_2 - \frac{1}{2} \bar{\xi}_1
 \end{aligned}$$

We recall eq. 81 , repeating it below

$$\begin{aligned}
 (99) \quad \bar{\xi}_1 &= \frac{1}{\sqrt{2}} (\bar{x}_1 - \bar{x}_2), \quad \bar{\xi}_2 = \frac{1}{\sqrt{6}} (\bar{x}_1 + \bar{x}_2 - 2\bar{x}_3) \\
 \bar{\xi}_3 &= \frac{1}{\sqrt{3}} (\bar{x}_1 + \bar{x}_2 + \bar{x}_3) = \sqrt{3} \bar{X}_{c.m.} \rightarrow 0
 \end{aligned}$$

Substituting the expressions on the first line in eq. 99 in eq. 98 we obtain

$$\begin{aligned}
 (100) \quad \bar{\xi}'_2 &= -\frac{1}{2\sqrt{6}} (\bar{x}_1 + \bar{x}_2 - 2\bar{x}_3) - \frac{\sqrt{3}}{2\sqrt{2}} (\bar{x}_1 - \bar{x}_2) \\
 \bar{\xi}'_1 &= \frac{\sqrt{3}}{2\sqrt{6}} (\bar{x}_1 + \bar{x}_2 - 2\bar{x}_3) - \frac{1}{2\sqrt{2}} (\bar{x}_1 - \bar{x}_2)
 \end{aligned}$$



and arranging the factors yielding the result

$$(101) \quad \begin{aligned} \bar{\xi}'_2 &= -\frac{1}{2\sqrt{6}} [(\bar{x}_1 + \bar{x}_2 - 2\bar{x}_3) + 3(\bar{x}_1 - \bar{x}_2)] \\ \bar{\xi}'_1 &= \frac{1}{2\sqrt{2}} [(\bar{x}_1 + \bar{x}_2 - 2\bar{x}_3) - (\bar{x}_1 - \bar{x}_2)] \end{aligned}$$

The final result is compared with the initial choice of barycentric variables in eq. 99

$$(102) \quad \begin{aligned} \bar{\xi}'_2 &= \frac{1}{\sqrt{6}} (\bar{x}_2 + \bar{x}_3 - 2\bar{x}_1) & \leftarrow & \quad \bar{\xi}_2 = \frac{1}{\sqrt{6}} (\bar{x}_1 + \bar{x}_2 - 2\bar{x}_3) \\ \bar{\xi}'_1 &= \frac{1}{\sqrt{2}} (\bar{x}_2 - \bar{x}_3) & \leftarrow & \quad \bar{\xi}_1 = \frac{1}{\sqrt{2}} (\bar{x}_1 - \bar{x}_2) \end{aligned}$$

The choice , marked by 'or' in eq. 93, is revealed inspecting the substitution of the $\bar{\xi}_j$ indices from the right hand - to the left hand side of eq. 102 , corresponding to the cyclic permutation associated with the actions of Z and $Z^{-1} \equiv Z^2$

$$(103) \quad \begin{aligned} Z &: \zeta \longrightarrow Z\zeta \quad \simeq \quad \pi \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \\ Z^2 &: \zeta \longrightarrow Z^2\zeta \quad \simeq \quad \pi \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix} \end{aligned}$$



3-rec - Reconstruction of the two-dimensional irreducible unitary representation of S_3 from 1 spacelike dimension

The actions of Z and $Z^{-1} \equiv Z^2$, defined in eq. 103 allows to associate two 2×2 representation matrices with the corresponding permutations, using eq. 98

$$\begin{aligned}
 & 2) \quad \pi \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \quad + \quad \rightarrow Z \\
 (104) \quad & 3) \quad \pi \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \quad + \quad \rightarrow Z^2 \equiv Z^{-1} \\
 & 4) \quad \pi \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad - \quad \rightarrow T_{12}
 \end{aligned}$$

In eq. 104 the last column denotes a shorthand name for the 3 constructed elements of S_3 , while the signs in the second last column are $+$ for even and $-$ for odd permutations respectively, equal to the determinant of the 2×2 representation matrices.

The assignment of the barycentric variables to the numbering of the associated – 1-dimensional – coordinates $\bar{x}_{1,2,3}$ as given in eq. 102 singles out the representation of the transposition $T_{12} \simeq 1 \leftrightarrow 2$ as being associated with the diagonal Pauli matrix σ_3 .



3-17

The remaining 3 elements including the identity permutation, denoted $\mathbb{1}$, also for its representing 2×2 unit matrix, can be found from multiplications of the 3 elements defined in eq. 104

$$(105) \quad \pi \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \longrightarrow \mathbb{1} = Z^3 = (Z^{-1})^3 = (T_{12})^2$$

$$1) \quad \pi \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \longrightarrow \mathbb{1}$$

The numbering 2) - 4) in eq. 104 and 1) in eq. 105 serves to segregate the subgroup of even permutations - $A_3 \simeq Z_3$ - corresponding to the entries numbered 1), 2), 3) from the odd ones, to be completed next.

This we do step by step. First we determine the product, using the symbol \circ : 2) \circ 4)

$$(106) \quad 2) \circ 4) = \pi \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \circ \pi \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \pi \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \rightarrow T_{23}$$



Thus we multiply the 2×2 matrices associated with the elements 2) and 4) in eq. 104

$$(107) \quad 2) \rightarrow \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} ; \quad 4) \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rightarrow$$

$$2) \circ 4) \rightarrow \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

and check compatibility of eqs. 106 and 107 implied by the 2-dimensional unitary representation of S_3 . This is done in an analogous way to the substitutions relative to A_3 applied in eqs. 102 and 103

$$(108) \quad \zeta \rightarrow \bar{\xi}_2 + i\bar{\xi}_1 \rightarrow \begin{pmatrix} \bar{\xi}_2 \\ \bar{\xi}_1 \end{pmatrix} ; \quad 2) \circ 4) : \zeta'' \rightarrow \begin{pmatrix} \bar{\xi}_2'' \\ \bar{\xi}_1'' \end{pmatrix}$$

$$\begin{pmatrix} \bar{\xi}_2'' \\ \bar{\xi}_1'' \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \bar{\xi}_2 \\ \bar{\xi}_1 \end{pmatrix}$$

with the identifications (eqs. 98 , 101)



3-19

$$\begin{aligned}
 (109) \quad \bar{\xi}_1 &= \frac{1}{\sqrt{2}} (\bar{x}_1 - \bar{x}_2) \quad , \quad \bar{\xi}_2 = \frac{1}{\sqrt{6}} (\bar{x}_1 + \bar{x}_2 - 2\bar{x}_3) \\
 \bar{\xi}_3 &= \frac{1}{\sqrt{3}} (\bar{x}_1 + \bar{x}_2 + \bar{x}_3) = \sqrt{3} \bar{X}_{c.m.} \rightarrow 0 \\
 \text{and} \quad \bar{\xi}_j &\rightarrow \bar{\xi}_j'' \quad ; \quad \bar{x}_j \rightarrow \bar{x}_j'' \quad ; \quad j = 1, 2, 3
 \end{aligned}$$

Then eq. 108 becomes

$$\begin{aligned}
 (110) \quad \bar{\xi}_2'' &= -\frac{1}{2} \bar{\xi}_2 + \frac{\sqrt{3}}{2} \bar{\xi}_1 \quad ; \quad \bar{\xi}_1'' = \frac{\sqrt{3}}{2} \bar{\xi}_2 + \frac{1}{2} \bar{\xi}_1 \rightarrow \\
 \bar{\xi}_2'' &= -\frac{1}{2\sqrt{6}} (\bar{x}_1 + \bar{x}_2 - 2\bar{x}_3) + \frac{\sqrt{3}}{2\sqrt{2}} (\bar{x}_1 - \bar{x}_2) \\
 \bar{\xi}_1'' &= +\frac{\sqrt{3}}{2\sqrt{6}} (\bar{x}_1 + \bar{x}_2 - 2\bar{x}_3) + \frac{1}{2\sqrt{2}} (\bar{x}_1 - \bar{x}_2)
 \end{aligned}$$

Rendering the factors commensurable in the last 2 lines of eq. 110 we obtain

$$\begin{aligned}
 (111) \quad \bar{\xi}_2'' &= \frac{1}{2\sqrt{6}} (2\bar{x}_3 - (\bar{x}_1 + \bar{x}_2) + 3(\bar{x}_1 - \bar{x}_2)) \\
 &= \frac{1}{\sqrt{6}} (\bar{x}_3 + \bar{x}_1 - 2\bar{x}_2) \\
 \bar{\xi}_1'' &= \frac{1}{2\sqrt{2}} ((\bar{x}_1 + \bar{x}_2 - 2\bar{x}_3) + (\bar{x}_1 - \bar{x}_2)) \\
 &= \frac{1}{\sqrt{2}} (\bar{x}_1 - \bar{x}_3)
 \end{aligned}$$



It remains to substitute the variables $\bar{\xi}_{1,2}''$ on the left hand side of eq. 111 to identify the associated permutation

$$(112) \quad \begin{aligned} \bar{\xi}_2'' &= \frac{1}{\sqrt{6}} \left(\bar{x}_1'' + \bar{x}_2'' - 2\bar{x}_3'' \right) = \frac{1}{\sqrt{6}} \left(\bar{x}_1 + \bar{x}_3 - 2\bar{x}_2 \right) \\ \bar{\xi}_1'' &= \frac{1}{\sqrt{2}} \left(\bar{x}_1'' - \bar{x}_2'' \right) = \frac{1}{\sqrt{2}} \left(\bar{x}_1 - \bar{x}_3 \right) \end{aligned}$$

which follows from the ordering of the indices of the barycentric variables \bar{x}_j ; $j = 1, 2, 3$ appearing on the rightmost side of eq. 112 as

$$(113) \quad 2) \circ 4) \rightarrow \pi \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \rightarrow T_{23} \quad (\checkmark)$$

Hence we can consistently identify the second odd permutation, which becomes the fifth constructed permutation according to the definition $2) \circ 4) = 5)$ in accordance with eqs. 107 and 113

$$(114) \quad 5) \quad \pi \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \rightarrow T_{23}$$



3-21

It remains to construct the third odd permutation and its associated 2×2 unitary representation matrix. We choose among several paths to consider the multiplication $3) \circ 4)$ closely follow the previous multiplication $2) \circ 4)$ in eq. 106

$$(115) \quad 3) \circ 4) = \pi \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \circ \pi \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \pi \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \rightarrow T_{13}$$

The resulting remaining odd permutation T_{13} as result of the $3) \circ 4)$ multiplication is not surprising. Thus we multiply the 2×2 matrices associated with the elements 3) and 4) analogous to eq. 107 for $2) \circ 4)$

$$(116) \quad 3) \rightarrow \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} ; \quad 4) \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rightarrow$$

$$3) \circ 4) \rightarrow \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$



The analog for 3) o 4) to eq. 108 for 2) o 4) becomes

$$(117) \quad \zeta \rightarrow \bar{\xi}_2 + i\bar{\xi}_1 \rightarrow \begin{pmatrix} \bar{\xi}_2 \\ \bar{\xi}_1 \end{pmatrix} ; 3) \circ 4) : \zeta''' \rightarrow \begin{pmatrix} \bar{\xi}_2''' \\ \bar{\xi}_1''' \end{pmatrix}$$

$$\begin{pmatrix} \bar{\xi}_2''' \\ \bar{\xi}_1''' \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \bar{\xi}_2 \\ \bar{\xi}_1 \end{pmatrix}$$

Next we adapt eqs. 109 - 111 relative to 2) o 4) to 3) o 4) , which yields

$$(118) \quad \bar{\xi}_1 = \frac{1}{\sqrt{2}} (\bar{x}_1 - \bar{x}_2) , \quad \bar{\xi}_2 = \frac{1}{\sqrt{6}} (\bar{x}_1 + \bar{x}_2 - 2\bar{x}_3)$$

$$\bar{\xi}_3 = \frac{1}{\sqrt{3}} (\bar{x}_1 + \bar{x}_2 + \bar{x}_3) = \sqrt{3} \bar{X}_{c.m.} \rightarrow 0$$

and $\bar{\xi}_j \rightarrow \bar{\xi}_j''' ; \bar{x}_j \rightarrow \bar{x}_j''' ; j = 1, 2, 3$

as well as

$$(119) \quad \bar{\xi}_2''' = -\frac{1}{2} \bar{\xi}_2 - \frac{\sqrt{3}}{2} \bar{\xi}_1 ; \quad \bar{\xi}_1''' = -\frac{\sqrt{3}}{2} \bar{\xi}_2 + \frac{1}{2} \bar{\xi}_1 \rightarrow$$

$$\bar{\xi}_2''' = -\frac{1}{2\sqrt{6}} (\bar{x}_1 + \bar{x}_2 - 2\bar{x}_3) - \frac{\sqrt{3}}{2\sqrt{2}} (\bar{x}_1 - \bar{x}_2)$$

$$\bar{\xi}_1''' = -\frac{\sqrt{3}}{2\sqrt{6}} (\bar{x}_1 + \bar{x}_2 - 2\bar{x}_3) + \frac{1}{2\sqrt{2}} (\bar{x}_1 - \bar{x}_2) \rightarrow$$

The third equation (eq. 111) relative to 2) o 4) is replaced for 3) o 4) by

$$\begin{aligned}
 \bar{\xi}_2''' &= \frac{1}{2\sqrt{6}} (2\bar{x}_3 - (\bar{x}_1 + \bar{x}_2) - 3(\bar{x}_1 - \bar{x}_2)) \\
 &= \frac{1}{\sqrt{6}} (\bar{x}_3 + \bar{x}_2 - 2\bar{x}_1) \\
 \bar{\xi}_1''' &= \frac{1}{2\sqrt{2}} ((-\bar{x}_1 - \bar{x}_2 + 2\bar{x}_3) + (\bar{x}_1 - \bar{x}_2)) \\
 &= \frac{1}{\sqrt{2}} (\bar{x}_3 - \bar{x}_2)
 \end{aligned}
 \tag{120}$$

Substituting the variables $\bar{\xi}_{1,2}'''$ on the left hand side of eq. 120 we identify the permutation associated with 3) o 4) , in analogy to eq. 112 for 2) o 4)

$$\begin{aligned}
 \bar{\xi}_2''' &= \frac{1}{\sqrt{6}} (\bar{x}_1''' + \bar{x}_2''' - 2\bar{x}_3''') = \frac{1}{\sqrt{6}} (\bar{x}_3 + \bar{x}_2 - 2\bar{x}_1) \\
 \bar{\xi}_1''' &= \frac{1}{\sqrt{2}} (\bar{x}_1''' - \bar{x}_2''') = \frac{1}{\sqrt{2}} (\bar{x}_3 - \bar{x}_2)
 \end{aligned}
 \tag{121}$$

which follows from the ordering of the indices of the barycentric variables $\bar{x}_j ; j = 1, 2, 3$ appearing on the rightmost side of eq. 121 in continuing the analogy tp eqs. 112 - 114 for 2) o 4) = 5) \rightarrow

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$$(122) \quad 3) \circ 4) \rightarrow \pi \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \rightarrow T_{13} \quad (\checkmark)$$

Hence we can consistently identify the third odd permutation, which becomes the six'th and last constructed permutation according to the definition $3) \circ 4) = 6)$ in accordance with eqs. 116 and 122 analogous with eqs. 107 and 113 for the product $2) \circ 4) = 5)$

$$(123) \quad 6) \quad \pi \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \longrightarrow T_{13}$$

3-res - Results on the reconstruction of the two-dimensional irreducible unitary representation of S_3 from 1 spacelike dimension

In this subsection we collect the results first in the representation of the permutation group elements as in eqs.

104 (2), 3), 4)), 105 (1)), 114 (5)), 123 (6)), in the order 1) - 6). \longrightarrow

(124)

$$\begin{aligned}
 1) \quad \pi \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} &\longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} &+ &\rightarrow \mathbb{I} \\
 2) \quad \pi \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} &\longrightarrow \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} &+ &\rightarrow Z \\
 3) \quad \pi \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} &\longrightarrow \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} &+ &\rightarrow Z^2 \equiv Z^{-1} \\
 4) \quad \pi \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} &\longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} &- &\rightarrow T_{12} \\
 5) \quad \pi \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} &\longrightarrow \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} &- &\rightarrow T_{23} \\
 6) \quad \pi \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} &\longrightarrow \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} &- &\rightarrow T_{13}
 \end{aligned}$$



The main ingredients to the construction of irreducible representations of finite groups , here S_3 , leading – in summary – to eq. 124 , are to my knowledge due to Issai Schur , as displayed also in ref. [22-2006] .

3-res-2 - Choosing a complex basis for transforming the basis derived in eq. 124 for the two-dimensional irreducible unitary representation of S_3 from 1 spacelike dimension

Here we invert the decomposition of the complex numbers ζ , $\bar{\zeta}$ into real and imaginary parts , as displayed (e.g.) in eqs. 98 and 108

$$\begin{aligned}
 \zeta &\longrightarrow Z \zeta = \zeta' : \\
 \zeta &= \begin{pmatrix} \bar{\xi}_2 \\ \bar{\xi}_1 \end{pmatrix} , \quad \zeta' = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \bar{\xi}_2 \\ \bar{\xi}_1 \end{pmatrix} \longrightarrow \\
 \bar{\xi}'_2 &= -\frac{1}{2} \bar{\xi}_2 - \frac{\sqrt{3}}{2} \bar{\xi}_1 \\
 \bar{\xi}'_1 &= \frac{\sqrt{3}}{2} \bar{\xi}_2 - \frac{1}{2} \bar{\xi}_1
 \end{aligned}
 \tag{125}$$

repeated above and on next page below



$$(126) \quad \zeta \rightarrow \bar{\xi}_2 + i\bar{\xi}_1 \rightarrow \begin{pmatrix} \bar{\xi}_2 \\ \bar{\xi}_1 \end{pmatrix} ; 2) \circ 4) : \zeta'' \rightarrow \begin{pmatrix} \bar{\xi}_2'' \\ \bar{\xi}_1'' \end{pmatrix}$$

$$\begin{pmatrix} \bar{\xi}_2'' \\ \bar{\xi}_1'' \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \bar{\xi}_2 \\ \bar{\xi}_1 \end{pmatrix}$$

back to the original complex- and complex conjugate variables ζ , $\bar{\zeta}$, as defined on the left hand side of the first relation in eq 126

$$\begin{pmatrix} \bar{\xi}_2 \\ \bar{\xi}_1 \end{pmatrix} \rightarrow \begin{pmatrix} \zeta = \bar{\xi}_2 + i\bar{\xi}_1 \\ \bar{\zeta} = \bar{\xi}_2 - i\bar{\xi}_1 \end{pmatrix} = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} \bar{\xi}_2 \\ \bar{\xi}_1 \end{pmatrix} = \mathcal{M} \begin{pmatrix} \bar{\xi}_2 \\ \bar{\xi}_1 \end{pmatrix}$$

$$\mathcal{M}^\dagger = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \rightarrow \mathcal{M}\mathcal{M}^\dagger = \mathcal{M}^\dagger\mathcal{M} = 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{2 \times 2}$$

(127)

Thus the unitary 2 x 2 matrix

$$(128) \quad u = \frac{1}{\sqrt{2}} \mathcal{M}$$

is a unitary 2 x 2 matrix which generates the similarity transformation \rightarrow

through the following steps , denoting by $D_\pi ; \pi \in S_3$ the six 2 x 2 unitary matrices in the basis given in eq. 124 and likewise by $d_\pi ; \pi \in S_3$ the six transformed 2 x 2 unitary matrices , associated with the basis as described in eq. 127

$$\pi \rightarrow d_\pi : \begin{pmatrix} \zeta \\ \bar{\zeta} \end{pmatrix} \longrightarrow d_\pi \begin{pmatrix} \zeta \\ \bar{\zeta} \end{pmatrix} = d_\pi \mathcal{M} \begin{pmatrix} \bar{\xi}_2 \\ \bar{\xi}_1 \end{pmatrix}$$

$$(129) \quad \begin{pmatrix} \zeta \\ \bar{\zeta} \end{pmatrix} = \mathcal{M} \begin{pmatrix} \bar{\xi}_2 \\ \bar{\xi}_1 \end{pmatrix}$$

$$\pi \rightarrow D_\pi : \begin{pmatrix} \bar{\xi}_2 \\ \bar{\xi}_1 \end{pmatrix} \longrightarrow D_\pi \begin{pmatrix} \bar{\xi}_2 \\ \bar{\xi}_1 \end{pmatrix}$$

Next we multiply the last relation in eq. 129 by \mathcal{M} from the left

$$(130) \quad \pi \rightarrow D_\pi : \begin{pmatrix} \zeta \\ \bar{\zeta} \end{pmatrix} \longrightarrow \mathcal{M} D_\pi \begin{pmatrix} \bar{\xi}_2 \\ \bar{\xi}_1 \end{pmatrix} = \mathcal{M} D_\pi \mathcal{M}^{-1} \begin{pmatrix} \zeta \\ \bar{\zeta} \end{pmatrix}$$



Comparing eq. 130 with the first relation in eq. 129 we obtain the sought similarity transformation

$$d_{\pi} = \mathcal{M} D_{\pi} \mathcal{M}^{-1} = u D_{\pi} u^{-1}$$

(131)

$$u = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}, \quad u^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$$

The detailed calculations of the matrix product associated with the sixth permutation representation matrices d_{π} from the basis formed by D_{π} given in eq. 124 are performed in Appendix 3 . The collection of 2 x 2 representation matrices $d_{\pi}; \pi = 1, \dots, 6$ is displayed in eq. 132 below →

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$$\begin{aligned}
 1) \ d_{\pi=1} \quad \pi \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} &\longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} &+ &\rightarrow \mathbb{1} \\
 2) \ d_{\pi=2} \quad \pi \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} &\longrightarrow \begin{pmatrix} e^{+i(2\pi/3)} & 0 \\ 0 & e^{-i(2\pi/3)} \end{pmatrix} &+ &\rightarrow Z \\
 3) \ d_{\pi=3} \quad \pi \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} &\longrightarrow \begin{pmatrix} e^{-i(2\pi/3)} & 0 \\ 0 & e^{+i(2\pi/3)} \end{pmatrix} &+ &\rightarrow Z^2 \equiv Z^{-1} \\
 4) \ d_{\pi=4} \quad \pi \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} &\longrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} &- &\rightarrow T_{12} \\
 5) \ d_{\pi=5} \quad \pi \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} &\longrightarrow \begin{pmatrix} 0 & e^{+i(2\pi/3)} \\ e^{-i(2\pi/3)} & 0 \end{pmatrix} &- &\rightarrow T_{23} \\
 6) \ d_{\pi=6} \quad \pi \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} &\longrightarrow \begin{pmatrix} 0 & e^{-i(2\pi/3)} \\ e^{+i(2\pi/3)} & 0 \end{pmatrix} &- &\rightarrow T_{13}
 \end{aligned}$$

(132)



3-31-3

3-res-3 - Extending 1 spatial dimension to 3

We go back to the subsection on page 3-pmodes-1 :

Modes of a pair of onedimensional oscillators – pairmodes and the complex plane

which for 3 spatial dimension becomes 

Modes of 3 pairs of onedimensional oscillator-pairmodes and the complex 3-plane

The 1 complex variable ζ associated with the 1 pairmode as appropriate for 1 space dimension and defined in eq. 82 , for 3 space-time dimensions with axes (X) , (Y) , (Z) , thus becomes a complex three vector

$$(133) \quad \vec{\zeta} = (\zeta^{(X)} , \zeta^{(Y)} , \zeta^{(Z)})$$

The notation (X) , (Y) , (Z) for the three orthogonal axes of the 3-dimensional configuration space in the c.m. system is chosen in order to prevent confusing these with the 1-dimensional quantities denoted $x \dots$, $y \dots$, $z \dots$, as introduced for 1 spatial dimension and defined in eqs. 82 and 81 , which become 3-vectors for 3 space dimensions .

The extension of the various space variables from 1 to 3 dimensions we shall do in segmented steps :



3-32-3

1-1) The center of mass position variables in 1 spatial dimension

These variables appear (last) in eq. 75 repeated below

$$\langle \Psi^{(2)} | \Psi^{(1)} \rangle = \lambda^6 \int \prod_{i=1}^3 d^3 \bar{\xi}_i \delta^3 \left(\bar{\xi}_3 \right) \times \\ \times \Psi^{*(2)} \left(\bar{x}_1, \bar{x}_2, \bar{x}_3 \right) \Psi^{(1)} \left(\bar{x}_1, \bar{x}_2, \bar{x}_3 \right)$$

$$\bar{\xi}_1 = \frac{1}{\sqrt{2}} \left(\bar{x}_1 - \bar{x}_2 \right) , \quad \bar{\xi}_2 = \frac{1}{\sqrt{6}} \left(\bar{x}_1 + \bar{x}_2 - 2\bar{x}_3 \right)$$

$$\bar{\xi}_3 = \frac{1}{\sqrt{3}} \left(\bar{x}_1 + \bar{x}_2 + \bar{x}_3 \right) = \sqrt{3} \bar{X}_{c.m.} \rightarrow 0$$

$$z_j = \lambda \bar{z}_j , \quad x_j = \lambda \bar{x}_j ; \quad j = 1, 2, 3 ; \quad X_{c.m.} = \lambda \bar{X}_{c.m.} = \frac{1}{3} \sum_{i=1}^3 \bar{x}_i$$

(134)

1-3) Extension of the last three relations in eq. 134 to 3 spatial dimensions

The extension takes the form

$$x_j \rightarrow \vec{x}_j \quad \text{with} \quad \vec{x}_j = \left(x_j^{(X)} , x_j^{(Y)} , x_j^{(Z)} \right) ; \quad j = 1, 2, 3$$

$$X_{c.m.} \rightarrow \vec{X}_{c.m.} \quad \text{with} \quad \vec{X}_{c.m.} = \left(X_{c.m.}^{(X)} , X_{c.m.}^{(Y)} , X_{c.m.}^{(Z)} \right) = 0$$

(135)

Again note that $\vec{X}_{c.m.}$ and the axis superfix (X) denote very different objects .



3-33-3

1-3) (continued) The configuration space 3-vectors $\vec{x}_{1,2,3}$, $\vec{X}_{c.m.}$ in eq. 135 have dimension $[\text{mass}^{-1}]$ in rational units. They can be reduced to dimensionless configuration space variables, as given in eqs. 20 - 26 for 1 spatial dimension and in the case of 3 spatial dimensions follows straightforwardly from eqs. 20, 24 and 28

$$(136) \quad \vec{\bar{x}}_j = \lambda^{-1} \vec{x}_j ; \quad j = 1, 2, 3 ; \quad \lambda^{-1} = \left(\frac{\Lambda}{K_{N_c}} \right)^{1/2}$$

$$K_{N_c} = N_c / (N_c - 1) \quad \left(= \frac{3}{2} \text{ here} \right) ; \quad 2\Lambda = 1 / \alpha' \sim 1.06 \text{ GeV}^2$$

2-1) Dimensionless barycentric coordinates in 1 space dimension

We recall the definition of the dimensionless barycentric coordinates associated with the dimensionless quantities $\bar{x}_{1,2,3}$, $\bar{X}_{c.m.} = 0$ for 1 spatial dimension in eqs. 81 and 135 in point 1-1)



3-34-3

2-1) (continued)

$$\bar{\xi}_1 = \frac{1}{\sqrt{2}} (\bar{x}_1 - \bar{x}_2) \quad , \quad \bar{\xi}_2 = \frac{1}{\sqrt{6}} (\bar{x}_1 + \bar{x}_2 - 2\bar{x}_3)$$

$$\bar{\xi}_3 = \frac{1}{\sqrt{3}} (\bar{x}_1 + \bar{x}_2 + \bar{x}_3) = \sqrt{3} \bar{X}_{c.m.} \rightarrow 0$$

$$x_j = \lambda \bar{x}_j \quad ; \quad j = 1, 2, 3 \quad ; \quad X_{c.m.} = \lambda \bar{X}_{c.m.} = \frac{1}{3} \sum_{i=1}^3 \bar{x}_i = 0$$

(137)

$$\bar{x}_1 = \frac{1}{\sqrt{2}} \bar{\xi}_1 + \frac{1}{\sqrt{6}} \bar{\xi}_2$$

$$\bar{x}_2 = -\frac{1}{\sqrt{2}} \bar{\xi}_1 + \frac{1}{\sqrt{6}} \bar{\xi}_2$$

$$\bar{x}_3 = -\frac{2}{\sqrt{6}} \bar{\xi}_2$$

2-3) Extension of the dimensionless variables in eq. 137 to 3 spatial coordinates

The extension of the configuration variables $\bar{x}_{1,2,3} \rightarrow \vec{\bar{x}}_{1,2,3}$ from $d = 1$ to $d=3$ dimensions is defined in point 1-3) . Similarly the 2 barycentric coordinates for $d = 1$ become three vectors for $d = 3$.



3-35-3

2-3) (continued)

$$(138) \quad \begin{aligned} \vec{\xi}_1 &= \frac{1}{\sqrt{2}} \left(\vec{x}_1 - \vec{x}_2 \right) , \quad \vec{\xi}_2 = \frac{1}{\sqrt{6}} \left(\vec{x}_1 + \vec{x}_2 - 2\vec{x}_3 \right) \\ \vec{\xi}_3 &= \frac{1}{\sqrt{3}} \left(\vec{x}_1 + \vec{x}_2 + \vec{x}_3 \right) = \sqrt{3} \vec{X}_{c.m.} \rightarrow 0 \end{aligned}$$

$$\vec{\xi}_j = \left(\vec{\xi}_j^{(X)} , \vec{\xi}_j^{(Y)} , \vec{\xi}_j^{(Z)} \right) ; \quad j = 1.2.3$$

The suffix numbering the different barycentric 3-vectors (in 3 spacelike dimension) are displayed in boldface style in order to emphasize that the numerals labelling $\vec{\xi}_j$; $j = 1.2.3$ are logically quite distinct from those numerals labelling $\vec{x}_{1,2,3}$; $j = 1.2.3$, displayed in cursive mode . The inverse relations expressing $\vec{x}_{1,2,3}$; $j = 1.2.3$ as a function of the 2 independent, barycentric dimensionless 3-vectors become

$$(139) \quad \begin{aligned} \vec{x}_1 &= \frac{1}{\sqrt{2}} \vec{\xi}_1 + \frac{1}{\sqrt{6}} \vec{\xi}_2 \\ \vec{x}_2 &= -\frac{1}{\sqrt{2}} \vec{\xi}_1 + \frac{1}{\sqrt{6}} \vec{\xi}_2 \\ \vec{x}_3 &= -\frac{2}{\sqrt{6}} \vec{\xi}_2 \end{aligned}$$

The relations in eq. 139 elucidate the different meaning of the suffix labels in cursive mode and boldface mode .



3-36-3

3-1) The *linear* oscillator basis and mode excitation numbers n_1, n_2 for 1 and 3 spatial dimensions
In order to recall the meaning of the originally adopted labels 1,2 we refer back to subsection 2-4-1 comprising eqs 16 - 28 . First we adapt from these equations eq. 19 from the barycentric coordinates $z_1 z_2$, defined in ref. [1-1980] , to the dimensionless ones $\bar{\xi}_1, \bar{\xi}_2$ as used here for 1 spatial dimension , with the identification as in eq. 137 , together with their relative canonical momenta , denoted $\bar{\pi}_1, \bar{\pi}_2$ (as displayed in eqs. 20 - 23)

$$\bar{\xi}_1, \bar{\xi}_2 \longleftrightarrow \bar{\pi}_1, \bar{\pi}_2$$

$$\bar{\pi}_\mu = \frac{1}{i} \partial_{\bar{\xi}_\mu} ; \mu = 1, 2$$

(140)

$$\bar{\xi}_1 = \frac{1}{\sqrt{2}} (\bar{x}_1 - \bar{x}_2) , \quad \bar{\xi}_2 = \frac{1}{\sqrt{6}} (\bar{x}_1 + \bar{x}_2 - 2\bar{x}_3)$$

The associated (universal) oscillator vector-variables derive from the relation on the last line in eq. 24

$$a_{\mu k} = \frac{1}{\sqrt{2}} \left(i \bar{\pi}_{\mu k} + \bar{\xi}_{\mu k} \right) ; \quad a_{\mu k}^\dagger = \frac{1}{\sqrt{2}} \left(-i \bar{\pi}_{\mu k} + \bar{\xi}_{\mu k} \right)$$

$$\mu = 1, 2 ; \quad k = 1, 2, 3 ; \quad i \bar{\pi}_{\mu k} = \partial_{\bar{\xi}_{\mu k}}$$

(141)

which in turn yields the relativistic structure of the oscillatory \mathcal{M}^2 operator



3-1) (continued)

substituting in eqs. 20 , 22 and 24

$$(142) \quad \mathcal{M}^2 = (2\Lambda) \sum_{\nu=1, k=1}^{\nu=2, k=3} \left[\bar{a}_{\nu k}^\dagger \bar{a}_{\nu k} + \frac{1}{2} \right] ; \quad 2\Lambda = 1/\alpha'$$

The contribution of the zero mode oscillations , $\frac{1}{2}$ for each dimension of configuration space (= 3) is inherent to the classical limiting form of oscillatory motion , and is accompanied in the sense of a long range approximation (especially given finite quark masses) by a constant correction .The latter remains non zero also in the limit of vanishing quark masses , as discussed in ref. [1-1980] . In eq. 142 $1/\alpha'$ denotes the inverse of the Regge slope.

The last relation in eq. 141 determines the oscillator basis corresponding to the decomposition into *linear* oscillatory modes , which straightforwardly extend from 1 to 3 spatial dimensions

$$(143) \quad a_{\mu k} = \frac{1}{\sqrt{2}} \left(\partial \bar{\xi}_{\mu k} + \bar{\xi}_{\mu k} \right) ; \quad a_{\mu k}^\dagger = \frac{1}{\sqrt{2}} \left(-\partial \bar{\xi}_{\mu k} + \bar{\xi}_{\mu k} \right)$$

$$\mu = 1, 2 ; \quad k = 1, 2, 3$$

Eq. 143 refers to 3 spatial dimensions while the case d = 1 would correspond to limit the suffix k in eqs. 141 - 143 to k = 1 .



3-3) Extending to 3 spatial dimensions and circular oscillatory pair-modes and wave function basis
Pair-modes have been discussed for 1 spatial dimension in subsection

”Modes of a pair of onedimensional oscillators – pairmodes and the complex plane”

which comprizes eqs. 82 - 103 .

For 1 spatial dimension we have 2 independent oscillators , which exhibit , whence the flavor & spin degrees of freedom are decoupled according to the decomposition in eq. 63 , an intrinsic SU2 symmetry of the 2 barycentric oscillatory modes . This symmetry allows to choose – always for 1 spacial dimension – a basis , henceforth called circular mode basis , in which the two oscillators take the form given in eq. 82 repeated in adapted notation below

$$(144) \quad \begin{aligned} \zeta &= \frac{1}{\sqrt{2}} \left(\bar{\xi}_2 + i \bar{\xi}_1 \right) & ; & \quad x = \bar{\xi}_2 , y = \bar{\xi}_1 \\ a_1 &= \frac{1}{\sqrt{2}} \left(\partial_\zeta + \bar{\zeta} \right) & ; & \quad a_2 = \frac{1}{\sqrt{2}} \left(\partial_{\bar{\zeta}} + \zeta \right) \\ a_1^\dagger &= \frac{1}{\sqrt{2}} \left(-\partial_{\bar{\zeta}} + \zeta \right) & ; & \quad a_2^\dagger = \frac{1}{\sqrt{2}} \left(-\partial_\zeta + \bar{\zeta} \right) \end{aligned}$$

$$[a_1, a_2] = [a_1, a_2^\dagger] = [a_2, a_1^\dagger] = 0$$



3-3) (continued)

From here (eq. 144) extension from 1 to 3 space dimensions follows 'stroke by stroke' .

To demonstrate this we repeat eq. 133 in adapted form below

$$(145) \quad \zeta \rightarrow \zeta \longrightarrow \vec{\zeta} = (\zeta^{(X)} , \zeta^{(Y)} , \zeta^{(Z)})$$

which entails the extension of the circular mode oscillators in eq. 144

$$(146) \quad a_{\mu} \rightarrow \vec{a}_{\mu} ; \mu = 1, 2 \rightarrow \mu = 1, 2$$

$$\vec{a}_{\mu} = \left(a_{\mu}^{(X)} , a_{\mu}^{(Y)} , a_{\mu}^{(Z)} \right) ; \mu = 1, 2$$

Eq. 144 extended to 3 space dimensions becomes

$$\zeta_j = \frac{1}{\sqrt{2}} \left(\bar{\xi}_{2j} + i \bar{\xi}_{1j} \right) ; x_j = \bar{\xi}_{2j}, y_j = \bar{\xi}_{1j}$$

$$a_{1j} = \frac{1}{\sqrt{2}} \left(\partial_{\zeta_j} + \bar{\zeta}_j \right) ; a_{2j} = \frac{1}{\sqrt{2}} \left(\partial_{\bar{\zeta}_j} + \zeta_j \right)$$

$$(147) \quad a_{1j}^{\dagger} = \frac{1}{\sqrt{2}} \left(-\partial_{\bar{\zeta}_j} + \zeta_j \right) ; a_{2j}^{\dagger} = \frac{1}{\sqrt{2}} \left(-\partial_{\zeta_j} + \bar{\zeta}_j \right)$$

$$\left[a_{1j}, a_{2k} \right] = \left[a_{1j}, a_{2k}^{\dagger} \right] = \left[a_{2k}, a_{1j}^{\dagger} \right] = 0 ; j, k = (X), (Y), (Z)$$



3-3) (continued)

It is appropriate here – now for 3 space dimensions , and using the circular pair mode oscillator basis – to restate the vector nature of oscillator absorption and creation operators (as derived in eqs. 146 and 147) in vector- and component notation

$$(148) \quad \begin{aligned} \vec{a}_\mu &\leftrightarrow a_{\mu \mathbf{k}} = \left(\vec{a}_\mu \right)_{\mathbf{k}} \\ \vec{a}_\mu^\dagger &\leftrightarrow a_{\mu \mathbf{k}}^\dagger = \left(\vec{a}_\mu^\dagger \right)_{\mathbf{k}} \end{aligned} ; \quad \mu = 1,2, \quad \mathbf{k} = (X), (Y), (Z)$$

The components – 6 each for creation- and annihilation operators – $a_{\mu \mathbf{j}}, a_{\nu \mathbf{k}}^\dagger$, interpreted in the circular pair mode oscillator basis , satisfy the nontrivial *commutation* relations

$$(149) \quad \left[a_{\mu \mathbf{j}}, a_{\nu \mathbf{k}}^\dagger \right] = \delta_{\mu\nu} \delta_{\mathbf{j}\mathbf{k}} \mathbb{1} ; \quad \left\{ \begin{array}{l} \mu, \nu \\ \mathbf{j}, \mathbf{k} \end{array} \right\} = \left\{ \begin{array}{l} 1,2 \\ (X), (Y), (Z) \end{array} \right\}$$

all other commutators vanish

The commutation relations in eq. 149 as such do not depend on the oscillator basis (allowing an SU6 invariant structure) , but the operator realization is particularly adapted , whence circular oscillator basis is chosen . This is done in the next step



3-3) (continued)

Extending the mode structure in the circular oscillator basis from 1 to 3 space dimensions

From here on new material , elaborated in 2013 , shapes the discussion of counting oscillatory modes of quarks in baryons .

We repeat the form of the wave function in the circular oscillatry mode basis corresponding for 1 pair of oscillators , resulting from the associated structure of the pair of creation operators (eqs. 144 and 88 below

$$\begin{aligned}
 \zeta &= \frac{1}{\sqrt{2}} \left(\bar{\xi}_2 + i \bar{\xi}_1 \right) & ; & \quad x = \bar{\xi}_2 , y = \bar{\xi}_1 \\
 a_1 &= \frac{1}{\sqrt{2}} \left(\partial_\zeta + \bar{\zeta} \right) & ; & \quad a_2 = \frac{1}{\sqrt{2}} \left(\partial_{\bar{\zeta}} + \zeta \right) \\
 a_1^\dagger &= \frac{1}{\sqrt{2}} \left(-\partial_{\bar{\zeta}} + \zeta \right) & ; & \quad a_2^\dagger = \frac{1}{\sqrt{2}} \left(-\partial_\zeta + \bar{\zeta} \right)
 \end{aligned}
 \tag{150}$$

$$\begin{aligned}
 [a_1, a_2] &= [a_1, a_2^\dagger] = [a_2, a_1^\dagger] = 0 \\
 \sqrt{2} a_1^\dagger &= \bar{\zeta} - \partial_\zeta = \exp(\zeta \bar{\zeta}) (-\partial_\zeta) \exp(-\zeta \bar{\zeta}) \\
 \sqrt{2} a_2^\dagger &= \zeta - \partial_{\bar{\zeta}} = \exp(\zeta \bar{\zeta}) (-\partial_{\bar{\zeta}}) \exp(-\zeta \bar{\zeta})
 \end{aligned}
 \tag{151}$$



3-3) (continued)

Thus as shown in eqs. 89 and 90 for 1 space dimensions the wave function in the circular oscillator basis – also corresponding to 1 space dimension takes the form repeated below

$$\begin{aligned}
 \psi_{n_1, n_2}(\zeta, \bar{\zeta}) &= \\
 &= \mathcal{N} 2^{-\frac{1}{2}(n_1 + n_2)} \exp(\zeta \bar{\zeta}) (-\partial_\zeta)^{n_1} (-\partial_{\bar{\zeta}})^{n_2} \exp(-2\zeta \bar{\zeta}) \\
 &= \mathcal{N} 2^{\frac{1}{2}(n_1 + n_2)} \bar{\zeta}^{n_1} \zeta^{n_2} \exp(-\zeta \bar{\zeta})
 \end{aligned}$$

(152)

and

$$\begin{aligned}
 \psi_{n_1, n_2}(\zeta, \bar{\zeta}) &= \\
 &= \left(\frac{2^{(n_1 + n_2)}}{\pi (n_1!) (n_2!)} \right)^{\frac{1}{2}} \exp(i(n_2 - n_1)\varphi) [\varrho^{(n_1 + n_2)} \exp(-\varrho^2)]
 \end{aligned}$$

$$\varrho = |\zeta| ; \varphi = \arg(\zeta)$$

(153)



3-3) (continued)

The further extension from 1 to 3 space dimensions consists in assigning to the single integers $n_{1,2}$ determining the wave function in eqs. 152 and 153 vectors, denoted $\mathbf{n}_{1,2}$, which in components become

$$(154) \quad \mathbf{n}_{\mu} = \left(\mathbf{n}_{\mu}^{(X)}, \mathbf{n}_{\mu}^{(Y)}, \mathbf{n}_{\mu}^{(Z)} \right) ; \quad \mu = 1,2$$

The wave function for 1 space dimension, displayed in eq. 153 becomes for 3 such

$$(155) \quad \psi_{\mathbf{n}_1, \mathbf{n}_2} \left(\vec{\zeta}, \vec{\zeta} \right) = \prod_{\mathbf{k}} \left(\frac{2 \binom{\mathbf{k}}{n_1^{\mathbf{k}} + n_2^{\mathbf{k}}}}{\pi \left(n_1^{\mathbf{k}}! \right) \left(n_2^{\mathbf{k}}! \right)} \right)^{\frac{1}{2}} \exp \left(i \left(n_2^{\mathbf{k}} - n_1^{\mathbf{k}} \right) \varphi_{\mathbf{k}} \right) \times \left[\varrho_{\mathbf{k}}^{\left(n_1^{\mathbf{k}} + n_2^{\mathbf{k}} \right)} \exp \left(-\varrho_{\mathbf{k}}^2 \right) \right]$$

$$\varrho_j = \left| \zeta_j \right| ; \quad \varphi_j = \arg \left(\zeta_j \right) ; \quad \mathbf{j} = (X), (Y), (Z)$$



3-44-3

3-3) (continued)

The imeasure in the scalar product in eq. 134 repeated below

$$\langle \Psi^{(2)} | \Psi^{(1)} \rangle = \lambda^6 \int \prod_{i=1}^3 d^3 \bar{\xi}_i \delta^3 \left(\bar{\xi}_3 \right) \times \\ \times \Psi^{*(2)} \left(\bar{x}_1, \bar{x}_2, \bar{x}_3 \right) \Psi^{(1)} \left(\bar{x}_1, \bar{x}_2, \bar{x}_3 \right)$$

$$\bar{\xi}_1 = \frac{1}{\sqrt{2}} \left(\bar{x}_1 - \bar{x}_2 \right) , \quad \bar{\xi}_2 = \frac{1}{\sqrt{6}} \left(\bar{x}_1 + \bar{x}_2 - 2\bar{x}_3 \right)$$

$$\bar{\xi}_3 = \frac{1}{\sqrt{3}} \left(\bar{x}_1 + \bar{x}_2 + \bar{x}_3 \right) = \sqrt{3} \bar{X}_{c.m.} \rightarrow 0$$

$$z_j = \lambda \bar{z}_j , \quad x_j = \lambda \bar{x}_j ; \quad j = 1, 2, 3 ; \quad X_{c.m.} = \lambda \bar{X}_{c.m.} = \frac{1}{3} \sum_{i=1}^3 \bar{x}_i$$

(156)

is already adapted to 3 space dimensions.

This concludes all extensions from 1 to 3 space dimensions.



4-1

4 - Partial countings of oscillatory modes of quarks in baryons

In this section the mode countings are done in a quite different way than in section 3 , hereby correcting incorrect results therein .

4-1 - Characteristic transformation properties of the basis functions

$$\psi_{n_1, n_2} \left(\vec{\zeta}, \vec{\bar{\zeta}} \right) \text{ under the permutation group } S_3$$

It is enough to select a partial subset (not a subgroup of S_3 for the sought transformation properties namely the elements

$$(157) \text{ setmin} = \{ T_{12}, T_{13}, T_{23} \text{ and } Z_3 : \text{the abelian subgroup of cyclic permutations} \}$$

The notation of the elements of S_3 ; Z_3 as well as the associated (2 one-dimensional and 1 two-dimensional unitary irreducible representations) are discussed in subsection 3-rec and for the *circular pair -mode basis* in subsections 3-res , 3-res-2 and 3-res-3 and eq. 132 .

First we invert the decomposition of the complex numbers $\zeta_j, \bar{\zeta}_j$; $j = (X), (Y), (Z)$ \longrightarrow

into real and imaginary parts , as displayed in eq. 147

$$\begin{aligned}
 \zeta_j &= \frac{1}{\sqrt{2}} \left(\bar{\xi}_{2j} + i \bar{\xi}_{1j} \right) \quad ; \quad x_j = \bar{\xi}_{2j}, y_j = \bar{\xi}_{1j} \\
 a_{1j} &= \frac{1}{\sqrt{2}} \left(\partial_{\zeta_j} + \bar{\zeta}_j \right) \quad ; \quad a_{2j} = \frac{1}{\sqrt{2}} \left(\partial_{\bar{\zeta}_j} + \zeta_j \right) \\
 (158) \quad a_{1j}^\dagger &= \frac{1}{\sqrt{2}} \left(-\partial_{\bar{\zeta}_j} + \zeta_j \right) \quad ; \quad a_{2j}^\dagger = \frac{1}{\sqrt{2}} \left(-\partial_{\zeta_j} + \bar{\zeta}_j \right)
 \end{aligned}$$

$$\left[a_{1j}, a_{2k} \right] = \left[a_{1j}, a_{2k}^\dagger \right] = \left[a_{2k}, a_{1j}^\dagger \right] = 0 \quad ; \quad j, k = (X), (Y), (Z)$$

Next we adapt the form of the induced representation with argument the cyclic permutation

$$(159) \quad U \left[\pi \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \right] \longleftrightarrow Z = e^{i \frac{2\pi}{3}}$$

associated with the third root of 1 : $Z = e^{i \frac{2\pi}{3}}$ acting on the basis functions $\psi_{n_1, n_2} \left(\vec{\zeta}, \vec{\bar{\zeta}} \right)$

→

from 1 space time dimension as displayed in eq. 95 , to 3

$$(160) \quad \left(U \left[\pi \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \right] \psi_{\mathbf{n}_1, \mathbf{n}_2} \right) (\zeta_j, \bar{\zeta}_j) = \\ = D_{\mathbf{n}_1 \mathbf{n}_2} (Z) \psi_{\mathbf{n}_1, \mathbf{n}_2} (\zeta_j, \bar{\zeta}_j) = \psi_{\mathbf{n}_1, \mathbf{n}_2} (Z^{-1} \zeta_j, Z \bar{\zeta}_j)$$

$$D_{\mathbf{n}_1 \mathbf{n}_2} (Z) = Z^{\left(\sum_{\mathbf{m}} (\mathbf{n}_1^{\mathbf{m}} - \mathbf{n}_2^{\mathbf{m}}) \right)} ; \quad \mathbf{j}, \mathbf{k}, \mathbf{m} = (X), (Y), (Z)$$

**4-1-1 - Transformation properties of the basis functions $\psi_{\mathbf{n}_1, \mathbf{n}_2} (\vec{\zeta}, \vec{\bar{\zeta}})$
under the 3 transpositions T_{12}, T_{23}, T_{13}**

Here we need the properties of the abstract permutation group S_3 worked out in subsection 3-res and displayed in eq. 124 partially reproduced below →

4-4

regarding the odd permutations in S_3

$$4) \quad \pi \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad - \quad \rightarrow T_{12}$$

$$(161) \quad 5) \quad \pi \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \quad - \quad \rightarrow T_{23}$$

$$6) \quad \pi \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \quad - \quad \rightarrow T_{13}$$

Next we recall the multiplication table of S_3 using to the numbering of elements introduced in eq. 124

$$1) \quad \pi \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad + \quad \rightarrow \mathbb{1}$$

$$(162) \quad 2) \quad \pi \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \quad + \quad \rightarrow Z$$

$$3) \quad \pi \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \quad + \quad \rightarrow Z^2 \equiv Z^{-1}$$



4-5

4-C – Consolidation of transformation properties of S_3 as elaborated in subsection 3-rec

The minimal discussion of products of elements of S_3 in subsection 3-rec , eqs. 106 - 123 prove insufficient . Thus we construct the partial multiplication table

(163)

	2)	3)	4) = T_{12}	5) = T_{23}	6) = T_{13}
2)	3)	¶	5)	6)	4)
3)	¶	2)	6)	4)	5)
4)	6)	5)	¶	3)	2)
5)	4)	6)	2)	¶	3)
6)	5)	4)	3)	2)	¶

We also recall the association of permutations 2) and 3) from eq. 132

(164)

$$\begin{aligned}
 2) \ d_{\pi=2} \quad \pi \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} &\longrightarrow \begin{pmatrix} e^{+i(2\pi/3)} & 0 \\ 0 & e^{-i(2\pi/3)} \end{pmatrix} + \rightarrow Z \\
 3) \ d_{\pi=3} \quad \pi \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} &\longrightarrow \begin{pmatrix} e^{-i(2\pi/3)} & 0 \\ 0 & e^{+i(2\pi/3)} \end{pmatrix} + \rightarrow Z^2 \equiv Z^{-1}
 \end{aligned}$$



4-5-1

We note here the structure of the similarity transformations relating the three transpositions

$$(165) \quad \begin{aligned} & 2) \circ 4) \circ \{2)\}^{-1} = 6) \quad ; \quad 3) \circ 4) \circ \{3)\}^{-1} = 5) \\ & 2) \circ 5) \circ \{2)\}^{-1} = 4) \quad ; \quad 3) \circ 5) \circ \{3)\}^{-1} = 6) \\ & 2) \circ 6) \circ \{2)\}^{-1} = 5) \quad ; \quad 3) \circ 6) \circ \{3)\}^{-1} = 4) \end{aligned}$$

In conjunction with the conjugacy classes of a finite group \mathcal{G} it is a good place to remember the original notions due to Frobenius [21-2013] of conjugacy classes and their invariants, the traces over arbitrary finite unitary representations $\mathcal{D}(\mathcal{G})$

$$(166) \quad \begin{aligned} & \text{Cjclass}(g; \mathcal{G}) = g \mathcal{G} g^{-1}; \quad g \in \mathcal{G} \\ \hline & \mathcal{D}(\mathcal{G}) = \left\{ \bigcup_g \mathcal{D}(g) \mid \mathcal{D}(g_2) \mathcal{D}(g_1) = \mathcal{D}(g_2 \circ g_1) \right\} \\ & g, g_1, g_2 \in \mathcal{G} \end{aligned}$$

4-C-1 - Schur's lemma [23-1973]

I follow here the proof of Schur's lemma laid out in ref. [22-2006] with slight modifications to adjust to the notation used in this outline. →

4-5-2

We consider two arbitrary *irreducible unitary representations* of \mathcal{G} , denoted $\mathcal{D}^{(1)}$ and $\mathcal{D}^{(2)}$, and form the quantity

$$(167) \quad \left(Y^{(1)(2)} \right)_{M N}^{m n} = \sum_g \left(\mathcal{D}^{(1)}(g) \right)_{M m} \left(\mathcal{D}^{(2)}(g) \right)_{N n}$$

The group structure then implies that $Y^{(1)(2)}$ is a group invariant

$$\begin{aligned} & \left(\mathcal{D}^{(1)}(\hat{h}) \right)_{M M'} \left(\mathcal{D}^{(2)}(\hat{h}) \right)_{N N'} \left(Y^{(1)(2)} \right)_{M' N'}^{m n} = \left(Y^{(1)(2)} \right)_{M N}^{m n} = \\ & = \sum_g \left(\mathcal{D}^{(1)}(\hat{h}g) \right)_{M m} \left(\mathcal{D}^{(2)}(\hat{h}g) \right)_{N n} ; \quad \forall \hat{h} \in \mathcal{G} \longrightarrow \end{aligned}$$

$$(168) \quad k = \hat{h}g \quad ; \quad g = \hat{h}^{-1}k \quad ; \quad \sum_g \equiv \sum_k$$

In eq. 168 we have chosen as group transformation, justifying the hat symbol for \hat{h} , left-multiplication

$$(169) \quad \hat{h} : g \rightarrow \hat{h} \circ g$$

The relation in the first line of eq. 168 →

4-5-3

using matrix notation with respect to the indices $M M'$, N , N' for fixed m, n becomes

$$\mathcal{M}_{MN} \mid_{m, n \text{ fixed}} \rightarrow \mathcal{M}_{MN} \rightarrow \mathcal{M}$$

$$(170) \quad \mathcal{D}^{(1)}(\hat{h}) \mathcal{M} \left(\mathcal{D}^{(2)}(\hat{h}) \right)^T = \mathcal{D}^{(1)}(\hat{h}) \mathcal{M} \left(\mathcal{D}^{(2)}(\hat{h}) \right)^{-1} = \mathcal{M}$$

$$M, M' = 1, \dots, \dim \mathcal{D}^{(1)} ; N, N' = 1, \dots, \dim \mathcal{D}^{(2)}$$

and multiplying the relation on the second line in eq. 170 with $\mathcal{D}^{(2)}(\hat{h})$ from the right it follows

$$(171) \quad \mathcal{D}^{(1)}(\hat{h}) \mathcal{M} = \mathcal{M} \mathcal{D}^{(2)}(\hat{h}) ; \forall \hat{h} \{in \mathcal{G}\}$$

$$\mathcal{M} : \dim \mathcal{D}^{(1)} \times \dim \mathcal{D}^{(2)} \text{ matrix}$$

Schur's lemma states that the *only* solution of eq. 171 compatible with the representations $\mathcal{D}^{(1)}$ and $\mathcal{D}^{(2)}$ being irreducible and unitary, except $\mathcal{M} \rightarrow M = 0$, demands

$$(172) \quad \dim \mathcal{D}^{(1)} = \dim \mathcal{D}^{(2)} \text{ and } \det \mathcal{M} \rightarrow \det M \neq 0 \text{ and}$$

$$\mathcal{D}^{(2)} = M^{-1} \mathcal{D}^{(1)} M \Leftrightarrow \mathcal{D}^{(1)} = M \mathcal{D}^{(1)} M^{-1}$$

The proof makes use of 'elementary' properties of linear algebra .

For the original paper(s) see ref. [23-1973] .



4-6

The multiplication table of S_3 in eq. 163 shall be checked *here* , since it provides the bridge its regular representation based on the group algebra over the group elements

$$(173) \quad \mathcal{A} = \left\{ a \mid a = \sum_{i=1}^6 a^i g_i \text{ with } g_i \in S_3 ; a^i \text{ real} \right\}$$

$$a = (a^1, a_2, \dots, a^6) \in \mathbb{C}^6$$

The regular representation consists of two representations , acting on the left and on the right in a commuting way

$$(174) \quad L : g_i \longrightarrow a \rightarrow g_i \circ a = \sum_{k=1}^6 a^k g_i \circ g_k$$

$$R : g_i \longrightarrow a \rightarrow a \circ (g_i)^{-1} = \sum_{k=1}^6 a^k g_k \circ (g_i)^{-1}$$

From eq. 174 and the multiplication table of S_3 here , the two regular representations are defined

$$(175) \quad L : (g_i \circ a)^j = \left((D_L)^i_j (g_i) \right) a^j = a^k g_{ik \rightarrow j}$$

$$g_{ik \rightarrow j} = g_i \circ g_k$$

$$R : \left(a \circ (g_i)^{-1} \right)^j = \left((D_R)^i_j (g_i) \right) a^j = a^k \tilde{g}_{ki \rightarrow j}$$

$$\tilde{g}_{ki \rightarrow j} = g_k \circ (g_i)^{-1}$$



The L , R representations derived in eqs. 173 - 175 are of course classical cornerstones of the theory of finite groups . In ref. [20-2013] they are attributed to Ferdinand Georg Frobenius .

Ferdinand Frobenius + 26 October 1849 in Berlin-Charlottenburg , Prussia , Germany
 † 3 August 1917 in Berlin , Prussia , Germany

cited from ref. [21-2013] .

Eqs. 174 and 175 shall be expressed using matrix notation

$$(176) \quad \widehat{g_{iL}} a = D_L (g_i) a = g_i \circ a$$

$$\widehat{g_{jR}} a = D_R (g_j) a = a \circ (g_j)^{-1}$$

In eq. 176 the symbols $\widehat{g_{iL}}$, $\widehat{g_{iR}}$ denote the operator nature of the substitutions associated with the group multiplications on its uttermost right hand side .

From the relations in eq. 176a the bilateral representation structure following the pattern

$$(177) \quad \mathcal{D} (\mathcal{G} \times \mathcal{G}) = \mathcal{D}_L (\mathcal{G}) \otimes \mathcal{D}_R (\mathcal{G})$$

follows

$$(178) \quad D_L (g_k) D_L (g_i) = D_L (g_k \circ g_i)$$

$$D_R (g_k) D_R (g_i) = D_R (g_k \circ g_i)$$



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and using (6×6) matrix notation

$$(179) D_R(g_k) D_L(g_i) = D_L(g_i) D_R(g_k) \quad \forall g_k, g_i ; k, j = 1, \dots, 6$$

With this description of the regular representation all basic group- and oscillatory mode- properties are assembled , which allow to perform all necessary checks straightforwardly.

We proceed to the direct count of these modes , using the circular oscillatory wave function basis and the symmetry properties of associated permutation group representations in the next subsection .

5 – Counting oscillatory modes using the circular oscillatory wave function basis enforcing overall Bose symmetry under the combined permutations of $SU6$ ($fl \times spin$) \times barycentric coordinates

5-1 – Traces of irreducible representations of S_3

Not being able to give a clearcut reference to the original publication(s) yielding knowledge with respect to what became known as 'Schur's lemma , as formulated here for unitary irreducible representations in eq. 172 the present subsection is added here.

It follows directly within the hypothesis discussed in the neighbourhood of eq. 172 that the only nontrivial relation between two unitary irreducible linear representations of S_3 , $\mathcal{D}^{(1)}$ and $\mathcal{D}^{(2)}$ is the unitary equivalence

$$(180) \quad \mathcal{D}^{(2)} = M^{-1} \mathcal{D}^{(1)} M \Leftrightarrow \mathcal{D}^{(1)} = M \mathcal{D}^{(1)} M^{-1} ; M : \text{unitary}$$



The matrix M – establishing unitary equivalence of the already assumed linear *unitary* irreducible representations \mathcal{D}^1 and \mathcal{D}^2 – as it appears in eq. 180 is unitary but not unique and not to be identified with the quantity \mathcal{M} as it is defined in eqs. 167 and 170 , with its additional indices $m n$.

Remark

The explicit derivations are not displayed in this outline , for compactness of presentation .

They can be found *in extenso* in ref. [22-2006] .

It is a general procedure within wide fields of mathematics to attempt a fully reducible construction of invariants , here with respect to the equivalence class of unitary similarity transformations as given in eq. 180 , to consider invariants for this class .

Such invariants are the traces of any given unitary irreducible representation $\mathcal{D} (\mathcal{G})$ as functions over a finite group \mathcal{G} , here S_3

$$(181) \quad \chi (h, \mathcal{D}) = tr (\mathcal{D} (h)) \quad ; \quad h \in \mathcal{G}$$

The sought invariance follows from the relation for the trace of a product of $dim \mathcal{D} \times dim \mathcal{D}$ matrices

$$(182) \quad tr (A \circ B) = tr (B \circ A) \longrightarrow tr (M \circ \mathcal{D} (h) \circ M^{-1}) = tr (\mathcal{D} (h))$$

for any given unitary $dim \mathcal{D} \times dim \mathcal{D}$ matrix M and unitary irreducible representation \mathcal{D} . \longrightarrow

5-3

It follows for characters , which are the same for equivalent representations (eq. 182)

$$\sum_g \bar{\chi}(g, \mathcal{D}') \chi(g, \mathcal{D}) = \Pi(\mathcal{G}) \delta_{\mathcal{D}' \mathcal{D}} ; \delta_{\mathcal{D}' \mathcal{D}} = \begin{cases} 1 & \text{for } \mathcal{D}' \simeq \mathcal{D} \\ 0 & \text{else} \end{cases}$$

\forall irreducible $\mathcal{D}'(\mathcal{G}) , \mathcal{D}(\mathcal{G})$

(183)

In eq. 183 $\bar{\chi}$ is the complex conjugate of χ and $\Pi(\mathcal{G})$ the number of elements of the finite group \mathcal{G} .
This ends my account of the pertinent messages from Issai Schur (1875-1941).

The characters form a complete orthonormal – modulo the factor $\Pi(\mathcal{G})$ – set of functions on \mathcal{G} . This had been proven by Peter and Weyl [24-1927] for compact Lie groups , but clearly applies also to the simpler case of finite groups .

5-2 – Aligning statistics between the u, d , s SU_6 ($fl \times spin$) group and oscillator modes in 6 barycentric configuration space variables

We resume our main theme recalling the three Young tableaux induced by the statistics of 3 valence quarks u, d, s →

5-4

from the group SU_6 ($fl \times spin$) group in subsection 3-1-3a , eq. 63 repeated below for coherence of presentation

$$\begin{aligned}
 & 1 : \dim \left(\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \right) = \frac{4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}{5!} = 56 \\
 (184) \quad & 2 : \dim \left(\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array} \right) = \frac{2 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{4!5!} = 70 \\
 & 3 : \dim \left(\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \right) = \frac{2 \cdot 4 \cdot 5 \cdot 6 \cdot 3 \cdot 4 \cdot 5 \cdot 2 \cdot 3 \cdot 4}{3!4!5!} = 20
 \end{aligned}$$

5-2-1 – Associating symmetric and antisymmetric representations of SU_6 ($fl \times spin$) to oscillator mode wave functions in the circular pair-mode basis

This corresponds to the Young tableaux 1 and 3 in eq. 184 . The associated (inequivalent) irreducible representations of S_3 are both one dimensional : the identity for case 1 and the antisymmetric representation , assigning $+1$ for the even - and -1 for the odd permutations . →

5-5

We verify eq. 183 for the two (1-dim) inequivalent representations of S_3 , denoted $\boxed{1^+}$ and $\boxed{1^-}$

g	$\chi \left(g, \boxed{1^+} \right)$	$\chi \left(g, \boxed{1^-} \right)$
1)	+1	+1
2)	+1	+1
3)	+1	+1
4)	+1	-1
5)	+1	-1
6)	+1	-1
$\sum_g \chi^2$	6	6

(185)

g	1)	2)	3)	4)	5)	6)
1)	1)	2	3)	4)	5)	6)
2)	2)	3)	↯	5)	6)	4)
3)	3)	↯	2)	6)	4)	5)
4)	4)	6)	5)	↯	3)	2)
5)	5)	4)	6)	2)	↯	3)
6)	6)	5)	4)	3)	2)	↯



5-6

The left panel in eq. 185 gives the full multiplication table for S_3 recalling eq. 163 from subsection 4-C .
 From the right panel in eq. 185 we also verify the nondiagonal orthogonality relation given in eq. 183

$$(186) \quad \sum_g \bar{\chi} \left(g, \boxed{1^-} \right) \chi \left(g, \boxed{1^+} \right) = 0$$

We turn to the remaining irreducible (2-dim) representation of S_3 , which shall be denoted $\boxed{2}$ and is displayed in the circular pair-mode associated basis in eq. 132, shown in eq. 188 below

From the $\mathcal{D}_{\boxed{2}}$ traces we find

$$(187) \quad \begin{aligned} \sum_g \bar{\chi} \left(g, \boxed{2} \right) \chi \left(g, \boxed{2} \right) &= 6 \\ \sum_g \bar{\chi} \left(g, \boxed{2} \right) \chi \left(g, \boxed{1^-} \right) &= \sum_g \bar{\chi} \left(g, \boxed{2} \right) \chi \left(g, \boxed{1^+} \right) = 0 \end{aligned}$$

verifying all of eq. 183 .

$\mathcal{D}_{\boxed{2}}$ corresponds *uniquely* to Young tableau 2 in eq. 184 .

This exhaust – modulo similarity transformation – all conjugation invariant functions over S_3 . →

(188)

g	$\mathcal{D}_{\boxed{2}}$	$\chi(g, \boxed{2})$
1) $d_{\pi=1} = \pi \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	2
2) $d_{\pi=2} = \pi \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$	$\begin{pmatrix} e^{+i(2\pi/3)} & 0 \\ 0 & e^{-i(2\pi/3)} \end{pmatrix}$	-1
3) $d_{\pi=3} = \pi \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$	$\begin{pmatrix} e^{-i(2\pi/3)} & 0 \\ 0 & e^{+i(2\pi/3)} \end{pmatrix}$	-1
4) $d_{\pi=4} = \pi \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	0
5) $d_{\pi=5} = \pi \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$	$\begin{pmatrix} 0 & e^{+i(2\pi/3)} \\ e^{-i(2\pi/3)} & 0 \end{pmatrix}$	0
6) $d_{\pi=6} = \pi \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & e^{-i(2\pi/3)} \\ e^{+i(2\pi/3)} & 0 \end{pmatrix}$	0



After this detour to include the basic conjugation invariant functions over S_3 , assembled in eqs. 185 and 188, we return to the main theme of this subsection : 5-2-1. This necessitates the insertion regarding reducible traces, from direct product representations of S_3 in the next subsection

a **5-ins – Reducible traces from direct product representations of S_3**

The definition of traces in eq. 181 does not imply to restrict a given representation $\mathcal{D}(\mathcal{G})$ to an irreducible one. In the environment of eqs. 183 - 188 however, only the 3 inequivalent *irreducible* unitary representations of S_3 were considered.

The counting of oscillatory modes (of u, d, s quarks in baryons) involves intrinsically multiple direct product representations. Thus we are led to extend the catalog of basic traces over the direct (simple) products of these irreducible representations.

The general reduction of the trace of the direct product \mathcal{D} of unitary but otherwise still arbitrary factor representations $\mathcal{D}^{(\alpha)}$ and $\mathcal{D}^{(\beta)}$ is

$$(189) \quad \mathcal{D} = \mathcal{D}^{(\alpha)} \otimes \mathcal{D}^{(\beta)} \longrightarrow \chi(g, \mathcal{D}) = \chi(g, \mathcal{D}^{(\alpha)}) \times \chi(g, \mathcal{D}^{(\beta)}) \longrightarrow$$

a **It is an obligation to refer to and thank here Kurt Schütte [25-1961] (+ 14. October 1909 in Salzwedel ; † 18. August 1998 in Munich, Germany). It was probably during the WS 1961/62, when I followed as an undergraduate student his lectures on 'Algebra', including a thorough discussion of properties of finite groups. He was then guest professor at the ETH in Zurich.**

5-9

and hence immaterial of the order of the direct product factors . As a consequence (inequivalent) products of the base , irreducible representations $\boxed{1^+}$ and $\boxed{1^-}$ of S_3 , amount to

$$(190) \quad \begin{aligned} \chi \left(g, \boxed{1^+} \otimes \boxed{1^+} \right) &= \left(\chi \left(g, \boxed{1^+} \right) \right)^2 = \chi \left(g, \boxed{1^+} \right) \\ \chi \left(g, \boxed{1^+} \otimes \boxed{1^-} \right) &= \chi \left(g, \boxed{1^+} \right) \chi \left(g, \boxed{1^-} \right) = \chi \left(g, \boxed{1^-} \right) \end{aligned}$$

The single nontrivial direct product is $\boxed{2} \otimes \boxed{2}$

$$(191) \quad \chi \left(g, \boxed{2} \otimes \boxed{2} \right) = \left(\chi \left(g, \boxed{2} \right) \right)^2 =$$

$\chi \boxed{2} \otimes \boxed{2}$	g
4	1)
1	2)
1	3)
0	4)
0	5)
0	6)



5-10

From eq. 191 using the primitive traces of the irreducible representations $\boxed{1^+}$, $\boxed{1^-}$ in eq. 185 and $\boxed{2}$ in eq. 188 we find the decomposition

$$(192) \quad \begin{array}{c|c} g & \chi_{\boxed{2} \otimes \boxed{2}} \\ \hline 1) & 4 \\ 2) & 1 \\ 3) & 1 \\ 4) & 0 \\ 5) & 0 \\ 6) & 0 \end{array} = \begin{array}{c} \chi_{\boxed{2}} \\ \hline 2 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \end{array} + \begin{array}{c} \chi_{\boxed{1^+}} \\ \hline 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array} + \begin{array}{c} \chi_{\boxed{1^-}} \\ \hline 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \end{array}$$

$$\longrightarrow \boxed{2} \otimes \boxed{2} = \boxed{2} \oplus \boxed{1^+} \oplus \boxed{1^-}$$

The direct product decomposition constructed here in its basic form in eqs. 190 - 192 can systematically be extended to arbitrary numbers of direct product factors . It is logically similar to the decomposition of natural numbers into products of prime ones. →

5-2-2 – Back to subsection 5-2-1 : Associating symmetric and antisymmetric representations of SU_6 ($fl \times spin$) to oscillator mode wave functions in the circular pair-mode basis

Having completed the two insertions – in subsections 5-1 and 5-ins – dealing with the properties of traces of general and irreducible unitary representations of a finite group \mathcal{G} and their direct products , adapting to the case relevant here $\mathcal{G} \rightarrow S_3$, we resume the 'counting' of oscillatory modes of u, d , s light flavored valence quarks in baryons .

The symmetric and antisymmetric representations of S_3 correspond to simple conditions on the wave functions formed from the basis in eq. 155 repeated below

$$(193) \quad \psi_{\mathbf{n}_1, \mathbf{n}_2} \left(\vec{\zeta}, \vec{\bar{\zeta}} \right) =$$

$$= \prod_{\mathbf{k}} \left(\frac{2 \binom{\mathbf{n}_1 + \mathbf{n}_2}{\mathbf{k}}}{\pi \binom{\mathbf{n}_1}{\mathbf{k}} \binom{\mathbf{n}_2}{\mathbf{k}}} \right)^{\frac{1}{2}} \exp \left(i \left(n_2^{\mathbf{k}} - n_1^{\mathbf{k}} \right) \varphi_{\mathbf{k}} \right) \times$$

$$\times \left[\varrho_{\mathbf{k}}^{\binom{\mathbf{n}_1 + \mathbf{n}_2}{\mathbf{k}}} \exp \left(- \varrho_{\mathbf{k}}^2 \right) \right]$$

$$\varrho_{\mathbf{j}} = \left| \zeta_{\mathbf{j}} \right| ; \varphi_{\mathbf{j}} = \arg \left(\zeta_{\mathbf{j}} \right) ; \mathbf{j} = (X), (Y), (Z)$$



5-12

These conditions are twofold , expressed in the quantum numbers $n_1 ; n_2$ as defined in eqs. 155 and equivalently 193 . It is convenient to define the nonnegative integer quantities $N_{1,2}$ associated with $n_{1,2}$ respectively as well as their signed difference

$$(194) \quad \begin{aligned} N_1 &= \sum_{\mathbf{k}} n_{\mathbf{k}}^1 ; & N_2 &= \sum_{\mathbf{k}} n_{\mathbf{k}}^2 \quad \text{with} \quad N = N_1 + N_2 \\ \Delta &= N_1 - N_2 \end{aligned}$$

The first condition is common to both cases 1 and 3 induced by the statistics of 3 valence quarks u, d, s from the group SU_6 ($fl \times spin$) in subsection 3-1-3a , eqs. 63 and 63 , repeated below

$$(195) \quad \begin{aligned} 1 : \quad dim \left(\begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} \right) &= \frac{4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}{5!} &= 56 \\ \\ 2 : \quad dim \left(\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \right) &= \frac{2 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{4!5!} &= 70 \\ \\ 3 : \quad dim \left(\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \right) &= \frac{2 \cdot 4 \cdot 5 \cdot 6 \cdot 3 \cdot 4 \cdot 5 \cdot 2 \cdot 3 \cdot 4}{3!4!5!} &= 20 \end{aligned}$$



The two conditions for cases 1 and 3 in eq. 195 are

Condition 1

$$(196) \quad \Delta = 0 \pmod{3} \text{ for cases } \mathbf{1 \text{ and } 3}$$

Condition 2

$$(197) \quad \psi_{\mathbf{n}_1, \mathbf{n}_2} \left(\vec{\zeta}, \vec{\bar{\zeta}} \right) \longrightarrow \begin{cases} \psi_{\mathbf{n}_1, \mathbf{n}_2}^{(+)} = \mathcal{N}^{+\frac{1}{2}} \{ \psi_{\mathbf{n}_1, \mathbf{n}_2} + \psi_{\mathbf{n}_2, \mathbf{n}_1} \} & \text{for case } \mathbf{1} \\ \psi_{\mathbf{n}_1, \mathbf{n}_2}^{(-)} = \mathcal{N}^{-\frac{1}{2}} \{ \psi_{\mathbf{n}_1, \mathbf{n}_2} - \psi_{\mathbf{n}_2, \mathbf{n}_1} \} & \text{for case } \mathbf{3} \end{cases}$$

In eq. 197 \mathcal{N}^{\pm} denote normalization constants .

The two conditions defined in eqs. 196 and 197 determine the counting in accordance with overall Bose symmetry for the complete bosonic wave function respecting $SU6$ ($fl \times spin$) and oscillatory modes in 6 barycentric coordinates – for the two cases considered in this subsection . →

5-2-3 – Associating the mixed 70-representation of $SU_6 (fl \times spin)$ to oscillator mode wave functions in the circular pair-mode basis

It turns out that the association of the symmetric and antisymmetric representations of $SU_6 (fl \times spin)$ show the way to determine the Bose statistics association of the mixed 70-representation of $SU_6 (fl \times spin)$ to the oscillatory modes in the circular pair-mode basis , i.e. in the remaining case **2** in eq. 195 .

In fact there is only one condition (modulo a restriction by a factor of $\frac{1}{2}$) determining the sought association , complementing cases 1 and 3 in eqs. 196 and 197

Condition for case 2

$$(198) \quad \Delta = 1 \& 2 \pmod{3} \text{ for case } \mathbf{2}$$

For the counting the powers of the two sets

$$(199) \quad \Delta = 1 \pmod{3} ; \Delta = 2 \pmod{3}$$

are the same , allowing the required (multiple) realization of the $\boxed{2}$ representation of S_3 on the wave functions in the circular pair-mode basis – intervening in case **2** . Given this realization the number of oscillator modes satisfying the condition in eq. 198 are then paired with the corresponding $\boxed{2}$ in the 70-plet of $SU_6 (fl \times spin)$ in eq. 195 as shown in the direct product decomposition in eq. 192 . Hence overall Bose symmetry restricts the number count of wave functions satisfying the condition in eq. 198 by a factor of $\frac{1}{2}$.



This ends the *theoretical embedding* of (counting) oscillatory modes of valence u, d, s quarks in baryons and in this subsections 5-2-1 – 5-2-3 and section 5 .

What remains to be done is the actual counting , according to the conditions formulated in eqs. 196 and 197 for cases 1 and 3 and eqs. 198 and 199 for case 2 . This is best done for a finite number of main oscillator quantum numbers N in a dedicated computer program , left to be worked out soon .

(\longrightarrow)

A1-1

Appendix 1 : Some checks on results in subsection 3-1-1b The binomial reduction formula

$$I (K , p + 1) \rightarrow I (K , p)$$

We perform simple checks on the validity of the binomial reduction formula , derived in subsection 3-1-1b .

To this end we repeat eq. 58 as starting point in eq. 200 below

$$(200) \quad I (K , p + 1) =$$
$$= \left[(p + 2)^{-1} (K + 1)^{p+2} - \right.$$
$$\left. - (p + 2)^{-1} \sum_{q=1}^{p+1} \binom{p + 2}{p + 1 - q} I (K , p + 1 - q) \right]$$

$$I (K , 0) = K + 1 ; \quad \binom{p + 1}{0} = 1 \quad \text{for } p \geq 0$$

We first set $p = 0$ in eq. 200

$$(201) \quad I (K , 1) = \frac{1}{2} (K + 1)^2 - \frac{1}{2} \binom{2}{0} I (K , 0)$$
$$= \frac{1}{2} K^2 + K + \frac{1}{2} - \frac{1}{2} K - \frac{1}{2} = \frac{1}{2} K (K + 1) \quad (\checkmark)$$

Eq. 201 anchors the recursion from $p = 0$ to $p = 1$.



A1-2

We proceed to calculate $I(K, 2)$ setting $p = 1$ in eq. 200, which yields

$$\begin{aligned} (202) \quad I(K, 2) &= \left[\frac{1}{3} (K+1)^3 - \right. \\ &\quad \left. - \frac{1}{3} \sum_{q=1}^2 \binom{3}{2-q} I(K, 2-q) \right] \\ &= \frac{1}{3} (K+1)^3 - I(K, 1) - \frac{1}{3} I(K, 0) \\ &= \frac{1}{3} (K+1)^3 - \frac{1}{2} (K+1)K - \frac{1}{3} (K+1) \\ &= \frac{1}{3} (K+1) \left[(K+1)^2 - \frac{3}{2}K - 1 \right] \end{aligned}$$

We check this for $K = 2, 3$

$$\begin{aligned} (203) \quad I(2, 2) &= 5 = (9 - 3 - 1) \quad (\checkmark) \\ I(3, 2) &= 14 = \frac{4}{3} \left(16 - \frac{9}{2} - 1 \right) \\ &= \frac{2}{3} (32 - 11) = 2 \times 7 = 14 \quad (\checkmark) \end{aligned}$$



A2-1

Appendix 2 : A Fortran program to calculate $\wp (N , R)$, the power of the set of states of the oscillatory modes of u, d, s - valence quarks in baryons , displayed in eqs. 45 , 59

An alternative and as it turns out quite manageable and exact way to evaluate the sum of integers as displayed in eqs. 45 = 59 and repeated below as eq. 204 consists in making a Fortran program execute the sum for the quantity $\wp (N , R)$ displayed in eq. 204

$$\wp (N , R) =$$

$$(204) \left\{ \sum_{N \geq N_1 \geq \dots \geq N_{R-1} \geq 0} \right\} \left[\begin{aligned} & (N - N_1 + 1) (N_1 - N_2 + 1) \times \dots \\ & \times (N_{R-2} - N_{R-1} + 1) \times \\ & \times (N_{R-1} - N_R + 1) \end{aligned} \right]$$

$N_0 = N , N_R = 0$ fixed end points ; summation variables : N_1 , \dots , N_{R-1}

The quantity $\wp (N , R)$ in eq. 204 represents the number of states of oscillatory modes of three valence u,d,s - quarks in baryons with oscillatory modes smaller or equal to the main quantum number $N = N_0$ with

$$(205) \quad N^* = \sum_{\kappa=1}^{35} N_{\kappa} \leq N$$

as derived in subsection 3-1-1 .



To evaluate the sum of products over the integers according to eq. 204

$$(206) \quad N \geq N_1 \geq \cdots \geq N_{R-1} \geq 0$$

integer variables : $(N_1 , N_2 , \cdots N_{R-2} , N_{R-1})$ with $R = 36$

where the summation goes over the variables $(N_1 , N_2 , \cdots N_{R-2} , N_{R-1})$ shown in eq. 206 , subject to the condition for fixed N in the first line of eq. 206 , requires a nested but otherwise conditionless sequence of 35 do-loops .

This was done – at present – in two Fortran programs :

sumbar2013.f for N = 2 and sumbar2013n3.f for N = 3 .

While a more complete program can eventually be constructed from these two, I describe sumbar2013.f here in extenso . It contains all basic definitions and operations in the integer logic of the Fortran code.



A2-3

A2-f - The Fortran code of the file sumbar2013.f for N = 2 with extended details in comment statements

```
program sumbaryon
```

```
implicit double precision (a-d,x-z)
```

```
implicit integer (e-w)
```

```
integer Nvar,lvar
```

```
dimension Nvar(37)
```

```
common/var/Nvar
```

```
c common
```

```
c common
```

```
Nsumvar=35
```

```
c-----
```

```
nzero=0
```

```
one=1
```

```
two=2
```

```
three=3
```

```
four=4
```

```
five=5
```

```
six=6
```

```
ten=10
```



```
c ahalf=one/two
```

```
api=four*atan(1.)
```

```
c *****
```

```
c here we start summation
```

```
call evalsum
```

```
stop
```

```
end
```

```
c *****
```

```
c *****
```

```
subroutine evalsum
```

```
implicit double precision (a-d,x-z)
```

```
implicit integer (e-w)
```

```
integer Nvar,lvar
```

```
character*200 filename
```

```
character*200 filenam1
```

```
character*200 filenam2
```

```
character*200 filenam3
```

```
character*200 filenam4
```

```
character*200 filenam5
```



character*200 filenam6
character*200 filenam7
character*300 filenam201
character*300 filenam202
character*300 filenam203
character*300 filenam204
character*300 filenam205
character*300 filenam206
dimension Nvar(37),lvar(37)

common

c ————— **contents of Nvar array**

R=36

c Theassignment of the elements of the array 37 dimensional

c array Nvar(37) to the quantities

c ($N = N_0, N_1, N_2, \dots, N_{R-1}, N_R$) in eqs. 204 , 206 are

c $Nvar = (Nvar(1), Nvar(2), \dots, Nvar(R-1) ; Nvar(36) = NN, Nvar(37) = 0)$

c each Nvar(k) takes values between 0 and $NN = N$

c the values of Nvar(1) are summed over and are steered by the integer

c value I1 and so on vor Nvar(1) a function of I1



A2-6

```
c beyond Nvar(35),l35 for completeness we set
c Nvar(36) = N ( = NN in the code )
c filenam201 = "gibbs-dens-with-pvac-versus-T.data"
c filenam202 = "gibbs-dens-without-pvac-versus-T.data"
c filenames from another Fortran program serving as exemples
filenam201 = "sumbaryon-fl6.data"
filenam202= "sumbaryonN3-fl6.data"
nzero=0
one=1
two=2
three=3
four=4
five=5
six=6
ten=10
api=four*atan(1.)
c NN is the order of oscimodes
NN=2
```



c initialization

Nvar(36)=NN

Nvar(37)=nzero do i=1,35

Nvar(i)= nzero

enddo

c — m here I start the sumbar nested do-loop summation

sumbar=nzero

c print*,Nvar(36),Nvar(37)

c Here begins the 35-fold nested loop

Do 101 I1=Nvar(37),Nvar(36)

Nvar(1)=I1

101 fact=Nvar(36)-Nvar(1)+one

c in this Do loop : Do 101 , the first factor of the product in eq. 204 is set : $N - N_1 + 1$

c This factor is reset again in the next Do-loop : Do 1 , as prod1 , because of an initial error message ,

c which however was due to an erroneous statement, now corrected



Do 1 I1=Nvar(37),Nvar(36)

Nvar(1)=I1

prod1=Nvar(36)-Nvar(1)+one

c as said after Do-loop 101 , the variable fact , no more used in the following , is here redefined as prod1

c $prod1 (\equiv fact) = N - N_1 + 1$

c in this Do loop : Do 1 , the first factor of the product in eq. 204 is set : $prod1 = N - N_1 + 1$

Do 2 I2=Nvar(37),I1

Nvar(2)=I2

Do 3 I3=Nvar(37),I2

Nvar(3)=I3

Do 4 I4=Nvar(37),I3

Nvar(4)=I4

Do 5 I5=Nvar(37),I4

Nvar(5)=I5

Do 6 I6=Nvar(37),I5

Nvar(6)=I6

Do 7 I7=Nvar(37),I6

Nvar(7)=I7



Do 8 I8=Nvar(37),I7

Nvar(8)=I8

Do 9 I9=Nvar(37),I8

Nvar(9)=I9

Do 10 I10=Nvar(37),I9

Nvar(10)=I10

Do 11 I11=Nvar(37),I10

Nvar(11)=I11

Do 12 I12=Nvar(37),I11

Nvar(12)=I12

Do 13 I13=Nvar(37),I12

Nvar(13)=I13

Do 14 I14=Nvar(37),I13

Nvar(14)=I14

Do 15 I15=Nvar(37),I14

Nvar(15)=I15

Do 16 I16=Nvar(37),I15

Nvar(16)=I16



A2-10

Do 17 I17=Nvar(37),I16

Nvar(17)=I17

Do 18 I18=Nvar(37),I17

Nvar(18)=I18

Do 19 I19=Nvar(37),I18

Nvar(19)=I19

Do 20 I20=Nvar(37),I19

Nvar(20)=I20

Do 21 I21=Nvar(37),I20

Nvar(21)=I21

Do 22 I22=Nvar(37),I21

Nvar(22)=I22

Do 23 I23=Nvar(37),I22

Nvar(23)=I23

Do 24 I24=Nvar(37),I23

Nvar(24)=I24

Do 25 I25=Nvar(37),I24

Nvar(25)=I25



A2-11

Do 26 I26=Nvar(37),I25

Nvar(26)=I26

Do 27 I27=Nvar(37),I26

Nvar(27)=I27

Do 28 I28=Nvar(37),I27

Nvar(28)=I28

Do 29 I29=Nvar(37),I28

Nvar(29)=I29 Do 30 I30=Nvar(37),I29

Nvar(30)=I30

Do 31 I31=Nvar(37),I30

Nvar(31)=I31

Do 32 I32=Nvar(37),I31

Nvar(32)=I32

Do 33 I33=Nvar(37),I32

Nvar(33)=I33

Do 34 I34=Nvar(37),I33

Nvar(34)=I34



Do 35 I35=Nvar(37),I34

Nvar(35)=I35

c fact=(Nvar(36)-Nvar(35)+one)

c as said after Do-loop 101 , the variable fact , no more used in the following , is here redefined as prod1

c $prod1 (\equiv fact) = N - N_1 + 1$

c in the previous Do loop : Do 1 , the first factor of the product in eq. 204 was set :

$prod1 = N - N_1 + 1$

prod=prod1

c Here the initial value of the full product in eq. 204 is set equal to prod1 taken over from Do-loop Do 1

Do 100 I=1,34

prod=prod*(Nvar(I)-Nvar(I+1)+one)

100 continue

c In Do-loop Do 100 the product in eq. 204 is performed , without the last factor , so up to here , after

c Do-loop 100 we have :

c $prod = (N - N_1 + 1) (N_1 - N_2 + 1) \cdots (N_{R-2} - N_{R-1} + 1)$ →

A2-13

prod=prod*(Nvar(35)-Nvar(37)+one)

**c Here the last factor is multiplied into the product in eq. 204 so at this stage we have the full expression
c in the variable prod :**

c *prod* =

c $(N - N_1 + 1) (N_1 - N_2 + 1) \cdots (N_{R-2} - N_{R-1} + 1) (N_{R-1} - N_R + 1)$

c with $R = 36 ; N_R = 0$

sumbar=sumbar+prod

c Here the sum of the products (prod) is performed , the sum is anchored with the statement :

**c sumbar = nzero (= 0) before D0-loop 1 . When the nested Do-loops Do 1 , ... , Do 35 have ended after
the statement 1 continue the variable sumbar gives the final result**

c *sumbar* = $\wp (N , R)$, the power of the set of u, d, s - baryon states with

c $N^* = \sum_{\kappa=1}^{35} N_{\kappa} \leq N$ as defined in eq. 205

35 continue

c Here the innermost of the nested Do-loops : Do 35 , ends continuing with the statements :

c 34 continue , ... , 1 continue .

34 continue

33 continue

32 continue

31 continue



A2-14

30 continue

29 continue

28 continue

27 continue

26 continue

25 continue

24 continue

23 continue

22 continue

21 continue

20 continue

19 continue

18 continue

17 continue

16 continue

15 continue

14 continue

13 continue

12 continue



A2-15

11 continue

10 continue

9 continue

8 continue

7 continue

6 continue

5 continue

4 continue

3 continue

2 continue

1 continue

c Here the full 35-fold nested Do-loop ends . At this stage the variable sumbar takes the value

c $sumbar = \wp (N , R)$, the power of the set of u, d, s - baryon states with

c $N^* = \sum_{\kappa=1}^{35} N_{\kappa} \leq N$, as defined in eqs. 204 and 205

open (unit = 201 , file = filenam201)

c write structure

write(201,*) NN,sumbar

c Here he paramer N = NN and the resulting sumbar , 2 integers are written on unit 201



A2-16

```
c write(202,*) NN,sumbar
```

```
c typical write statement
```

```
c write (16,*) T= ',temp
```

```
return
```

```
end
```

```
c — end of sbroutine evalsum
```

```
c We note that all comment statements included here are not on the Fortran assigned position 1 ,
```

```
c for simplicity of presentation, but together with all executable statements on position 7.
```

```
c Futhermore the statement numbers are for the most part also on position 7 instead of positions 1 - 5 .
```

```
c _____
```

The results are

(207)

N	$\varphi (N , 36)$
2	2628
3	64824

A3-1

Appendix 3 : The 2 x 2 representation of S_3 in the basis denoted by the collection d_π in eq. 131

First we repeat eq. 131 below , defining the similarity equation

$$d_\pi = \mathcal{M} D_\pi \mathcal{M}^{-1} = u D_\pi u^{-1}$$

(208)
$$u = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}, \quad u^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$$

Next we substitute a general (real) 2 x 2 matrix for D_π , yielding the transformation

(209)
$$u \rightarrow A \longrightarrow E = u A u^\dagger ; \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; \quad E = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

The substitution in eq. 209 becomes

(210)
$$E = \frac{1}{2} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$$

and proceeding step by step



A3-2

$$\begin{aligned}
 E &= \frac{1}{2} \begin{pmatrix} a + ic & b + id \\ a - ic & b - id \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \\
 (211) \quad &= \frac{1}{2} \begin{pmatrix} a + d - i(b - c) & a - d + i(b + c) \\ a - d - i(b + c) & a + d + i(b - c) \end{pmatrix}
 \end{aligned}$$

$A =$ real matrix , by inspection of the collection D_q in eq. 124

The matrix E (in eqs. 209 - 211) has complex matrix elements , which can be parametrized as follows

$$(212) \quad E = \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} e & f \\ f^* & e^* \end{pmatrix} ; \quad h = e^* , \quad g = f^*$$

$$e = \frac{1}{2} [a + d - i(b - c)] , \quad f = \frac{1}{2} [a - d + i(b + c)]$$

In eq. 212 e.g. e^* denotes the complex conjugate number relative to e .

We note here that $tr E = tr A$ is a unitary invariant , independent of the transformation matrix u .



A3-3

Associating the collections D_π in eq. 124 and d_π in eq. 131 for elements beyond unity of the 2 x 2 representation of S_3

We repeat the first 2 entries of the D_π collection in eq. 124 below in eq. 213

$$\begin{aligned}
 (213) \quad & 1) \quad \pi \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \rightarrow \mathbb{I} \\
 & 2) \quad \pi \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} + \rightarrow Z
 \end{aligned}$$

Thus we fill the $A(a, b, c, d)$ pattern belonging to $A = A_2 = D_{\pi=2}$

$$\begin{aligned}
 (214) \quad & \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \\
 & \hline
 & \frac{1}{2}(a + d) = -\frac{1}{2} \quad , \quad \frac{1}{2}(b - c) = -\frac{\sqrt{3}}{2} \\
 & \frac{1}{2}(a - d) = 0 \quad , \quad \frac{1}{2}(b + c) = 0
 \end{aligned}$$

We shall not repeat the index 2 . No confusion arises before the next index (3) is addressed . →

A3-4

Also the symbol e , identified with a matrix element is not to be confused with Eulers constant, even though the symbols are the same.

It follows from eqs. 212 and 214 (for $d_{\pi=2}$)

$$(215) \quad \begin{aligned} e &= \frac{1}{2} [a + d - i(b - c)] = -\frac{1}{2} + i \frac{\sqrt{3}}{2} = e^{i(2\pi/3)} \\ f &= \frac{1}{2} [a - d + i(b + c)] = 0 \end{aligned}$$

Finally we obtain for $d_{\pi=2}$

$$(216) \quad 2) \quad d_{\pi=2} \quad \pi \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} e^{+i(2\pi/3)} & 0 \\ 0 & e^{-i(2\pi/3)} \end{pmatrix} + \rightarrow Z$$

From $d_{\pi=2}$ to $d_{\pi=3}$

The algebraic as well as complex conjugation properties of the d_{π} collection, restricted to even permutations, allows to derive the (diagonal form) of the representation matrix $d_{\pi=3}$ directly from the element $d_{\pi=2}$

$$(217) \quad 3) \quad d_{\pi=3} \quad \pi \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} e^{-i(2\pi/3)} & 0 \\ 0 & e^{+i(2\pi/3)} \end{pmatrix} + \rightarrow Z^2 \equiv Z^{-1} \rightarrow$$

A3-5

The representation matrix for the transposition $d_{\pi=4}$

We repeat the fourth entry of the D_{π} collection in eq. 124 below in eq. 218

$$(218) \quad 4) \quad \pi \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \longrightarrow T_{12}$$

Here we follow the same procedure as in the subsection on $d_{\pi=2}$ in eqs. 214 - 215

$$(219) \quad \begin{pmatrix} a4 & b4 \\ c4 & d4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\frac{1}{2}(a + d) = 0 \quad , \quad \frac{1}{2}(b - c) = 0$$

$$\frac{1}{2}(a - d) = 1 \quad , \quad \frac{1}{2}(b + c) = 0$$

The analog to eq. 215 becomes

$$(220) \quad e = \frac{1}{2} [a + d - i(b - c)] = 0$$

$$f = \frac{1}{2} [a - d + i(b + c)] = 1$$



A3-6

So we arrive with the help of eq. 212 at the result

$$(221) \quad 4) \ d_{\pi=4} \ \pi \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad - \quad \rightarrow T_{12}$$

The representation matrices for transpositions $d_{\pi=5}$ and $d_{\pi=6}$

We repeat entries 5) and 6) of the D_{π} collection in eq. 124 below in eq. 222

$$(222) \quad 5) \ \pi \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \quad - \quad \rightarrow T_{23}$$

$$6) \ \pi \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \quad - \quad \rightarrow T_{13}$$

For $D_{\pi=5}$ the equation analogous to eq. 219 reads



A3-7

$$(223) \quad \begin{pmatrix} a_5 & b_5 \\ c_5 & d_5 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

$$\begin{aligned} \frac{1}{2}(a + d) &= 0, & \frac{1}{2}(b - c) &= 0 \\ \frac{1}{2}(a - d) &= -\frac{1}{2}, & \frac{1}{2}(b + c) &= \frac{\sqrt{3}}{2} \end{aligned}$$

The analog to eq. 220 becomes

$$(224) \quad \begin{aligned} e &= \frac{1}{2} [a + d - i(b - c)] = 0 \\ f &= \frac{1}{2} [a - d + i(b + c)] = -\frac{1}{2} + i\frac{\sqrt{3}}{2} = e^{+i(2\pi/3)} \end{aligned}$$

The entry for $d_{\pi=5}$ becomes

$$(225) \quad 5) d_{\pi=5} \quad \pi \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & e^{+i(2\pi/3)} \\ e^{-i(2\pi/3)} & 0 \end{pmatrix} \quad - \quad \rightarrow T_{23}$$



A3-8

The transition from $d_{\pi=5}$ to $d_{\pi=6}$, corresponding to the substitution $T_{23} \rightarrow T_{13}$ is achieved by complex conjugation of the matrix elements of $d_{\pi=5}$

$$(226) \quad 6) \quad d_{\pi=6} \quad \pi \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & e^{-i(2\pi/3)} \\ e^{+i(2\pi/3)} & 0 \end{pmatrix} \quad - \quad \rightarrow T_{13}$$



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