Counting of oscillatory modes of valence quarks forming qqq baryons for 3 quark flavors u , d , s

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We present the unique properties of oscillatory modes of $N_{fl} = 3$ light quarks, $- - u, d, s - -$, using the SU $(2 N_{fl} = 6)$ $\times$ SO3 $\left(\vec{L} \right)$ broken symmetry classification. $\vec{L} = \sum_{n=1}^{N_{fl}} \vec{L}_n$ stands for the space rotation group generated by the sum of the three individual angular momenta of quarks in their c.m. system. The baryonic multiplets are shown to emerge from the picture of oscillating quarks in 3 space dimensions in the center of mass system of the baryons. All oscillatory modes are fully relativistic with a finite number of oscillators and this is forming the unique harmonic oscillator with these properties. The density of states as a function of mass-square is calculated. This estimate is of relevance for the accounting of the missing states of unobserved hadrons, as the here estimated baryonic multiplets include both the observed and the unobserved (or "missing") hadrons. The estimate is conceptually different from Hagedorn's model and is based on field theory of QCD.

1. Introduction

The underlying field theory of strong interaction, QCD, assigns a local spin 1/2 field to every valence quark and antiquark with the quantum numbers color and flavor, interacting locally with spin 1 color octet gauge fields, called gluons. The gauge interactions of the gluons are exactly conserved gauge transformations based on the local gauge group SU(3) of color. The field theoretic concepts were developed as core topics of an in depth investigation together with the oscillatory modes of quarks suggested by linear Regge trajectories[1] .
The linear Regge trajectories (e.g. ref.[1] and [2]) with exception of the pomeron, suggest oscillatory behaviour, whereas the difficulty to describe them is due to the fact that the field theory variables fall into the perturbatively non accessible domain. Lattice QCD in principle is set to solve the non-perturbative problems ref. [3] .
Within the theory of Quantum Chromo Dynamics the mesons and baryons are arranged in multiplets while it was already found since the mid-1960'ies that many states are missing refs. [4] , [5] , [6] in the sense that they have not been detected by experiments even though they are predicted by the quark model. The non observation of some states can be due for example to resonance states overlapping or being very wide. The missing states remain missing up to the present.
In this paper we are constructing the full theoretically expected multiplets of baryons including those measured and those that are missing, and count their states, based on the oscillatory modes for quarks and antiquarks.

The oscillatory modes of valence quarks and antiquarks in mesons and baryons were developed by one of us (P.M.) in CALTECH from 1975 onwards and published in ref. [1] , ref. [7] , ref. [8] . In reference [8] only non strange hadrons have been considered, which have been compared to a good compilation of missing states of non strange baryons, as can be found in PDG 1980. However there are missing states also among the strange hadrons ref. [9] or ref. [10] .
The main idea of this approach is to use relative coordinates and associated canonically conjugate momenta in the center of mass of $q\overline{q}^{\,\prime}$ and $q\;q^{\,\prime}\;q^{\,\prime\prime}$ systems to describe harmonic oscillations of the valence quarks and antiquarks, with full special relativistic invariance using the zero width approximation for resonances.
We are separating quark antiquark spins combining them to baryonic sub-multiplets.
Oscillator modes mean by definition gaussian wavefunction in the ground state, either in configuration space or in momentum space.
With respect to light quark masses u,d,s we take the chiral limit. Further the limit considered is the long range approximation between quarks and antiquarks which involves large main oscillatory quantum number N.
The model is valid strictly for large N. For small and finite N the multiplets are not degenerate for different I, J, P, C quantum numbers of particles, deviating from the large N asymptoytic behaviour.
Many modes are unstable against strong decays and are approximated in the zero width limit.
A perturbatively renormalizable field theory does not allow an exponential growth of mass- and/or masssquare-density of such states because it would lead to an infinite number of free constants, which render the theory non-renormalizable. In string theory there is an infinite number of independent oscillators (like many springs) one each for the ground frequency ω, 2ω, 3ω etc. This cannot be described by a field theory which contains only a finite number of degrees of freedom. The string theory allows a spectrum of density growing exponentially with mass(square).
The prediction of Hagedorn's bootstrap model [11] , was an exponential spectrum of hadron resonances , which later was shown to be realized in string theory , refs. [12] , [13] , [14] , [15] . In addition the limiting temperature resulting from the exponential hadron resonance spectrum in Hagedorn's model was associated with the critical temperature of the QCD phase transition between hadrons and partons [16] . However if a limiting temperature exists from the hadron spectrum this is not nessecarily associated with a phase transition.
In string theory there is a limiting temperature à la Hagedorn, which is determined from the string tension i.e. of the order of the Planck mass and it is not nessecarily associated with a phase transition.
An exponentially growing spectrum of hadron resonances would not agree with the expectation for an underlying field theory of strong interactions.

The approximate use of a broken extended symmetry combining $SU6 = SU\left(2\left(\sum_q \vec{S}_q\right)\times 3_{\;fl}\right)$ with $SO3\left(\sum_q \vec{L}_q\right)$ was vigorously pursued up to the end of the 60-ies, whence the three valence quarks wave function is conceived in their c.m. frame. Other orbital $q\overline{q}\,'$ and 3-quark wave functions were also considered, as e.g. the $p_{\,z} \rightarrow \infty$ states, where a boosted $SU6_{\,w}$ symmetry is introduced instead of $SU6$. In this respect the main quantum number N was not related to genuine oscillator degrees of freedom but instead inferred from recurrences along Regge trajectories from the relation

$$\alpha\,' M^2 \,(\,J\,) \;=\; J \,+\, J_{\,0} \;\text{ with }\; \alpha\,'\Delta\,M^{\,2} \;=\; \Delta\,J \;=\; \Delta\,N \tag{1}$$

In eq. 1 $\vec{J} \,=\, \vec{L} \,+\, \vec{S}$ denotes total angular momentum.
There is some literature quoted in ref. [8], which is however very limited, and among the many papers on that matter we mention here few representative ones by J. Schwinger [4], R. Dashen and M. Gell-Mann [5] and by G. Zweig [6], illustrating the search for local field variables underlying the strong interactions at that time.
Other reference frames, alternative to the qqq overall rest system can also be used, e.g. the $p_z \rightarrow \infty-$ frame, involving a boosted $SU6_{\,w}$ symmetry replacing $SU3_{\,fl}$.
Historically, one result from the investigations and efforts towards identifying the known baryon resonances, was the finding that several states were missing. The counting of oscillatory modes of baryons with light flavored u, d, s quarks that we report here is a new investigation.
The estimate and counting of oscillatory modes of mesons with light flavored u, d, s quarks and corresponding antiquarks has been published previously in ref.[17]. In the following we will elaborate on the oscillating modes of baryons and compare with the measured ones.

2. Oscillatory modes for u,d,s baryons

2.1. Modes of quarks in baryons according to the $SU\,(\,2\,N_{\,fl} \;=\; 6\,) \times SO3\left(\,\vec{L}\,\right)$ *broken symmetry classification*

In this section we discuss general properties of valence quark modes in baryons,and we will proceed in the next section to the special features of oscillatory modes.
Historically valence quark modes in baryons followed the pioneers of the quark model, whome we quote[18–20].
Oscillatory modes for baryons and mesons have been developed for the first time by one of us (P.M.) in[8] for non strange quarks.
This revealed the existence of missing states in the measured baryon spectrum.
The extension to include the strange quark in baryon oscillatory modes has been published in[21] and will be elaborated here for the first time in a coherent fashion.
We quote here a paper by S. Brodsky et al. in which Regge trajectory forming modes have been derived from light-front holography in superconformal algebras.[22]
Oscillatory modes of quarks in baryons reveal many properties going beyond the quark model, e.g. the numbers of modes with fixed main quantum number N.

In the following we discuss the technique of Young tableaux applied to the classification of representations of the group SU $\left(2 \ N_{fl} \ = \ 6 \right) \ \times$ SO3 $\left(\ \vec{L} \ \right)$.

2.2. The reduction of an SUN group R-fold product representation through the symmetric group S_R $\longleftrightarrow$ *Young tableaux* Y_R^N

The three irreducible Young [a] tableaux in figures 1 - 3 below , arise as tensors of rank 3 within a symmetry group of SU $\left(6 \ = \ \text{SU2}_{spin} \ \times \text{SUN}_{fl} \ = 3 \right)$. Its broken character is discussed below.
Rank three corresponds to the wave function of a baryon formed from three valence quarks, confined with respect to color, exclusively . In the construction of this wave function the width of the clearly involved resonance shall be set approximately to zero .
We turn towards the functions asociated with these Young tableaux , depending on 6 integer arguments

$$\begin{aligned}
& D_3^6 \ (m_6 , m_5 , \cdots , m_1) \ = \ D \ (m_6 , m_5 , \cdots , m_1) \\
& m_6 \ > \ m_5 \ \cdots \ > \ m_1 \ > \ 0 \ : \text{integers} \\
& D \ (m_6 , m_5 , \cdots , m_1) \ = \ \Pi_{j=2}^{6} \ \Pi_{k=1}^{k=j-1} \ (m_j \ - \ m_{j-k})
\end{aligned} \tag{2}$$

The D - functions for symmetric , mixed and antisymmetric representations of SU6 will be discussed after the 3 figures below .

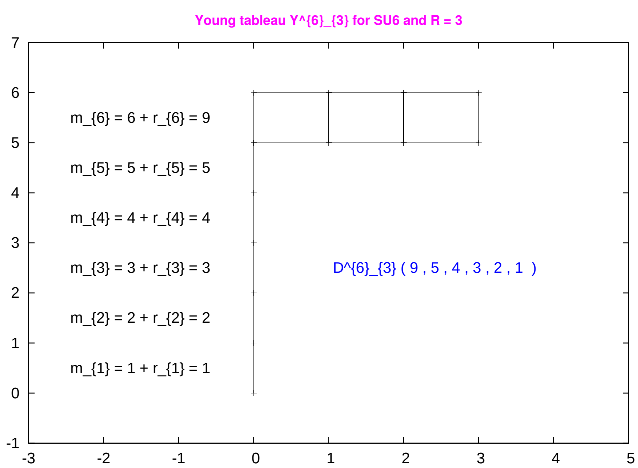

Fig 1 : The symmetric Young tableau for SU6 and rank $R \ = \ 3$

[a] Alfred Young + 16 April 1873 in Widnes , Lancashire , England
† 15 December 1940 in Birdbrook , Essex , England

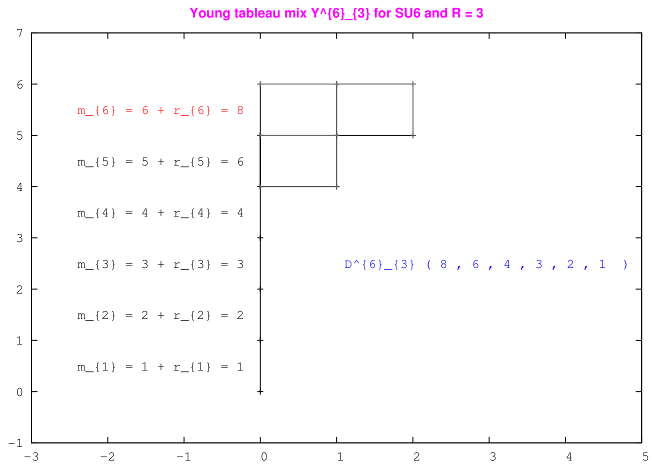

Fig 2 : The mixed Young tableau for SU6 and rank $R = 3$

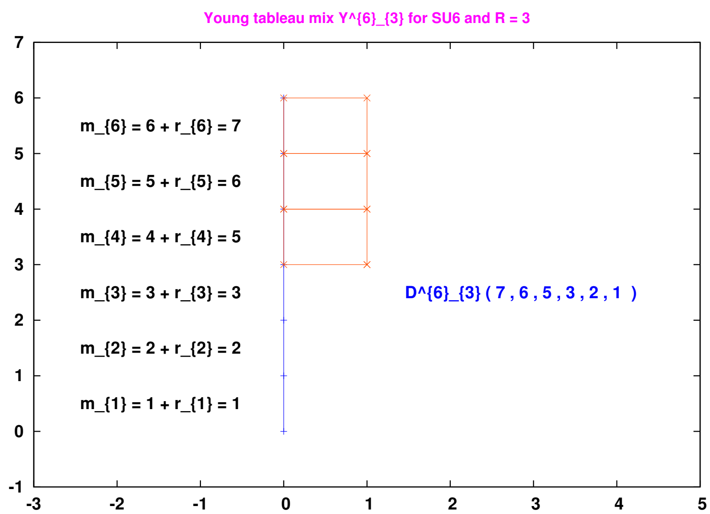

Fig 3 : The antisymmetric Young tableau for SU6 and rank $R = 3$

The significance of Young tableaux with respect to an SUN transformation group lies in the one to one correspondence of irreducible representations of this group

formed by the R-fold tensor product of the defining one , symmetrized first along the rows and antisymmetrized thereafter along the columns associated with any given Young tableau , forming a representaion of the symmetric group S $_R$,
The three Young tableaux in figures 1 - 3 each determine such an irreducible representation of SU6 . The D functions , defined in eq. 2 , determine the respective dimensions of these representations .
We list the three D functions in eq. 3 below

$$
\begin{array}{ll}
\square\square\square : & D \ (\ 9 \ , \ 5 \ , \ 4 \ , \ 3 \ , \ 2 \ , \ 1 \ ) \ = \ 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot d_{\ 5} \\
\begin{smallmatrix} \square & \square \\ \square & \end{smallmatrix} : & D \ (\ 8 \ , \ 6 \ , \ 4 \ , \ 3 \ , \ 2 \ , \ 1 \ ) \ = \ 2 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot d_{\ 4} \\
\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix} : & D \ (\ 7 \ , \ 6 \ , \ 5 \ , \ 3 \ , \ 2 \ , \ 1 \ ) \ = \ 2 \cdot 4 \cdot 5 \cdot 6 \cdot 3 \cdot 4 \cdot 5 \cdot 2 \cdot 3 \cdot 4 \cdot d_{\ 3} \\
 & d_{\ j} \ = \ D \ (\ j \ , \ j-1 \ , \ \cdots \ , \ 1 \ ) \ = \ (\ j-1 \ ) \ ! \ d_{\ j-1} \ = \ \Pi_{\ k=1}^{\ j-1} \ k \ ! \\
 & d_1 = d_2 = 1, d_3 = 2, d_4 = 12, d \ 5 = 24 \cdot 12, d_6 = 120 \cdot 24 \cdot 12
\end{array} \tag{3}
$$

This concludes the discussion of the main premises contained in Young tableaux . The dimensions of irreducible SUN representations belonging to a specific Young tableau are given by

$$
dim \ (\text{ Y-t }) \ = \ \frac{D \ (\ m_{\ R} \ , \ m_{\ R-1} \ , \ \cdots \ , \ m_{\ 1} \ )}{D \ (\ R \ , \ R-1 \ , \ \cdots \ , \ 1 \ )} \tag{4}
$$

Substituting eq. 3 in eq. 4 it follows , always for SU6

$$
\begin{array}{lll}
dim \ (\square\square\square) = & \dfrac{4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}{5 \ !} & = 56 \\
dim \left(\begin{smallmatrix} \square & \square \\ \square & \end{smallmatrix} \right) = & \dfrac{2 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{4 \ ! \ 5 \ !} & = 70 \\
dim \left(\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix} \right) = & \dfrac{2 \cdot 4 \cdot 5 \cdot 6 \cdot 3 \cdot 4 \cdot 5 \cdot 2 \cdot 3 \cdot 4}{3 \ ! \ 4 \ ! \ 5 \ !} & = 20
\end{array} \tag{5}
$$

2.3. The main $\underline{N}^{\ P} \ = \ 56^{\ +}$ baryon valence quark configuration multiplet – the entry point

The most stable baryon multiplet , restricted to the light three flavors of u , d , s quarks , shall be labelled by the total number of states including spin and flavor , with multiplicity and parity denoted $\underline{N}$ and by the superfix ^{P} respectively .

The landmark pertaining to the $J^{P} = \frac{3}{2}^{+}$ states within this multiplet is the discovery picture of the Omega^{-} in 1964, shown in figure 4 below[23] for its discovery, ref. [24] .
J denotes the total angular momentum formed from spins and orbital angular momenta . The strangeness3 baryon was predicted in 1962 together with its mass region for a decuplet with $J = \frac{3}{2}$ in refs.[25] . The configuration space wave functions of the 56^{+} combined SU6 $\times$ SU2 $\left(\vec{L} \right)$ - ground state multiplet are (assumed) independent of angular relative momenta of the 3 valence quarks and all equal as a consequence , and thus form an octet of $J = \frac{1}{2}$ and a decuplet of $J = \frac{3}{2}$ i.e. a multiplet of overall 56 states of equal parity , positive by convention .
This concludes the entry point presentation of Young tableaux . The nontrivial higher configurations shall be discussed in subsequent sections .

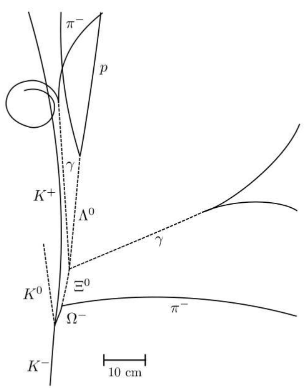

Fig 4 : The bubble chamber picture of the Omega^{-} in 1964 , see ref. [24] .

2.4. The first orbitally excited , negative parity , $N^{P} = 70^{-}$ based , $\underline{N}^{-} = 210^{-}$ baryon valence quark configuration multiplet – the confirmation from experimental spectroscopy ?

The $\underline{N}^{-} = 210^{-}$ negative parity u , d , s multiplet is well described in the current PDG review 'Quark Model' by C. Amsler , T. De Grand and B. Krusche in ref.[10] . The mixed $\text{SU6}_{spin \times fl}$ Young tableau yielding the 70 representation is combined with an SU2 $\left(\vec{L} \right)$; $L = 1$ orbital angular momentum wave function in a way compatible with – including color – overall fermion permutation antisymmetry of three valence quarks u, d , s .

$$\begin{array}{l} \vec{L} = \vec{L}_{1} + \vec{L}_{2} + \vec{L}_{3} \ ; \ L = L_{123} = 1 \ ; \ 2L + 1 = 3 \\ \underline{N} = (2L + 1) \, N = 210 \end{array} \tag{6}$$

The $N = 70$ representation of SU6 $_{spin \times fl}$ decomposes with respect to total quark spin multiplicity and (times) SU3 $_{fl}$ irreducible representations as

$$70 = 2 \times 10 + 2 \times 8 + 4 \times 8 + 2 \times 1 \tag{7}$$

This combines the respective total spins $\frac{1}{2}$ and $\frac{3}{2}$ to total angular momenta $J^{\,P}$

$$\begin{array}{l} \text{spin} = \frac{1}{2} : J^{\,P} = \frac{3}{2} \oplus \frac{1}{2} \\ \text{spin} = \frac{3}{2} : J^{\,P} = \frac{5}{2} \oplus \frac{3}{2} \oplus \frac{1}{2} \end{array} \tag{8}$$

Finally also combining total orbital angular momentum and total spin the $\underline{N}^{\,-} = 210^{\,-}$ multiplet decomposes according to

$$\underline{N}^{\,-} = 210^{\,-} = \left\{ \begin{array}{ll} 10 \,, \left(\frac{3}{2}\right)^{\,-} \oplus \left(\frac{1}{2}\right)^{\,-} + & \# \, 60 \\ 8 \,, \left(\frac{3}{2}\right)^{\,-} \oplus \left(\frac{1}{2}\right)^{\,-} + & \# \, 48 \\ 8 \,, \left(\frac{5}{2}\right)^{\,-} \oplus \left(\frac{3}{2}\right)^{\,-} \oplus \left(\frac{1}{2}\right)^{\,-} + & \# \, 96 \\ 1 \,, \left(\frac{3}{2}\right)^{\,-} \oplus \left(\frac{1}{2}\right)^{\,-} & \# \;\; 6 \\ \hline & 210 \end{array} \right. \tag{9}$$

$S = 0$ $I = \frac{1}{2}$	$S = 0$ $I = \frac{3}{2}$	$S = -1$ $I = 0$	$S = -1$ $I = 1$	$S = -2$ $I = \frac{1}{2}$	$S = -3$ $I = 0$
$J^P = \left(\frac{3}{2}\right)^-$ $M = 1520$	$J^P = \left(\frac{3}{2}\right)^-$ $M = 1700$	$J^P = \left(\frac{3}{2}\right)^-$ $M = 1690$	$J^P = \left(\frac{3}{2}\right)^-$ $M = 1670$		
$J^P = \left(\frac{1}{2}\right)^-$ $M = 1535$	$J^P = \left(\frac{1}{2}\right)^-$ $M = 1620$	$J^P = \left(\frac{1}{2}\right)^-$ $M = 1670$			
$J^P = \left(\frac{1}{2}\right)^-$ $M = 1650$		$J^P = \left(\frac{1}{2}\right)^-$ $M = 1405$	$J^P = \left(\frac{1}{2}\right)^-$ $M = 1750$		
$J^P = \left(\frac{3}{2}\right)^-$ $M = 1700$		$J^P = \left(\frac{3}{2}\right)^-$ $M = 1520$	$J^P = \left(\frac{3}{2}\right)^-$ $M = 1940$?	$J^P = \left(\frac{3}{2}\right)^-$ $M = 1820$	
$J^P = \left(\frac{5}{2}\right)^-$ $M = 1675$		$J^P = \left(\frac{5}{2}\right)^-$ $M = 1830$	$J^P = \left(\frac{5}{2}\right)^-$ $M = 1775$		

Table 1 : Candidate states belonging to the mixed $N = 70^-$ negative parity Young tableau ; M in MeV

(10)

The lowest in mass negative parity states , without heavy flavor content, according to the current PDG tables [9] , are listed in Table 1, eq. 10. We list the numbers for SU3 $_{fl}$ octets , decuplets and singlets in the $N = 70^{\,-}$; $\underline{N} = 210^{\,-}$ multiplet of states

$$\begin{array}{lllll} \text{octets} & J^{\,P} & = \left(\frac{1}{2}\right)^{-} & : 2\ \# & 32 \\ \text{octets} & J^{\,P} & = \left(\frac{3}{2}\right)^{-} & : 2\ \# & 64 \\ \text{octets} & J^{\,P} & = \left(\frac{5}{2}\right)^{-} & : 1\ \# & 48 \\ \text{decuplets} & J^{\,P} & = \left(\frac{1}{2}\right)^{-} & : 1\ \# & 20 \\ \text{decuplets} & J^{\,P} & = \left(\frac{3}{2}\right)^{-} & : 1\ \# & 40 \\ \text{singlets} & J^{\,P} & = \left(\frac{1}{2}\right)^{-} & : 1\ \# & 2 \\ \text{singlets} & J^{\,P} & = \left(\frac{3}{2}\right)^{-} & : 1\ \# & \underline{\ \ 4} \\ & & & \# & 210 \end{array} \tag{11}$$

Accepting all states listed in Table 1 and assigning $\Xi\left(\frac{3}{2}\right)^{-}$; $M = 1820$ to an octet , $\Sigma\left(\frac{3}{2}\right)^{-}$; $M = 1949$ to a decuplet and $\Lambda\left(\frac{3}{2}\right)^{-}$; $M = 1520$ and $\Lambda\left(\frac{1}{2}\right)^{-}$; $M = 1405$ to a singlet , the following states are either missing or corresponding quantum numbers cannot be assigned .

$$\left.\begin{array}{lll} \Omega\left(\frac{3}{2}\right)^{-} & \text{decuplet} & \#\ \ 4 \\ \Omega\left(\frac{1}{2}\right)^{-} & \text{decuplet} & \#\ \ 2 \\ \Xi\left(\frac{3}{2}\right)^{-} & \text{decuplet} & \#\ \ 8 \\ \Xi\left(\frac{1}{2}\right)^{-} & \text{decuplet} & \#\ \ 4 \\ \Xi\left(\frac{5}{2}\right)^{-} & \text{octet} & \#\ 12 \\ \Xi\left(\frac{3}{2}\right)^{-} & \text{octet} & \#\ \ 8 \\ \Xi\left(\frac{1}{2}\right)^{-} & \text{octet} & \#\ \ 8 \\ \Sigma\left(\frac{1}{2}\right)^{-} & \text{decuplet} & \#\ \ 6 \\ \Sigma\left(\frac{3}{2}\right)^{-} & \text{octet} & \#\ 12 \\ \Sigma\left(\frac{1}{2}\right)^{-} & \text{octet} & \#\ \ 6 \\ \Lambda\left(\frac{3}{2}\right)^{-} & \text{octet} & \#\ \ 4 \\ \Lambda\left(\frac{1}{2}\right)^{-} & \text{octet} & \#\ \ 2 \end{array}\right\}\ \#\ \text{missing states : 76 out of 210} \tag{12}$$

Only states with nonvanishing strangeness are among the missing or non-assignable. In order to put this number (76) in perspective we list among the $\underline{N}^{\,-} = 210^{\,-}$ states those with vanishing strangeness : a quartet per decuplet and a doublet per octet . Thus we have for the nonstrange number of states

$$\left.\begin{array}{lll} \Delta\left(\frac{3}{2}\right)^{-} & \text{decuplet} & \#\ 16 \\ \Delta\left(\frac{1}{2}\right)^{-} & \text{decuplet} & \#\ \ 8 \\ \mathcal{N}\left(\frac{5}{2}\right)^{-} & \text{octlet} & \#\ 12 \\ \mathcal{N}\left(\frac{3}{2}\right)^{-} & \text{octlet} & \#\ 16 \\ \mathcal{N}\left(\frac{1}{2}\right)^{-} & \text{octlet} & \#\ \ 8 \end{array}\right\}\ \#\ \text{nonstrange states : } 60 \tag{13}$$

$$\longrightarrow\ \#\ \text{strange states : } 150$$

Let us associate a quality factor relative to the spectroscopic recognition of the states with nonvanishishing strangeness , pertaining to the negative parity $N = 70^{-}$; $\underline{N} = 210^{-}$ multiplet of states , abbreviated by $\{ 70^{-} \}$ restricted also to three u, d , s flavored valence quark configurations, lowest in mass

$$Qf \ (\ S < 0 \ ) = 1 \ - \ \frac{\# \text{ correctly assigned strange states}}{\# \text{ all strange states} \{ \ 70^{-} \ \}} \ \forall \text{ nonstrange states} \in$$
$$= \frac{74}{150} = 0.493 \ \sim \ 49.3 \ \% \tag{14}$$

The result in eq. 14 prompts two remarks

1) The quality factor $Qf \ (\ S \ < \ 0 \ ) \ \sim \ 49.3$, defined in eqs. 12 is clearly insufficient.

2) This calls for explanations in the first instance by the experimenters having performed the pertinent experiments .

2.5. The first orbitally excited, negative parity , $N^{P} = 20^{-}$ based, $\underline{N}^{-} = 20^{-}$ baryon valence quark configuration multiplet – about $SU3_{fl}$ singlet baryons

First we study the following symmetric (pseudo-) tensor structure obtained from a triplet of configuration space vectors $\vec{x}_{1,2,3}$ subject to the c.m. restriction

$$T^{mn}_{L=2} = \left\{ \begin{array}{c} + \ (x_1)^m \ (x_2 \wedge x_3)^n \ + \ (x_1)^n \ (x_2 \wedge x_3)^m \\ + \ (x_3)^m \ (x_1 \wedge x_2)^n \ + \ (x_3)^n \ (x_1 \wedge x_2)^m \\ + \ (x_2)^m \ (x_3 \wedge x_1)^n \ + \ (x_3)^n \ (x_2 \wedge x_1)^m \\ - \ 2 \ \delta^{mn} \ Det \ (x_1 , \ x_2 , x_3) \end{array} \right\} \tag{15}$$
$$\sum_{i=1}^{3} \vec{x}_i = 0 \ \rightarrow \ = \ Det \ (x_1 , \ x_2 , x_3) \ = \ 0$$

Under rotations of the configuration vectors $\vec{x}_i \ \rightarrow \ R \ \vec{x}_i$; $i = 1, 2, 3$ the functions $T^{mn}_{L=2} \ (x_1 , x_2 , x_3)$ form the 5 orbital angular momentum wave functions with L = 2 .

The full wave functions, not restricted to oscillatory modes, as described in ref. [8] , but with quantum numbers specified according to the SU $(2 N_{fl} = 6) \times$ SO3 $\left(\vec{L} \right)$ broken symmetry classification , and associated Young tableau symmetry relations are then constructed as follows , where the orbital part – here $T^{mn}_{L=2}$

defined in eq. 15 – is a factor

$$\Psi^{\mu\nu}\,(x_1,x_2,x_3) \;=\; T^{mn}_{L=2}\,(x_1,x_2,x_3)\,\psi\,(x_1,x_2,x_3\;;\;L=2)$$

$$\psi\,(\,x_1,x_2,x_3\;;\;L=2\,)\;\longrightarrow\;\psi\,(\,x_1,x_2,x_3\,)\;:\;\text{to simplify notation}$$

$$\begin{array}{ll} \psi\,(\,x_1,x_2,x_3\,)\;=\;\mathcal{D}\,(\,L=2\,)\,\psi\,(\,Rx_1,Rx_2,Rx_3\,) & \text{orbital rotation symmetry} \\ \psi\,(\,x_{\,1}\,,\,x_{\,2}\,,\,x_{\,3}\,)\;=\psi\,(\,x_{\,j\,1}\,,\,x_{\,j\,2}\,,\,x_{\,j\,3}\,) & \forall\text{ permutations }\begin{pmatrix}1&2&3\\ j_{\,1}&j_{\,2}&j_{\,3}\end{pmatrix} \\ \sum_{i=1}^{3}\,\vec{x}_{\,i}\;=\;0 & \text{overall center of mass condition} \end{array} \tag{16}$$

Whence the rotational , combinatorial and c.m. related conditions , displayed in the last 3 lines of eq. 16 , are satisfied – they are broken by the asymmetries of the quark masses $m_{\,u}\,,\,m_{\,d}\,,\,m_{\,s}$ *and independently* by the breaking of $\mathrm{SU6}_{\,spin\,\times\,fl}$ – the remaining dynamical structure of QCD resides in the specific form of the residual wave function $\psi\,(\,x_{\,1}\,,\,x_{\,2}\,,\,x_{\,3}\,)$. It is a Gaussian function for the oscillatory ground mode [8].

2.6. The unitary scalar product adapted to rotational , combinatorial and c.m. related conditions of valence quark u,d,s modes in baryons

The present subsection fits nicely in the context of this chapter , even if it is a digression , preparing discussion of positive parity $\mathrm{SU}\,(\,2\,N_{\,fl}\;=\;6\,)\times\mathrm{SO3}\,\left(\,\vec{L}\,\right)$ broken symmetry baryon multiplets .

To this end we replace the specific wave function $\Psi^{\,\mu\nu}\,(\,x_{\,1}\,,\,,x_{\,2}\,,\,x_{\,3}\,)$ defined in eq. 16 by a collection of generic ones

$$\Psi^{\,\mu\nu}\,(\,x_{\,1}\,,\,,x_{\,2}\,,\,x_{\,3}\,)\;\longrightarrow\bigcup_{(\alpha)}\,\Psi^{\,(\alpha)}\,(\,x_{\,1}\,,\,,x_{\,2}\,,\,x_{\,3}\,) \tag{17}$$

The unique unitary scalar product , compatible with all rotational , combinatorial and c.m. related conditions is then of the form

$$\left\langle\,\Psi^{\,(2)}\,\middle|\,\Psi^{\,(1)}\,\right\rangle\;=\;\begin{array}{l}(\,3\,)^{\,3/2}\displaystyle\int\Pi_{\,i=1}^{\,3}\,d^{\,3}\,x_{\,i}\,\delta^{\,3}\,(\,x_{\,1}\,+\,x_{\,2}\,+\,x_{\,3}\,)\;\times\\ \quad\times\,\Psi^{\,*\,(2)}\,(\,x_{\,1}\,,\,,x_{\,2}\,,\,x_{\,3}\,)\,\Psi^{\,(1)}\,(\,x_{\,1}\,,\,,x_{\,2}\,,\,x_{\,3}\,)\end{array} \tag{18}$$

We calculate the the square norm of a Gaussian , using the barycentric coordinates for three valence quarks in the c.m. system

$$\begin{array}{c} z_{\,1}\;=\;\frac{1}{\sqrt{2}}\,(\,x_{\,1}\,-\,x_{\,2}\,)\;\;,\;\;z_{\,2}\;=\;\frac{1}{\sqrt{6}}\,(\,x_{\,1}\,+\,x_{\,2}\,-\,2\,x_{\,3}\,) \\ z_{\,3}\;=\;\frac{1}{\sqrt{3}}\,(\,x_{\,1}\,+\,x_{\,2}\,+\,x_{\,3}\,)\;=\;\sqrt{3}\,X_{\,c.m.}\;\;\rightarrow\;0 \end{array} \tag{19}$$

It is good to remember from ref.[8] that only 2 out of the three barycentric three-vector variables as well as their conjugate momenta , i.e.

$$z_{\,1}\,,\,z_{\,2}\;\longleftrightarrow\;\pi_{\,1}\,,\,\pi_{\,2} \tag{20}$$

exhibit oscillatory (confined) motion , wheres the c.m. related position and conjugate momentum are left in free motion . Thus the universal , i.e. N_c independent oscillatory frequency appears in the form of the induced $\mathcal{M}^2$ operator in the N_c ($=$ 3 here) dependent combination

$$\begin{aligned} &\mathcal{M}^2 = \sum_{\nu=1}^{N_c-1} \left[K_{N_c} \pi_\nu^2 + (K_{N_c})^{-1} \Lambda^2 z_\nu^2 \right] \\ &K_{N_c} = N_c / (N_c - 1) \ \left(= \tfrac{3}{2} \text{ here} \right) \\ &\pi_\mu = \tfrac{1}{i} \partial_{z_\mu} \ ; \ \mu = 1,2 \end{aligned} \tag{21}$$

Finally in the identificationi of oscillatory variables we reduce to the actual oscillator ones through a canonical transformation

$$z_\mu = \lambda \overline{z}_\mu \longleftrightarrow \pi_\mu = \lambda^{-1} \overline{\pi}_\mu \ ; \ \mu = 1,2 \tag{22}$$

under which $\mathcal{M}^2$ in eq. 21 becomes

$$\mathcal{M}^2 = \sum_{\nu=1}^{N_c-1} \left[\frac{K_{N_c}}{\lambda^2} \overline{\pi}_\nu^2 + \frac{\lambda^2 \Lambda^2}{K_{N_c}} \overline{z}_\nu^2 \right] \tag{23}$$

The quantity λ of dimension length , introduced in eqs. 22 and 23 , is to be chosen such that

$$\frac{K_{N_c}}{\lambda^2} = \frac{\lambda^2 \Lambda^2}{K_{N_c}} \rightarrow \lambda^4 = \left(\frac{K_{N_c}}{\Lambda} \right)^2 \rightarrow \lambda^2 = \frac{K_{N_c}}{\Lambda} \tag{24}$$

whereupon $\mathcal{M}^2$ in eqs. 21 , 23 assumes the reduced universal form

$$\begin{aligned} &\mathcal{M}^2 = \Lambda \sum_{\nu=1}^{N_c-1} \left[\overline{\pi}_\nu^2 + \overline{z}_\nu^2 \right] \\ &z_\mu = \lambda \overline{z}_\mu \longleftrightarrow \pi_\mu = \lambda^{-1} \overline{\pi}_\mu \ ; \ \mu = 1,2 \ ; \ \lambda^2 = \frac{K_{N_c}}{\Lambda} \\ &\overline{\pi}_\mu = \partial_{\overline{z}_\mu} \longleftrightarrow \overline{z}_\mu \ ; \ \mu = 1,2 \end{aligned} \tag{25}$$

The associated (universal) oscillator vector-variables derive from the relation on the last line in eq. 25

$$\begin{aligned} &a_{\mu k} = \tfrac{1}{\sqrt{2}} (i \overline{\pi}_{\mu k} + \overline{z}_{\mu k}) \ ; \ a_{\mu k}^\dagger = \tfrac{1}{\sqrt{2}} (- i \overline{\pi}_{\mu k} + \overline{z}_{\mu k}) \\ &\mu = 1,2 \ ; \ k = 1,2,3 \end{aligned} \tag{26}$$

which in turn yields the relativistic structure of the oscillatory $\mathcal{M}^2$ operator , substituting in eqs. 21 , 23 and 25

$$\mathcal{M}^2 = (2 \Lambda) \sum_{\nu=1,\ k=1}^{\nu=2\ ,k=3} \left[\overline{a}_{\nu k}^\dagger \overline{a}_{\nu k} + \tfrac{1}{2} \right] \ ; \ 2 \Lambda = 1 / \alpha' \tag{27}$$

The contribution of the zero mode oscillations , $\frac{1}{2}$ for each dimension of configuration space, ($= 3$) is inherent to the classical limiting form of oscillatory motion, and is accompanied in the sense of a long range approximation (especially given

finite quark masses) by a constant correction .The latter remains non zero also in the limit of vanishing quark masses , as discussed in ref.[8] .
In eq. 27 $1/\alpha'$ denotes the inverse of the Regge slope. If we determine it from the positive parity Λ trajectory from the present PDG tables [9] ,

$$\begin{array}{rllll} \Lambda\ ,\ J^{\,P} & : & \frac{1}{2}^{\,+} & \frac{5}{2}^{\,+} & \frac{9}{2}^{\,+} \\ M_{\,j} & : & 1.115683 & 1.820 & 2.350 \\ M_{\,j}^{\,2} & : & 1.2447485 & 3.3124 & 5.5225 \\ \frac{1}{2}\,\Delta\,M^{\,2} & : & & 1.034 & 1.105 \end{array} \tag{28}$$

and average the two half mass square difference entries in the last line of eq. 28 with weights two to one we obtain

$$1\,/\,\alpha^{\,\prime} \;=\; \frac{1}{3}\,\left(\,M_{\,2}^{\,2}\,-M_{\,1}^{\,2}\,\right)\,+\,\frac{1}{6}\,\left(\,M_{\,3}^{\,2}\,-M_{\,2}^{\,2}\,\right)\,\sim\,1.06\ \text{GeV}^{\,2} \tag{29}$$

We remark that in ref.[9] $\Lambda\frac{9}{2}^{\,+}$ has only three stars , and furthermore the trajectory contains only three entries. $\Lambda\,\frac{13}{2}^{\,+}$ would extrapolate to 2.755 GeV using eq. 29 .
We will come back to eigenvalues , number partitions and counting of 3 flavor- , 2 spin- and 6 orbital oscillator-states , deriving from the mass-square operator as characterized in eq. 27 in a subsequent subsection and return to the scalar product of the ground state oscillator wave function with itself .
The latter is constructed from the orbital oscillator operators specified in eq. 26 , neglecting here flavor and spin quantum numbers within the approximations discussed in this subsection .

3. From oscillatory modes to counting of states

The form of the mass square operator , as displayed in eqs.621 - 27 is – as a long distance approximation – not specified , in particular with respect to the oscillatory zero modes , as well as other constant contributions in the configuration space distances at large .
The next step can be inferred from the way the universal inverse Regge slope is determined in eqs. 28 - 29 and amounts to the parametrization starting with eq. 27 repeated below

$$\mathcal{M}^{\,2} \;=\; (\,2\,\Lambda\,)\,\sum_{\nu=1,\;k=1}^{\nu=2\;,k=3}\left[\,\overline{a}_{\,\nu\,k}^{\,\dagger}\,\overline{a}_{\,\nu\,k}\,+\,\frac{1}{2}\,\right]\;;\;2\,\Lambda\;=\;1\,/\,\alpha^{\,\prime} \tag{30}$$

From eqs. 27 , 30 we introduce the decomposition

$$\begin{array}{l} \mathcal{M}^{\,2} \;=\; \Delta\,\mathcal{M}^{\,2}\,+\,\mathcal{M}_{\,(0)}^{\,2}\;;\;\mathcal{M}_{\,(0)}^{\,2}\;=\;(\,2\,\Lambda\,)\;C_{\,(0)} \\ \Delta\,\mathcal{M}^{\,2} \;=\; (\,2\,\Lambda\,)\;\sum_{\nu=1,\;k=1}^{\nu=2\;,k=3}\Big|_{\text{extended to spin and flavor}}\left[\,\overline{a}_{\,\nu\,k}^{\,\dagger}\,\overline{a}_{\,\nu\,k}\,\right] \end{array} \tag{31}$$

3.2. The barycentric 6 spatial oscillatory variables and the set $\wp\,(\,N\,,\,R\,)$

It is preferable to work out first the barycentroc oscillatory variables as conditioned by the quark flavor Young tableau reduced flavor $\times$ spin multiplets as derived in eq. 5

$$\begin{array}{lll} dim \ (\square\square\square) = \dfrac{4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}{5 \ !} & & = 56 \\ dim \left(\begin{array}{c} \square\square \\ \square \end{array} \right) = \dfrac{2 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{4 \ ! \ 5 \ !} & & = 70 \\ dim \left(\begin{array}{c} \square \\ \square \\ \square \end{array} \right) = \dfrac{2 \cdot 4 \cdot 5 \cdot 6 \cdot 3 \cdot 4 \cdot 5 \cdot 2 \cdot 3 \cdot 4}{3 \ ! \ 4 \ ! \ 5 \ !} & & = 20 \end{array} \tag{32}$$

We recall the definition of the barycentric variables in eqs. 20 - 26 and 29

$$\begin{array}{l} \mathcal{M}^{\ 2} \ = \ \Lambda \ \sum_{\nu=1}^{N_{\ c}-1} \ \left[\ \overline{\pi}^{\ 2}_{\ \nu} \ + \ \overline{z}^{\ 2}_{\ \nu} \ \right] \\ z_{\ \mu} \ = \ \lambda \ \overline{z}_{\ \mu} \ \longleftrightarrow \pi_{\ \mu} \ = \ \lambda^{\ -1} \ \overline{\pi}_{\ \mu} \ ; \ \ \mu \ = \ 1,2 \ ; \ \ \lambda^{\ 2} \ = \ \dfrac{K_{\ N_{\ c}}}{\Lambda} \\ i \ \overline{\pi}_{\ \mu} \ = \ \partial_{\ \overline{z}_{\ \mu}} \ \ \longleftrightarrow \ \ \overline{z}_{\ \mu} \ ; \ \ \mu \ = \ 1,2 \\ \hline K_{\ N_{\ c}} \ = \ N_{\ c} \ / \ (\ N_{\ c} \ - \ 1 \ ) \ \left(= \ \tfrac{3}{2} \ \text{here} \ \right) \ ; \ 2 \ \Lambda \ = \ 1 \ / \ \alpha^{\ \prime} \ \sim \ 1.06 \ \text{GeV}^{\ 2} \end{array} \tag{33}$$

It is worthwhile to perform the calculation of the universal scale factors $\left(\ \lambda \ , \ \lambda^{\ -1} \ \right)$ step by step , using the GeV fm conversion factor

$$\hbar \ c \ = \ 0.1973269718(44) \ \text{GeV fm} \tag{34}$$

We add at this point the Regge slope determination from the mesonic ϱ trajectory , analogous to the Λ baryonic one in eq. 28

$$\begin{array}{rllll} \varrho \ , \ J^{\ P} & : & 1^{\ -} & 3^{\ -} & 5^{\ -} \\ M_{\ j} & : & 0.77549 \ \pm \ 0.00034 & 1.6888 \pm \ 0.0021 & 2.330 \ \pm \ 0.035 \\ M^{\ 2}_{\ j} & : & 0.6013847401 & 2.85204544 & 5.42890000 \\ \pm & : & 0.0005274488 & 0.00709737 & 0.16432500 \\ \tfrac{1}{2} \ \Delta \ M^{\ 2} & : & & 1.12533035 & 1.28842728 \\ \pm & : & & 0.00355847 & 0.08223910 \end{array} \tag{35}$$

For the inverse error-square weighted average of the root mean square of the two $\frac{1}{2} \ \Delta \ M^{\ 2}$ determinations in eq. 35 we obtain

$$\begin{array}{l} \tfrac{1}{2} \ \overline{\Delta \ M^{\ 2}} = (\ 1.125656787 \ \pm \ 0.007090759292 \ ) \ \text{GeV}^{\ 2} \ \sim \ 1 \ / \ \alpha^{\ \prime}_{\ \varrho} \\ \qquad \to (\ 1.126 \ \pm \ 0.007 \ ) \ \text{GeV}^{\ 2} \end{array} \tag{36}$$

The error in eq. 36 represents the statistical error only , it does not account for the systematic error caused by the widths of the resonances involved . The value of $1 \, / \, \alpha_{\varrho}^{\,\prime}$ can be compared wit the one obtained for the Λ trajectory in eq. 29 .
If we use the value for $1 \Big/ \alpha_{\varrho}^{\,\prime}$ also for the pion trajectory , we expect what in todays nomenclature is called $\pi(4)$ at a mass

$$m_{\pi\,(4)} \sim \sqrt{m_{\pi}^{2} + 4 \times 1.126} = 2.13 \text{ GeV} \tag{37}$$

yet no entry exists in the present PDG tables in ref.[9] , while a $J^{PC} = 4^{-+}$, $I = 1$ resonane was listed near this mass in earlier PDG tables . Here we must admit that we have never checked the quality of this $\pi(4)$ resonance in the mass range derived in eq. 37 .
After this digression we return to eqs. 33 and 34 , in order to complete the scale relation of configuration space variables to the dimensionless barycentric variables, defined in eq. 19 repeated below

$$\begin{array}{c} z_{1} = \frac{1}{\sqrt{2}} \left(x_{1} - x_{2} \right) \; , \; z_{2} = \frac{1}{\sqrt{6}} \left(x_{1} + x_{2} - 2\, x_{3} \right) \\ z_{3} = \frac{1}{\sqrt{3}} \left(x_{1} + x_{2} + x_{3} \right) = \sqrt{3}\, X_{c.m.} \rightarrow 0 \end{array} \tag{38}$$

Using rational units for which $\hbar = c = 1$ and choosing as energy and momentum units

$$[\, E \,] = [\, p \,] = 1 \text{ GeV} \tag{39}$$

the unit of length follows from eq. 34

$$[\, L \,] = 1 \text{ GeV}^{-1} = 0.1973269718(44) \text{ fm} \sim \tfrac{1}{5} \text{ fm} \tag{40}$$

From eq. 33 we obtain

$$\begin{array}{lll} z_{\mu} = \lambda\, \overline{z}_{\mu} \longleftrightarrow \pi_{\mu} = \lambda^{-1}\, \overline{\pi}_{\mu} \; ; \; \mu = 1,2 & & \\ \lambda \;\; = \sqrt{2\, K_{N_c}\, \alpha^{\prime}} & = 1.682316462 \text{ GeV}^{-1} & = 0.3319664131 \text{ fm} \\ \lambda^{-1} = & = 0.5944184834 \text{ GeV} & \\ \hline \text{for : } \left(\alpha^{\prime} \right)^{-1} = 1.06 \text{ GeV}^{2} & & \end{array} \tag{41}$$

3.3. The barycentric 6 spatial oscillatory variables and their symmetries with respect to the 3 quark positions

We extend the barycentric dimensionloss coordinates to include a general c.m. position

$$\begin{pmatrix} x_{1\,k} = \lambda\, \overline{x}_{1\,k} \\ x_{2\,k} = \lambda\, \overline{x}_{2\,k} \\ x_{3\,k} = \lambda\, \overline{x}_{3\,k} \end{pmatrix} \longrightarrow X_{k} = \lambda\, \overline{X}_{k} = \tfrac{1}{3} \left(x_{1\,k} + x_{2\,k} + x_{3\,k} \right) \tag{42}$$

$k = 1,2,3$: configuration space coordinate labels

In the following we will suppress the configuration space coordinate labels $k = 1, 2, 3$ displayed in eq. 42 and restrict configuration space three vectors to their dimensionless representatives $\overline{x}_{1,2,3}$ and functions thereof.
The first thing to do is to transform to universal dimensionless configuration space variables the scalar product defined in eqs. 18 , 19 repeated below

$$\begin{aligned} \left\langle \Psi^{(2)} \,\middle|\, \Psi^{(1)} \right\rangle \;=\; & (3)^{3/2} \int \Pi_{i=1}^{3} d^{3} x_{i} \, \delta^{3} (x_{1} + x_{2} + x_{3}) \times \\ & \times \Psi^{*\,(2)} (x_{1} , , x_{2} , x_{3}) \, \Psi^{(1)} (x_{1} , , x_{2} , x_{3}) \\ \hline z_{1} = \tfrac{1}{\sqrt{2}} (x_{1} - x_{2}) \; , \; & z_{2} = \tfrac{1}{\sqrt{6}} (x_{1} + x_{2} - 2 x_{3}) \\ z_{3} = \tfrac{1}{\sqrt{3}} (x_{1} + x_{2} + x_{3}) & = \sqrt{3} \, X_{c.m.} \;\rightarrow\; 0 \\ z_{j} = \lambda \, \overline{z}_{j} \; , \; x_{j} = \lambda \, \overline{x}_{j} \; ; \; & j = 1,2,3 \end{aligned} \tag{43}$$

The normalizing factor $(3)^{3/2}$ in the integral in eq. 18 and 43 , which were missing in earlier versions, are now included in order to generate a conventionally normalized 6 dimensional L_{2} space , which we do next step by step .
The scalar product in eq. 43 becomes

$$\begin{aligned} \left\langle \Psi^{(2)} \,\middle|\, \Psi^{(1)} \right\rangle \;=\; & \lambda^{6} \int \Pi_{i=1}^{3} d^{3} \overline{\xi}_{i} \, \delta^{3} \left(\overline{\xi}_{3} \right) \times \\ & \times \Psi^{*\,(2)} (\overline{x}_{1} , , \overline{x}_{2} , \overline{x}_{3}) \, \Psi^{(1)} (\overline{x}_{1} , \overline{x}_{2} , \overline{x}_{3}) \\ \hline \overline{\xi}_{1} = \tfrac{1}{\sqrt{2}} (\overline{x}_{1} - \overline{x}_{2}) \; , \; & \overline{\xi}_{2} = \tfrac{1}{\sqrt{6}} (\overline{x}_{1} + \overline{x}_{2} - 2 \overline{x}_{3}) \\ \overline{\xi}_{3} = \tfrac{1}{\sqrt{3}} (\overline{x}_{1} + \overline{x}_{2} + \overline{x}_{3}) & = \sqrt{3} \, \overline{X}_{c.m.} \;\rightarrow\; 0 \\ z_{j} = \lambda \, \overline{z}_{j} \; , \; x_{j} = \lambda \, \overline{x}_{j} \; ; \; j = 1,2,3 \; ; \; & X_{c.m.} = \lambda \, \overline{X}_{c.m.} = \tfrac{1}{3} \textstyle\sum_{i=1}^{3} \overline{x}_{i} \end{aligned} \tag{44}$$

In order to eliminate the scale factor λ we redefine the wave functions $\Psi^{(1),(2)}$ in eq. 44

$$\psi^{(1),(2)} = \lambda^{3} \, \Psi^{(1),(2)} \tag{45}$$

Using the dimensionless wave functions

$$\psi^{(1),(2)} (\overline{x}_{1} , \overline{x}_{2} , \overline{x}_{3}) \tag{46}$$

defined in eq. 45 the scalar product (eq. 44) becomes

$$\begin{aligned} \left\langle \psi^{(2)} \,\middle|\, \psi^{(1)} \right\rangle \;=\; & \int \Pi_{i=1}^{3} d^{3} \overline{\xi}_{i} \, \delta^{3} \left(\overline{\xi}_{3} \right) \times \\ & \times \psi^{*\,(2)} (\overline{x}_{1} , , \overline{x}_{2} , \overline{x}_{3}) \, \psi^{(1)} (\overline{x}_{1} , \overline{x}_{2} , \overline{x}_{3}) \end{aligned} \tag{47}$$

3.4. The barycentric 6 spatial oscillatory variables and their symmetries with respect to the 3 quark positions in dimensionless universal variables

The central properties under the 3 quark position permutation group S_3 can perfectly be discussed according to the dimensionless variables $\overline{x}_i \; ; \; i = 1,2,3$

$$\pi \begin{pmatrix} 1 & 2 & 3 \\ i_1 & i_2 & i_3 \end{pmatrix} \longrightarrow \begin{pmatrix} \overline{x}_1 & \overline{x}_2 & \overline{x}_3 \\ \overline{x}_{i_1} & \overline{x}_{i_2} & \overline{x}_{i_3} \end{pmatrix} \tag{48}$$

To this end we invert the linear relations in eq. 44

$$\begin{aligned} \overline{x}_1 &= \frac{1}{\sqrt{2}} \overline{\xi}_1 + \frac{1}{\sqrt{6}} \overline{\xi}_2 \\ \overline{x}_2 &= -\frac{1}{\sqrt{2}} \overline{\xi}_1 + \frac{1}{\sqrt{6}} \overline{\xi}_2 \\ \overline{x}_3 &= \qquad\qquad -\frac{2}{\sqrt{6}} \overline{\xi}_2 = \frac{1}{3} \left(2 \overline{x}_3 - \overline{x}_1 - \overline{x}_2 \right) \; (\surd) \end{aligned} \tag{49}$$

The relation on the last line of eq. 49 takes into account the vanishing of $\sum_{i=1}^{3} \overline{x}_i$ For completeness we give both barycentric variable transformations alongside (eqs. 44 and 49)

$$\begin{aligned} &\overline{\xi}_1 = \frac{1}{\sqrt{2}} \left(\overline{x}_1 - \overline{x}_2 \right) \;,\; \overline{\xi}_2 = \frac{1}{\sqrt{6}} \left(\overline{x}_1 + \overline{x}_2 - 2 \overline{x}_3 \right) \\ &\overline{\xi}_3 = \frac{1}{\sqrt{3}} \left(\overline{x}_1 + \overline{x}_2 + \overline{x}_3 \right) = \sqrt{3}\, \overline{X}_{c.m.} \rightarrow 0 \\ &z_j = \lambda \overline{z}_j \;,\; x_j = \lambda \overline{x}_j \;;\; j = 1,2,3 \;;\; X_{c.m.} = \lambda \overline{X}_{c.m.} = \frac{1}{3} \sum_{i=1}^{3} \overline{x}_i \end{aligned}$$

$$\begin{aligned} \overline{x}_1 &= \frac{1}{\sqrt{2}} \overline{\xi}_1 + \frac{1}{\sqrt{6}} \overline{\xi}_2 \\ \overline{x}_2 &= -\frac{1}{\sqrt{2}} \overline{\xi}_1 + \frac{1}{\sqrt{6}} \overline{\xi}_2 \\ \overline{x}_3 &= \qquad\qquad -\frac{2}{\sqrt{6}} \overline{\xi}_2 \end{aligned} \tag{50}$$

3.5. The barycentric coordinates dimension by dimension : cartesian and skew hexagonal coordinates as appropriate for 3 quark position variables

We shall sketch the hexagonal versus cartesion coordinate association simplifying first to one space dimesnion and two oscillator variables $(\xi_2 \,,\, \xi_1)$ in their two dimensional representation in figure 4 below

3.6. Modes of a pair of onedimensional oscillators – pairmodes and the complex plane

The even dimension already for just 1 space dimension : 2 , of the barycentric *relative* coordinates with vanishing values for the c.m. coordinate(s) derive from the

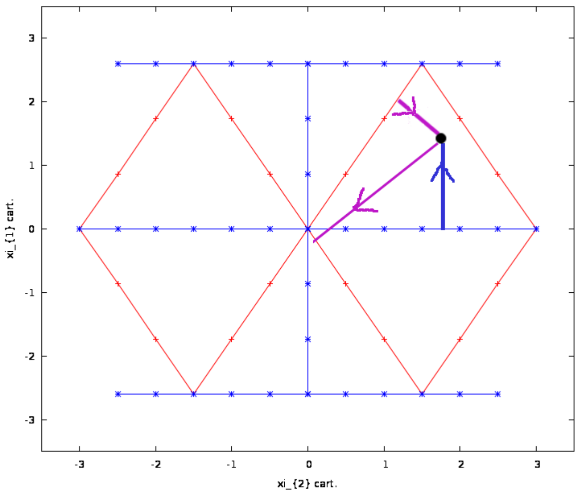

Fig 4 : The hexagonal logic in the (ξ_2 , ξ_1) plane $\longleftrightarrow$

3 base positions of the quarks bound in baryons, as discussed here in subsections 2-4 , and all subsections of section 3 .
The pairing mode allows to reveal explicity the hidden SU2 symmetry

$$\begin{array}{l} \zeta \ = \ \frac{1}{\sqrt{2}} \ (\ x \ + \ i \ y \ ) \ ; \ \ x = \xi_2 , y = \xi_1 \\ a = \frac{1}{\sqrt{2}} \ \left(\partial_\zeta \ + \ \overline{\zeta} \right) \ ; \ b = \frac{1}{\sqrt{2}} \ \left(\partial_{\overline{\zeta}} \ + \ \zeta \right) \\ \hline [\ a \ , \ b \ ] \ = \ 0 \end{array} \tag{51}$$

In eq. 51 a and b are two independent (commuting) , bosonic , absorption oscillator operators acting from the left on a wave function $\psi \ (\ \zeta \ , \ \overline{\zeta} \ )$. They can also be expressed in the real variables x and y as defined in eq. 51

$$\begin{array}{l} x \ = \ \frac{1}{\sqrt{2}} \ (\ \zeta \ + \ \overline{\zeta} \ ) \qquad ; y \ = \ - \ i \ \frac{1}{\sqrt{2}} \ (\ \zeta \ - \ \overline{\zeta} \ ) \\ \partial_{\ \zeta} \ = \ \frac{1}{\sqrt{2}} (\ \partial_{\ x} - i \ \partial_{\ y} \ ) ; \partial_{\ \overline{\zeta}} \ = \ \frac{1}{\sqrt{2}} \ (\ \partial_{\ x} \ + \ i \ \partial_{\ y} \ ) \end{array} \tag{52}$$

Thus we arrive at the $(\ x \ , \ y \ )$ representation of the paired oscillators $(\ a \ , \ b \ ) \equiv (\ a_{\ 1} \ , \ a_{\ 2} \ )$. We do this assembling the parts of $a_{\ 1,2}$ as defined in eq. 51 in the table-equations below

$\frac{1}{\sqrt{2}}\zeta = \frac{1}{2}(x + i\,y)$ $\frac{1}{\sqrt{2}}\partial_{\overline{\zeta}} = \frac{1}{2}(\partial_x + i\,\partial_y)$	$\frac{1}{\sqrt{2}}\overline{\zeta} = \frac{1}{2}(x - i\,y)$ $\frac{1}{\sqrt{2}}\partial_{\zeta} = \frac{1}{2}(\partial_x - i\,\partial_y)$
$\frac{1}{\sqrt{2}}\left(\zeta + \partial_{\overline{\zeta}}\right) = \frac{1}{2}\begin{bmatrix} x + \partial_x + \\ + i\,(+ \partial_y) \end{bmatrix}$	$\frac{1}{\sqrt{2}}\left(\overline{\zeta} + \partial_{\zeta}\right) = \frac{1}{2}\begin{bmatrix} x + \partial_x - \\ - i\,(y + \partial_y) \end{bmatrix}$

(53)

Further it follows for the adjoint operators from eq. 53

$$\begin{array}{l} a = \frac{1}{\sqrt{2}}\left(\zeta + \partial_{\overline{\zeta}}\right) \;;\; a^{\dagger} = \frac{1}{2}\begin{bmatrix} x - \partial_x - \\ - i\,(y - \partial_y) \end{bmatrix} = \frac{1}{\sqrt{2}}\left(\overline{\zeta} - \partial_{\zeta}\right) \\ b = \frac{1}{\sqrt{2}}\left(\overline{\zeta} + \partial_{\zeta}\right) \;;\; b^{\dagger} = \frac{1}{2}\begin{bmatrix} x - \partial_x + \\ + i\,(y - \partial_y) \end{bmatrix} = \frac{1}{\sqrt{2}}\left(\zeta - \partial_{\overline{\zeta}}\right) \end{array} \tag{54}$$

The polynomial basis of normalized wave function associated with the two paired absorption oscillators (a_1 , a_2) follows from the associated construction of the creation oscillators $a^{\dagger}_{1,2}$ in eq. 54

$$\begin{array}{l} \psi_{n_1 , n_2}\left(\zeta , \overline{\zeta} \right) = \mathcal{N}\, 2^{-\frac{1}{2}(n_1 + n_2)} \left(\overline{\zeta} - \partial_{\zeta}\right)^{n_1} \left(\zeta - \partial_{\overline{\zeta}}\right)^{n_2} \exp\left(- \zeta\,\overline{\zeta} \right) \\ \zeta\,\overline{\zeta} = \frac{1}{2}\left(x^2 + y^2 \right) \end{array} \tag{55}$$

In eq. 55 $\mathcal{N}$ denotes the normalization constant of the ground state with $N = 0$

$$\begin{array}{l} \mathcal{N}^{-2} = (n_1 !) (n_2 !) \int \frac{1}{2} \left| d\zeta \wedge d\overline{\zeta} \right| \exp\left(- 2\,\zeta\,\overline{\zeta} \right) \\ \frac{1}{2}\, d\zeta \wedge d\overline{\zeta} == \frac{1}{4} (dx + i\,dy) \wedge (dx - i\,dy) = \frac{1}{2\,i} (dx \wedge dy) \\ \frac{1}{2} \left| d\zeta \wedge d\overline{\zeta} \right| = d\zeta\, d\overline{\zeta} = dx\, dy \\ \hline \mathcal{N}^{-2} = (n_1 !) (n_2 !) \int dx\, dy\, \exp\left(- x^2 - y^2 \right) = \\ \quad = (n_1 !) (n_2 !)\, \pi \int_0^{\infty} d\varrho\, e^{-\varrho} = (n_1 !) (n_2 !)\, \pi \end{array} \tag{56}$$

Finally we come to the paired oscillator mode orthogonal polynomials , beeing not Hermite polynomials , which prevail for unpaired modes, but simple monomials. This structure is derived from substituting the two expressions for the creation operators $a^{\dagger}_{1,2}$: $\left(\overline{\zeta} - \partial_{\zeta} \right)^{n_1}$ and $\left(\zeta - \partial_{\overline{\zeta}}\right)^{n_2}$ in eq. 55 , as combined operators inside the powers , acting on the left on the given paired mode ground state

$$\begin{array}{lllll} \sqrt{2}\, a_{1}^{\dagger} & = \overline{\zeta} - \partial_{\zeta} & = \exp\left(\zeta\,\overline{\zeta}\right)\left(-\partial_{\zeta}\right)\exp\left(-\zeta\,\overline{\zeta}\right) \\ \sqrt{2}\, a_{2}^{\dagger} & = \zeta - \partial_{\overline{\zeta}} & = \exp\left(\zeta\,\overline{\zeta}\right)\left(-\partial_{\overline{\zeta}}\right)\exp\left(-\zeta\,\overline{\zeta}\right) \end{array} \tag{57}$$

The expression for the paired wave function $\psi_{n_1, n_2}\left(\zeta, \overline{\zeta}\right)$ in eq. 55 then takes the form

$$\begin{array}{l} \psi_{n_1, n_2}\left(\zeta, \overline{\zeta}\right) = \\ \quad = \mathcal{N}\, 2^{-\frac{1}{2}\left(n_1 + n_2\right)} \exp\left(\zeta\,\overline{\zeta}\right)\left(-\partial_{\zeta}\right)^{n_1}\left(-\partial_{\overline{\zeta}}\right)^{n_2} \exp\left(-2\,\zeta\,\overline{\zeta}\right) \\ \quad = \mathcal{N}\, 2^{\frac{1}{2}\left(n_1 + n_2\right)}\, \overline{\zeta}^{\,n_1}\, \zeta^{\,n_2} \exp\left(-\zeta\,\overline{\zeta}\right) \end{array} \tag{58}$$

We use polar coordinates , as they are representing finite rotations of the complex ζ - plane, leading to the wave function structure

$$\begin{array}{l} \psi_{n_1, n_2}\left(\zeta, \overline{\zeta}\right) = \\ = \left(\dfrac{2^{\left(n_1 + n_2\right)}}{\pi\left(n_1!\right)\left(n_2!\right)}\right)^{\frac{1}{2}} \exp\left(i\left(n_2 - n_1\right)\varphi\right)\left[\varrho^{\left(n_1 + n_2\right)} \exp\left(-\varrho^{2}\right)\right] \\ \hline \varrho = \left|\zeta\right| \; ; \; \varphi = arg\left(\zeta\right) \end{array} \tag{59}$$

The functions ψ_{n_1, n_2} in eq. 59 form a complete basis in the space $L_2\left(\zeta_1, \zeta_2\right)$. They are combined with the restrictions from overall Fermi statistics – including an overall color antisymmetric selection rule in conjunction with the three Young tableaux as displayed in eq. 32 .
Thus we study the action of the symmetric group S_3 on these base functions , as defined in eq. 48 .

$$\begin{array}{l} \left(U\left[\pi\begin{pmatrix} 1 & 2 & 3 \\ i_1 & i_2 & i_3 \end{pmatrix}\right]\psi_{n_1, n_2}\right)\left(\zeta, \overline{\zeta}\right) = \\ \quad = \psi_{n_1, n_2}\left(\left[\pi\begin{pmatrix} 1 & 2 & 3 \\ i_1 & i_2 & i_3 \end{pmatrix}\right]^{-1}\left(\zeta, \overline{\zeta}\right)\right) \end{array} \tag{60}$$

Eq. 60 needs to be elaborated, as follows

$$\left[\pi\begin{pmatrix} 1 & 2 & 3 \\ i_1 & i_2 & i_3 \end{pmatrix}\right]^{-1} = \pi\begin{pmatrix} i_1 & i_2 & i_3 \\ 1 & 2 & 3 \end{pmatrix} \tag{61}$$

Next we identify the subgroup of even permutations – $A_3 = Z_3$ – of S_3

$$\begin{aligned} &Z \quad = \pi \begin{pmatrix} 1\,2\,3 \\ 3\,1\,2 \end{pmatrix} \text{ or } \pi \begin{pmatrix} 1\,2\,3 \\ 2\,3\,1 \end{pmatrix} \\ &Z^{2} = \pi \begin{pmatrix} 1\,2\,3 \\ 2\,3\,1 \end{pmatrix} \text{ or } \pi \begin{pmatrix} 1\,2\,3 \\ 3\,1\,2 \end{pmatrix} \quad ; \quad Z^{3} = \P \\ &\rightarrow \; Z^{-1} = Z^{2} \; ; \; Z^{-2} = Z \end{aligned} \tag{62}$$

Here is the place to emphasize that the discussion within subsection 3-1-3d is for the time beeing restricted to oscillatory modes in *one* space dimension, to be generalized to three subsequently, but after finishing the selection rules staying with 1 space dimension for the time beeing.
We proceed to identify the permutation Z as defined in eq. 62 with a rotation of the ζ plane by 120 degrees , completing the action of S_3 displayed in eq. 60

$$\begin{aligned} &\left(U \left[\pi \begin{pmatrix} 1 & 2 & 3 \\ i_1\, i_2\, i_3 \end{pmatrix} \right] \psi_{n_1, n_2} \right) (\zeta , \overline{\zeta}) = \\ &= D_{m_1 m_2\, n_1 n_2} (\pi (\, . \,)) \, \psi_{m_1, m_2} (\zeta , \overline{\zeta}) \end{aligned} \tag{63}$$

and for the abelian cyclic subgroup A_3 of even permutations eq. 63 becomes

$$\begin{aligned} &\left(U \left[\pi \begin{pmatrix} 1\,2\,3 \\ 3\,1\,2 \end{pmatrix} \right] \psi_{n_1, n_2} \right) (\zeta , \overline{\zeta}) = \\ &= D_{n_1 n_2} (Z) \, \psi_{n_1, n_2} (\zeta , \overline{\zeta}) = \psi_{n_1, n_2} (Z^{-1} \zeta , Z \overline{\zeta}) \end{aligned} \tag{64}$$

Eqs. 59 and 64 imply

$$\begin{aligned} &D_{n_1 n_2} (Z) \, \psi_{n_1, n_2} (\zeta , \overline{\zeta}) = \\ &= \exp (i (n_1 - n_2) (2 \pi / 3)) \, \psi_{n_1, n_2} (\zeta , \overline{\zeta}) = \\ &= \psi_{n_1, n_2} (Z^{-1} \zeta , Z \overline{\zeta}) \end{aligned} \tag{65}$$

It follows from eq. 65

$$D_{n_1 n_2} (Z) = Z^{n_1 - n_2} \; , \; Z = \exp (i (2 \pi / 3)) \tag{66}$$

Next we decompose the action of Z on ζ into real and imaginary parts

$$\begin{aligned} &\zeta \longrightarrow Z \zeta = \zeta^{\prime} \; : \\ &\zeta = \begin{pmatrix} \overline{\xi}_2 \\ \overline{\xi}_1 \end{pmatrix} \; , \; \zeta^{\prime} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \overline{\xi}_2 \\ \overline{\xi}_1 \end{pmatrix} \longrightarrow \\ &\overline{\xi}^{\prime}_2 = -\frac{1}{2} \overline{\xi}_2 - \frac{\sqrt{3}}{2} \overline{\xi}_1 \\ &\overline{\xi}^{\prime}_1 = \frac{\sqrt{3}}{2} \overline{\xi}_2 - \frac{1}{2} \overline{\xi}_1 \end{aligned} \tag{67}$$

We recall eq. 50 , repeating it below

$$\begin{array}{l} \overline{\xi}_{1} = \frac{1}{\sqrt{2}} \left(\overline{x}_{1} - \overline{x}_{2} \right) \ , \ \overline{\xi}_{2} = \frac{1}{\sqrt{6}} \left(\overline{x}_{1} + \overline{x}_{2} - 2\,\overline{x}_{3} \right) \\ \overline{\xi}_{3} = \frac{1}{\sqrt{3}} \left(\overline{x}_{1} + \overline{x}_{2} + \overline{x}_{3} \right) = \sqrt{3}\ \overline{X}_{c.m.} \ \rightarrow \ 0 \end{array} \tag{68}$$

Substituting the expressions on the first line in eq. 68 in eq. 67 we obtain

$$\begin{array}{l} \overline{\xi}_{2}^{\ \prime} = -\frac{1}{2\sqrt{6}} \left(\overline{x}_{1} + \overline{x}_{2} - 2\,\overline{x}_{3} \right) - \frac{\sqrt{3}}{2\sqrt{2}} \left(\overline{x}_{1} - \overline{x}_{2} \right) \\ \overline{\xi}_{1}^{\ \prime} = \frac{\sqrt{3}}{2\sqrt{6}} \left(\overline{x}_{1} + \overline{x}_{2} - 2\,\overline{x}_{3} \right) - \frac{1}{2\sqrt{2}} \left(\overline{x}_{1} - \overline{x}_{2} \right) \end{array} \tag{69}$$

and arranging the factors yielding the result

$$\begin{array}{l} \overline{\xi}_{2}^{\ \prime} = -\frac{1}{2\sqrt{6}} \left[\left(\overline{x}_{1} + \overline{x}_{2} - 2\,\overline{x}_{3} \right) + 3 \left(\overline{x}_{1} - \overline{x}_{2} \right) \right] \\ \overline{\xi}_{1}^{\ \prime} = \frac{1}{2\sqrt{2}} \left[\left(\overline{x}_{1} + \overline{x}_{2} - 2\,\overline{x}_{3} \right) - \left(\overline{x}_{1} - \overline{x}_{2} \right) \right] \end{array} \tag{70}$$

The final result is compared with the initial choice of barycentric variables in eq. 68

$$\begin{array}{ll} \overline{\xi}_{2}^{\ \prime} = \frac{1}{\sqrt{6}} \left(\overline{x}_{2} + \overline{x}_{3} - 2\,\overline{x}_{1} \right) & \leftarrow \overline{\xi}_{2} = \frac{1}{\sqrt{6}} \left(\overline{x}_{1} + \overline{x}_{2} - 2\,\overline{x}_{3} \right) \\ \overline{\xi}_{1}^{\ \prime} = \frac{1}{\sqrt{2}} \left(\overline{x}_{2} - \overline{x}_{3} \right) & \leftarrow \overline{\xi}_{1} = \frac{1}{\sqrt{2}} \left(\overline{x}_{1} - \overline{x}_{2} \right) \end{array} \tag{71}$$

The choice , marked by 'or' in eq. 62, is revealed inspecting the substitution of the $\overline{\xi}_{j}$ indices from the right hand - to the left hand side of eq. 71 , corresponding to the cyclic permutation associated with the actions of Z and $Z^{-1} \equiv Z^{2}$

$$\begin{array}{l} Z \ \ : \zeta \longrightarrow Z\,\zeta \ \simeq \pi \begin{pmatrix} 1\,2\,3 \\ 2\,3\,1 \end{pmatrix} \\ Z^{2} : \zeta \longrightarrow Z^{2}\,\zeta \simeq \pi \begin{pmatrix} 1\,2\,3 \\ 3\,1\,1 \end{pmatrix} \end{array} \tag{72}$$

3.7. Reconstruction of the two-dimensional irreducible unitary representation of S_{3} from 1 spacelike dimension

The actions of Z and $Z^{-1} \equiv Z^{2}$, defined in eq. 72 allows to assoiate two 2×2 representation matrices with the corresponding permutations , using eq. 67

$$\begin{array}{l} 2)\ \pi \begin{pmatrix} 1\,2\,3 \\ 2\,3\,1 \end{pmatrix} \longrightarrow \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} + \rightarrow Z \\ 3)\ \pi \begin{pmatrix} 1\,2\,3 \\ 3\,1\,2 \end{pmatrix} \longrightarrow \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} + \rightarrow Z^{2} \equiv Z^{-1} \\ 4)\ \pi \begin{pmatrix} 1\,2\,3 \\ 2\,1\,3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ - \rightarrow T_{12} \end{array} \tag{73}$$

In eq. 73 the last column denotes a shorthand name for the 3 constructed elements of S_3 , while the signs in the second last column are $+$ for even and $-$ for odd permutations respectively , equal to the determinant of the 2×2 representation matrices .
The assignment of the barycentric variables to the numbering of the associated – 1-dimensional – coordinates $\overline{x}_{1,2,3}$ as given in eq. 71i, singles out the representation of the transposition $T_{12} \simeq 1 \leftrightarrow 2$ as beeing associated with the diagonal Pauli matrix σ_3 .
The remaining 3 elements including the identity permutation , denoted ¶ , also for its representing 2×2 unit matrix , can be found from multiplications of the 3 elements defined in eq. 73

$$\begin{array}{l} \pi \begin{pmatrix} 1\,2\,3 \\ 1\,2\,3 \end{pmatrix} \longrightarrow \P = Z^{3} = \left(Z^{-1} \right)^{3} = \left(T_{12} \right)^{2} \\ 1)\ \pi \begin{pmatrix} 1\,2\,3 \\ 1\,2\,3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1\ 0 \\ 0\ 1 \end{pmatrix} + \rightarrow \P \end{array} \tag{74}$$

The numbering 2) - 4) in eq. 73 and 1) in eq. 74 serves to segregate the subgroup of even permutations – $A_3 \simeq Z_3$ – corresponding to the entries numbered 1) , 2) , 3) from the odd ones , to be completed next .
This we do step by step . First we determine the product , using the symbol $\circ$: 2) $\circ$ 4)

$$2) \circ 4) = \pi \begin{pmatrix} 1\,2\,3 \\ 2\,3\,1 \end{pmatrix} \circ \pi \begin{pmatrix} 1\,2\,3 \\ 2\,1\,3 \end{pmatrix} = \pi \begin{pmatrix} 1\,2\,3 \\ 1\,3\,2 \end{pmatrix} \rightarrow T_{23} \tag{75}$$

Thus we multiply the 2×2 matrices associated with the elements 2) and 4) in eq. 73

$$\begin{array}{l} 2) \rightarrow \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \ ; \ 4) \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \longrightarrow \\ 2) \circ 4) \rightarrow \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \end{array} \tag{76}$$

and check compatibility of eqs. 75 and 76 implied by the 2-dimensional unitary representation of S_3.
This is done in an analogous way to the substitutions relative to A_3 applied in

eqs.71 and 72

$$\begin{aligned} &\zeta \to \overline{\xi}_2 + i\,\overline{\xi}_1 \to \begin{pmatrix} \overline{\xi}_2 \\ \overline{\xi}_1 \end{pmatrix} \;;\; 2) \circ 4) : \zeta'' \to \begin{pmatrix} \overline{\xi}''_2 \\ \overline{\xi}''_1 \end{pmatrix} \\ &\begin{pmatrix} \overline{\xi}''_2 \\ \overline{\xi}''_1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \overline{\xi}_2 \\ \overline{\xi}_1 \end{pmatrix} \end{aligned} \tag{77}$$

with the identifications (eqs. 67 , 70)

$$\begin{aligned} &\overline{\xi}_1 = \frac{1}{\sqrt{2}} (\overline{x}_1 - \overline{x}_2) \;,\; \overline{\xi}_2 = \frac{1}{\sqrt{6}} (\overline{x}_1 + \overline{x}_2 - 2\,\overline{x}_3) \\ &\overline{\xi}_3 = \frac{1}{\sqrt{3}} (\overline{x}_1 + \overline{x}_2 + \overline{x}_3) = \sqrt{3}\,\overline{X}_{c.m.} \to 0 \\ &\text{and} \quad \overline{\xi}_j \to \overline{\xi}''_j \;;\; \overline{x}_j \to \overline{x}''_j \;;\; j = 1,2,3 \end{aligned} \tag{78}$$

Then eq. 77 becomes

$$\begin{aligned} &\overline{\xi}''_2 = -\frac{1}{2}\overline{\xi}_2 + \frac{\sqrt{3}}{2}\overline{\xi}_1 \;;\; \overline{\xi}''_1 = \frac{\sqrt{3}}{2}\overline{\xi}_2 + \frac{1}{2}\overline{\xi}_1 \longrightarrow \\ &\overline{\xi}''_2 = -\frac{1}{2\sqrt{6}} (\overline{x}_1 + \overline{x}_2 - 2\,\overline{x}_3) + \frac{\sqrt{3}}{2\sqrt{2}} (\overline{x}_1 - \overline{x}_2) \\ &\overline{\xi}''_1 = +\frac{\sqrt{3}}{2\sqrt{6}} (\overline{x}_1 + \overline{x}_2 - 2\,\overline{x}_3) + \frac{1}{2\sqrt{2}} (\overline{x}_1 - \overline{x}_2) \end{aligned} \tag{79}$$

Rendering the factors commensurable in the last 2 lines of eq. 79, we obtain

$$\begin{aligned} \overline{\xi}''_2 &= \frac{1}{2\sqrt{6}} (2\,\overline{x}_3 - (\overline{x}_1 + \overline{x}_2) + 3 (\overline{x}_1 - \overline{x}_2)) \\ &= \frac{1}{\sqrt{6}} (\overline{x}_3 + \overline{x}_1 - 2\,\overline{x}_2) \\ \overline{\xi}''_1 &= \frac{1}{2\sqrt{2}} ((\overline{x}_1 + \overline{x}_2 - 2\,\overline{x}_3) + (\overline{x}_1 - \overline{x}_2)) \\ &= \frac{1}{\sqrt{2}} (\overline{x}_1 - \overline{x}_3) \end{aligned} \tag{80}$$

It remains to substitute the variables $\overline{\xi}''_{1,2}$ on the left hand side of eq. 80 to identify the associated permutation

$$\begin{aligned} \overline{\xi}''_2 &= \frac{1}{\sqrt{6}} \left(\overline{x}''_1 + \overline{x}''_2 - 2\,\overline{x}''_3\right) = \frac{1}{\sqrt{6}} (\overline{x}_1 + \overline{x}_3 - 2\,\overline{x}_2) \\ \overline{\xi}''_1 &= \frac{1}{\sqrt{2}} \left(\overline{x}''_1 - \overline{x}''_2\right) \qquad\qquad = \frac{1}{\sqrt{2}} (\overline{x}_1 - \overline{x}_3) \end{aligned} \tag{81}$$

which follows from the ordering of the indices of the barycentric variables $\overline{x}_j \,;\, j = 1,2,3$ appering on the rightmost side of eq. 81 as

$$2) \circ 4) \to \pi \begin{pmatrix} 1\,2\,3 \\ 1\,3\,2 \end{pmatrix} \to T_{23} \quad (\surd) \tag{82}$$

Hence we can consistently identify the second odd permutaion , which becomes the fifth constructed permutation according to the definition $2) \circ 4) = 5)$ in accordance with eqs. 76 and 82

$$5)\ \pi \begin{pmatrix} 1\,2\,3 \\ 1\,3\,2 \end{pmatrix} \longrightarrow \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} - \to T_{23} \tag{83}$$

It remains to construct the third odd permutation and its asociated 2×2 unitary representation matrix .

We choose among several paths to consider the multiplication $3) \circ 4)$ closely follow the previous multiplication $2) \circ 4)$ in eq. 75

$$3) \circ 4) = \pi \begin{pmatrix} 1\,2\,3 \\ 3\,1\,2 \end{pmatrix} \circ \pi \begin{pmatrix} 1\,2\,3 \\ 2\,1\,3 \end{pmatrix} = \pi \begin{pmatrix} 1\,2\,3 \\ 3\,2\,1 \end{pmatrix} \to T_{13} \tag{84}$$

The resulting remaining odd permutation T_{13} as result of the $3) \circ 4)$ multiplication is not surprising .

Thus we multiply the 2×2 matrices associated with the elements 3) and 4) analogous to eq. 76 for $2) \circ 4)$

$$\begin{array}{l} 3) \to \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \;;\; 4) \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \longrightarrow \\ 3) \circ 4) \to \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \end{array} \tag{85}$$

The analog for $3) \circ 4)$ to eq. 77 for $2) \circ 4)$ becomes

$$\begin{array}{l} \zeta \to \overline{\xi}_2 + i\,\overline{\xi}_1 \to \begin{pmatrix} \overline{\xi}_2 \\ \overline{\xi}_1 \end{pmatrix} \;;\; 3) \circ 4) : \zeta^{'''} \to \begin{pmatrix} \overline{\xi}_2^{'''} \\ \overline{\xi}_1^{'''} \end{pmatrix} \\ \begin{pmatrix} \overline{\xi}_2^{'''} \\ \overline{\xi}_1^{'''} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \overline{\xi}_2 \\ \overline{\xi}_1 \end{pmatrix} \end{array} \tag{86}$$

Next we adapt eqs. 78 - 80 relative to $2) \circ 4)$ to $3) \circ 4)$, which yields

$$\begin{array}{l} \overline{\xi}_1 = \frac{1}{\sqrt{2}} \left(\overline{x}_1 - \overline{x}_2 \right) \;,\; \overline{\xi}_2 = \frac{1}{\sqrt{6}} \left(\overline{x}_1 + \overline{x}_2 - 2\,\overline{x}_3 \right) \\ \overline{\xi}_3 = \frac{1}{\sqrt{3}} \left(\overline{x}_1 + \overline{x}_2 + \overline{x}_3 \right) = \sqrt{3}\,\overline{X}_{c.m.} \to 0 \\ \text{and} \quad \overline{\xi}_j \to \overline{\xi}_j^{'''} \;;\; \overline{x}_j \to \overline{x}_j^{'''} \;;\; j = 1,2,3 \end{array} \tag{87}$$

as well as

$$\begin{array}{l} \overline{\xi}_2^{'''} = -\frac{1}{2}\,\overline{\xi}_2 - \frac{\sqrt{3}}{2}\,\overline{\xi}_1 \;;\; \overline{\xi}_1^{'''} = -\frac{\sqrt{3}}{2}\,\overline{\xi}_2 + \frac{1}{2}\,\overline{\xi}_1 \longrightarrow \\ \overline{\xi}_2^{'''} = -\frac{1}{2\sqrt{6}} \left(\overline{x}_1 + \overline{x}_2 - 2\,\overline{x}_3 \right) - \frac{\sqrt{3}}{2\sqrt{2}} \left(\overline{x}_1 - \overline{x}_2 \right) \\ \overline{\xi}_1^{'''} = -\frac{\sqrt{3}}{2\sqrt{6}} \left(\overline{x}_1 + \overline{x}_2 - 2\,\overline{x}_3 \right) + \frac{1}{2\sqrt{2}} \left(\overline{x}_1 - \overline{x}_2 \right) \end{array} \tag{88}$$

The third equation (eq. 80) relative to $2) \circ 4)$ is replaced for $3) \circ 4)$ by

$$\begin{aligned} \overline{\xi}_{2}^{\,'''} &= \frac{1}{2\sqrt{6}} \left(2\,\overline{x}_{3} - (\overline{x}_{1} + \overline{x}_{2}) - 3\,(\overline{x}_{1} - \overline{x}_{2}) \right) \\ &= \frac{1}{\sqrt{6}} \left(\overline{x}_{3} + \overline{x}_{2} - 2\,\overline{x}_{1} \right) \\ \overline{\xi}_{1}^{\,'''} &= \frac{1}{2\sqrt{2}} \left((-\overline{x}_{1} - \overline{x}_{2} + 2\,\overline{x}_{3}) + (\overline{x}_{1} - \overline{x}_{2}) \right) \\ &= \frac{1}{\sqrt{2}} \left(\overline{x}_{3} - \overline{x}_{2} \right) \end{aligned} \tag{89}$$

Substituting the variables $\overline{\xi}_{1,2}^{\,'''}$ on the left hand side of eq. 89 we identify the permutation associated with $3) \circ 4)$, in analogy to eq. 81 for $2) \circ 4)$

$$\begin{aligned} \overline{\xi}_{2}^{\,'''} &= \frac{1}{\sqrt{6}} \left(\overline{x}_{1}^{\,'''} + \overline{x}_{2}^{\,'''} - 2\,\overline{x}_{3}^{\,'''} \right) = \frac{1}{\sqrt{6}} \left(\overline{x}_{3} + \overline{x}_{2} - 2\,\overline{x}_{1} \right) \\ \overline{\xi}_{1}^{\,'''} &= \frac{1}{\sqrt{2}} \left(\overline{x}_{1}^{\,'''} - \overline{x}_{2}^{\,'''} \right) \qquad\qquad = \frac{1}{\sqrt{2}} \left(\overline{x}_{3} - \overline{x}_{2} \right) \end{aligned} \tag{90}$$

which follows from the ordering of the indices of the barycentric variables $\overline{x}_{j}\,;\ j = 1,2,3$ appering on the rightmost side of eq. 90 in continuing the analogy tp eqs. 81 - 83 for $2) \circ 4) = 5)$

$$3) \circ 4) \ \rightarrow\ \pi \begin{pmatrix} 1\,2\,3 \\ 3\,2\,1 \end{pmatrix} \ \rightarrow\ T_{13} \quad (\surd) \tag{91}$$

Hence we can consistently identify the third odd permutation , which becomes the six'th and last constructed permutation according to the definition $3) \circ 4) = 6)$ in accordance with eqs. 85 and 91 analogous with eqs. 76 and 82 for the product $2) \circ 4) = 5)$

$$6)\ \pi \begin{pmatrix} 1\,2\,3 \\ 3\,2\,1 \end{pmatrix} \longrightarrow \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} - \rightarrow T_{13} \tag{92}$$

3.8. Results on the reconstruction of the two-dimensional irreducible unitary representation of S_3 from 1 spacelike dimension

In this subsection we collect the results first in the representation of the permutation group elements as in eqs. 73 (2) , 3) , 4)) , 74 (1)) , 83 (5)) , 92 (6)) , in the order 1) - 6) .

$$
\begin{array}{lll}
1)\ \pi \begin{pmatrix} 1\,2\,3 \\ 1\,2\,3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1\;0 \\ 0\;1 \end{pmatrix} & & + \rightarrow\ \P \\
2)\ \pi \begin{pmatrix} 1\,2\,3 \\ 2\,3\,1 \end{pmatrix} \longrightarrow \begin{pmatrix} -\ \frac{1}{2} & -\ \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\ \frac{1}{2} \end{pmatrix} & & + \rightarrow\ Z \\
3)\ \pi \begin{pmatrix} 1\,2\,3 \\ 3\,1\,2 \end{pmatrix} \longrightarrow \begin{pmatrix} -\ \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\ \frac{\sqrt{3}}{2} & -\ \frac{1}{2} \end{pmatrix} & & + \rightarrow\ Z^{\,2} \ \equiv\ Z^{\,-1} \\
4)\ \pi \begin{pmatrix} 1\,2\,3 \\ 2\,1\,3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -\ 1 \end{pmatrix} & & - \rightarrow\ T_{\,12} \\
5)\ \pi \begin{pmatrix} 1\,2\,3 \\ 1\,3\,2 \end{pmatrix} \longrightarrow \begin{pmatrix} -\ \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} & & - \rightarrow\ T_{\,23} \\
6)\ \pi \begin{pmatrix} 1\,2\,3 \\ 3\,2\,1 \end{pmatrix} \longrightarrow \begin{pmatrix} -\ \frac{1}{2} & -\ \frac{\sqrt{3}}{2} \\ -\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} & & - \rightarrow\ T_{\,13}
\end{array}
\tag{93}
$$

The main ingredients to the construction of irreducible representations of finite groups , here $S_{\,3}$, leading – in summary – to eq. 93 , are to my knowledge due to Issai Schur , as displayed also in ref. [26] .

3.9. Choosing a complex basis for transforming the basis derived in eq. 93 for the two-dimensional irreducible unitary representation of $S_{\,3}$ from 1 spacelike dimension

Here we invert the decomposition of the complex numbers ζ , $\overline{\zeta}$ into real and imaginary parts , as displayed (e.g.) in eqs. 67 and 77

$$
\begin{array}{l}
\zeta \ \longrightarrow\ Z\,\zeta \ =\ \zeta^{\,\prime}\ : \\
\zeta \ =\ \begin{pmatrix} \overline{\xi}_{\,2} \\ \overline{\xi}_{\,1} \end{pmatrix} \ ,\ \zeta^{\,\prime} \ =\ \begin{pmatrix} -\ \frac{1}{2} & -\ \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\ \frac{1}{2} \end{pmatrix} \begin{pmatrix} \overline{\xi}_{\,2} \\ \overline{\xi}_{\,1} \end{pmatrix} \ \longrightarrow \\
\overline{\xi}^{\,\prime}_{\,2} \ =\ -\ \frac{1}{2}\,\overline{\xi}_{\,2} \ -\ \frac{\sqrt{3}}{2}\,\overline{\xi}_{\,1} \\
\overline{\xi}^{\,\prime}_{\,1} \ =\ \frac{\sqrt{3}}{2}\,\overline{\xi}_{\,2} \ -\ \frac{1}{2}\,\overline{\xi}_{\,1}
\end{array}
\tag{94}
$$

repeated above and below

$$
\begin{array}{l}
\zeta \ \rightarrow\ \overline{\xi}_{\,2} \ +\ i\,\overline{\xi}_{\,1} \ \rightarrow\ \begin{pmatrix} \overline{\xi}_{\,2} \\ \overline{\xi}_{\,1} \end{pmatrix} \ ;\ 2)\ \circ\ 4)\ :\ \zeta^{\,\prime\prime} \ \rightarrow\ \begin{pmatrix} \overline{\xi}^{\,\prime\prime}_{\,2} \\ \overline{\xi}^{\,\prime\prime}_{\,1} \end{pmatrix} \\
\begin{pmatrix} \overline{\xi}^{\,\prime\prime}_{\,2} \\ \overline{\xi}^{\,\prime\prime}_{\,1} \end{pmatrix} \ =\ \begin{pmatrix} -\ \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \overline{\xi}_{\,2} \\ \overline{\xi}_{\,1} \end{pmatrix}
\end{array}
\tag{95}
$$

back to the original complex- and complex conjugate variables $\zeta \, , \, \overline{\zeta}$, as defined on the left hand side of the first relation in eq 95

$$\begin{array}{l} \begin{pmatrix} \overline{\xi}_{\,2} \\ \overline{\xi}_{\,1} \end{pmatrix} \rightarrow \begin{pmatrix} \zeta \; = \; \overline{\xi}_{\,2} \; + \; i \, \overline{\xi}_{\,1} \\ \overline{\zeta} \; = \; \overline{\xi}_{\,2} \; - \; i \, \overline{\xi}_{\,1} \end{pmatrix} \; = \; \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} \overline{\xi}_{\,2} \\ \overline{\xi}_{\,1} \end{pmatrix} \; = \; \mathcal{M} \begin{pmatrix} \overline{\xi}_{\,2} \\ \overline{\xi}_{\,1} \end{pmatrix} \\ \mathcal{M}^{\,\dagger} \; = \; \begin{pmatrix} 1 \, 1 \\ -i \;\, i \end{pmatrix} \longrightarrow \mathcal{M} \, \mathcal{M}^{\,\dagger} \; = \; \mathcal{M}^{\,\dagger} \, \mathcal{M} \; = \; 2 \; (\, \P \,)_{\,2 \, \times \, 2} \end{array} \tag{96}$$

Thus the unitary 2 x 2 matrix

$$u \; = \; \tfrac{1}{\sqrt{2}} \; \mathcal{M} \tag{97}$$

is a matrix which generates the similarity transformation
through the following steps , denoting by $D_{\,\pi} \; ; \; \pi \; \in \; S_{\,3}$ the six 2 x 2 unitary matrices in the basis given in eq. 93 and likewise by $d_{\,\pi} \; ; \; \pi \; \in \; S_{\,3}$ the six transformed 2 x 2 unitary matrices , associated with the basis as described in eq. 96

$$\begin{array}{l} \pi \; \rightarrow \; d_{\,\pi} \; : \; \begin{pmatrix} \zeta \\ \overline{\zeta} \end{pmatrix} \longrightarrow d_{\,\pi} \begin{pmatrix} \zeta \\ \overline{\zeta} \end{pmatrix} = d_{\,\pi} \, \mathcal{M} \begin{pmatrix} \overline{\xi}_{\,2} \\ \overline{\xi}_{\,1} \end{pmatrix} \\ \begin{pmatrix} \zeta \\ \overline{\zeta} \end{pmatrix} \; = \; \mathcal{M} \begin{pmatrix} \overline{\xi}_{\,2} \\ \overline{\xi}_{\,1} \end{pmatrix} \\ \hline \pi \; \rightarrow \; D_{\,\pi} \; : \; \begin{pmatrix} \overline{\xi}_{\,2} \\ \overline{\xi}_{\,1} \end{pmatrix} \; \longrightarrow \; D_{\,\pi} \begin{pmatrix} \overline{\xi}_{\,2} \\ \overline{\xi}_{\,1} \end{pmatrix} \end{array} \tag{98}$$

Next we multiply the last relation in eq. 98 by $\mathcal{M}$ from the left

$$\pi \; \rightarrow \; D_{\,\pi} \; : \; \begin{pmatrix} \zeta \\ \overline{\zeta} \end{pmatrix} \; \longrightarrow \; \mathcal{M} \; D_{\,\pi} \begin{pmatrix} \overline{\xi}_{\,2} \\ \overline{\xi}_{\,1} \end{pmatrix} \; = \; \mathcal{M} \; D_{\,\pi} \, \mathcal{M}^{\,-1} \begin{pmatrix} \zeta \\ \overline{\zeta} \end{pmatrix} \tag{99}$$

Comparing eq. 99 with the first relation in eq. 98 we obtain the sought similarity transformation

$$\begin{array}{l} d_{\,\pi} \; = \; \mathcal{M} \; D_{\,\pi} \, \mathcal{M}^{\,-1} \; = \; u \, D_{\,\pi} \, u^{\,-1} \\ \hline u \; = \; \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \; , \; u^{\,-1} \; = \; \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \, 1 \\ -i \;\, i \end{pmatrix} \end{array} \tag{100}$$

The detailed calculations of the matrix product associated with the sixth permutation representation matrices $d_{\,\pi}$ from the basis formed by $D_{\,\pi}$ given in eq.

93 are performed in Appendix 3 . The collection of 2 x 2 representation matrices $d_{\ \pi}$; $\pi \ = \ 1, \cdots, 6$ is displayed in eq. 101 below

$$
\begin{array}{llll}
1) \ d_{\ \pi=1} \ \pi & \begin{pmatrix} 1\,2\,3 \\ 1\,2\,3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1\ 0 \\ 0\ 1 \end{pmatrix} & & + \rightarrow \ \P \\
2) \ d_{\ \pi=2} \ \pi & \begin{pmatrix} 1\,2\,3 \\ 2\,3\,1 \end{pmatrix} \longrightarrow \begin{pmatrix} e^{\ +i(2\pi/3)} & 0 \\ 0 & e^{\ -i(2\pi/3)} \end{pmatrix} & & + \rightarrow \ Z \\
3) \ d_{\ \pi=3} \ \pi & \begin{pmatrix} 1\,2\,3 \\ 3\,1\,2 \end{pmatrix} \longrightarrow \begin{pmatrix} e^{\ -i(2\pi/3)} & 0 \\ 0 & e^{\ +i(2\pi/3)} \end{pmatrix} & & + \rightarrow Z^{\ 2} \ \equiv \ Z^{\ -1} \\
4) \ d_{\ \pi=4} \ \pi & \begin{pmatrix} 1\,2\,3 \\ 2\,1\,3 \end{pmatrix} \longrightarrow \begin{pmatrix} 0\ 1 \\ 1\ 0 \end{pmatrix} & & - \rightarrow \ T_{\ 12} \\
5) \ d_{\ \pi=5} \ \pi & \begin{pmatrix} 1\,2\,3 \\ 1\,3\,2 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & e^{\ +i(2\pi/3)} \\ e^{\ -i(2\pi/3)} & 0 \end{pmatrix} & & - \rightarrow \ T_{\ 23} \\
6) \ d_{\ \pi=6} \ \pi & \begin{pmatrix} 1\,2\,3 \\ 3\,2\,1 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & e^{\ -i(2\pi/3)} \\ e^{\ +i(2\pi/3)} & 0 \end{pmatrix} & & - \rightarrow \ T_{\ 13}
\end{array}
\qquad (101)
$$

3.10. Extending 1 spatial dimension to 3

We go back to the subsection on page 3-pmodes-1 :
Modes of a pair of onedimensional oscillators – pairmodes and the complex plane

which for 3 spatial dimension becomes ↘

Modes of *3 pairs* of onedimensional oscillator-pairmodes and the complex *3-plane*

The 1 complex variable ζ associated with the 1 pairmode as appropriate for 1 space dimension and defined in eq. 51 , for 3 space-time dimensions with axes (X) , (Y) , (Z) , thus becomes a complex three vector

$$\vec{\zeta} \ = \ \left(\ \zeta^{\ (X)} \ , \ \zeta^{\ (Y)} \ , \ \zeta^{\ (Z)} \ \right) \qquad (102)$$

The notation (X) , (Y) , (Z) for the three orthogonal axes of the 3-dimensional configuration space in the c.m. system is chosen in order to prevent confusing these with the 1-dimensional quantities denoted x ... , y ... , z ... , as introduced for 1 spatial dimension and defined in eqs. 50 and 51 , which become 3-vectors for 3 space dimensions .
The extension of the various space variables from 1 to 3 dimensions we shall do in segmented steps :

1-1) The center of mass position variables in 1 spatial dimension

These variables appear (last) in eq. 44 repeated below

$$\begin{aligned} \left\langle \Psi^{(2)} \,\middle|\, \Psi^{(1)} \right\rangle &= \lambda^6 \int \Pi_{i=1}^{3}\, d^3 \overline{\xi}_i\, \delta^3 \left(\overline{\xi}_3 \right) \times \\ &\quad \times \Psi^{*\,(2)} \left(\overline{x}_1, \overline{x}_2, \overline{x}_3 \right) \Psi^{(1)} \left(\overline{x}_1, \overline{x}_2, \overline{x}_3 \right) \end{aligned}$$

$$\begin{aligned} &\overline{\xi}_1 = \tfrac{1}{\sqrt{2}} \left(\overline{x}_1 - \overline{x}_2 \right) \ , \ \overline{\xi}_2 = \tfrac{1}{\sqrt{6}} \left(\overline{x}_1 + \overline{x}_2 - 2\,\overline{x}_3 \right) \\ &\overline{\xi}_3 = \tfrac{1}{\sqrt{3}} \left(\overline{x}_1 + \overline{x}_2 + \overline{x}_3 \right) = \sqrt{3}\, \overline{X}_{c.m.} \ \rightarrow 0 \\ &z_j = \lambda\, \overline{z}_j \ , \ x_j = \lambda\, \overline{x}_j \\ &j = 1,2,3 \ ; \ X_{c.m.} = \lambda\, \overline{X}_{c.m.} = \tfrac{1}{3} \sum_{i=1}^{3} \overline{x}_i \end{aligned} \tag{103}$$

1-3) Extension of the last three relations in eq. 103 to 3 spatial dimensions

The extension takes the form

$$\begin{aligned} &x_j \rightarrow \vec{x}_j \ \text{with}\ \vec{x}_j = \left(x_j^{(X)} , x_j^{(Y)} , x_j^{(Z)} \right) \ ; \ j = 1,2,3 \\ &X_{c.m.} \rightarrow \vec{X}_{c.m.} \ \text{with}\ \vec{X}_{c.m.} = \left(X_{c.m.}^{(X)} , X_{c.m.}^{(Y)} , X_{c.m.}^{(Z)} \right) = 0 \end{aligned} \tag{104}$$

Again note that $\vec{X}_{c.m.}$ and the axis superfix (X) denote very different objects .

The configuration space 3-vectors $\vec{x}_{1,2,3}$, $\vec{X}_{c.m.}$ in eq. 104 have dimension $\left[\text{mass}^{-1} \right]$ in rational units. They can be reduced to dimensionless configuration space variables , as given in eqs. 21 - 27 for 1 spatial dimesnsion and in the case of 3 spatial dimensions follows straightforwardly from eqs. 21 , 25 and 29

$$\begin{aligned} &\vec{\overline{x}}_j = \lambda^{-1}\, \vec{x}_j \ ; \ j = 1,2,3 \ ; \ \lambda^{-1} = \left(\frac{\Lambda}{K_{N_c}} \right)^{1/2} \\ &K_{N_c} = N_c \,/\, \left(N_c - 1 \right) \ \left(= \tfrac{3}{2}\ \text{here} \right) \ ; \ 2\,\Lambda = 1 \,/\, \alpha' \sim 1.06\ \text{GeV}^2 \end{aligned} \tag{105}$$

Dimensionless barycentric coordinates in 1 space dimension

We recall the definition of the dimensionless barycentric coordinates asociated with the dimensionless quantities $\overline{x}_{1,2,3}$, $\overline{X}_{c.m.} = 0$ for 1 spatial dimension in eqs. 50 and 104 in point 1-1)

$$\begin{aligned} &\overline{\xi}_1 = \tfrac{1}{\sqrt{2}} \left(\overline{x}_1 - \overline{x}_2 \right) \ , \ \overline{\xi}_2 = \tfrac{1}{\sqrt{6}} \left(\overline{x}_1 + \overline{x}_2 - 2\,\overline{x}_3 \right) \\ &\overline{\xi}_3 = \tfrac{1}{\sqrt{3}} \left(\overline{x}_1 + \overline{x}_2 + \overline{x}_3 \right) = \sqrt{3}\, \overline{X}_{c.m.} \ \rightarrow 0 \\ &x_j = \lambda\, \overline{x}_j \ ; \ j = 1,2,3 \ ; \ X_{c.m.} = \lambda\, \overline{X}_{c.m.} = \tfrac{1}{3} \sum_{i=1}^{3} \overline{x}_i = 0 \\ \hline &\overline{x}_1 = \ \ \tfrac{1}{\sqrt{2}}\, \overline{\xi}_1 + \tfrac{1}{\sqrt{6}}\, \overline{\xi}_2 \\ &\overline{x}_2 = -\tfrac{1}{\sqrt{2}}\, \overline{\xi}_1 + \tfrac{1}{\sqrt{6}}\, \overline{\xi}_2 \\ &\overline{x}_3 = \qquad\qquad\ \ - \tfrac{2}{\sqrt{6}}\, \overline{\xi}_2 \end{aligned} \tag{106}$$

2-3) Extension of the dimensionless variables in eq. 106 to 3 spatial coordinates
The extension of the configuration variables $\overline{x}_{1,2,3} \rightarrow \vec{\overline{x}}_{1,2,3}$ from d = 1 to d=3 dimensions is defined in point 1-3) . Similarly the 2 barycentric coordinates for d = 1 become three vectors
for d = 3 .

$$\begin{array}{l}
\vec{\overline{\xi}}_{\mathbf{1}} = \frac{1}{\sqrt{2}} \left(\vec{\overline{x}}_{1} - \vec{\overline{x}}_{2} \right) \ , \ \vec{\overline{\xi}}_{\mathbf{2}} = \frac{1}{\sqrt{6}} \left(\vec{\overline{x}}_{1} + \vec{\overline{x}}_{2} - 2 \vec{\overline{x}}_{3} \right) \\
\vec{\overline{\xi}}_{\mathbf{3}} = \frac{1}{\sqrt{3}} \left(\vec{\overline{x}}_{1} + \vec{\overline{x}}_{2} + \vec{\overline{x}}_{3} \right) = \sqrt{3} \ \vec{\overline{X}}_{c.m.} \rightarrow 0 \\
\hline
\vec{\overline{\xi}}_{\mathbf{j}} = \left(\overline{\xi}_{\mathbf{j}}^{(X)} , \overline{\xi}_{\mathbf{j}}^{(Y)} , \overline{\xi}_{\mathbf{j}}^{(Z)} \right) \ ; \ \mathbf{j} = 1.2.3
\end{array} \tag{107}$$

The suffix numbering the different barycentric 3-vectors (in 3 spacelike dimension) are displayed in boldface style in order to emphasize that the numerals labelling $\vec{\overline{\xi}}_{\mathbf{j}}$; $\mathbf{j} = 1.2.3$ are logically quite distinct from those numerals labelling $\overline{x}_{1,2,3}$; $j = 1.2.3$, displayed in cursive mode .
The inverse relations expressing $\overline{x}_{1,2,3}$; $j = 1.2.3$ as a function of the 2 independent, barycentric dimensionless 3-vectors become

$$\begin{array}{rl}
\vec{\overline{x}}_{1} = & \frac{1}{\sqrt{2}} \vec{\overline{\xi}}_{\mathbf{1}} + \frac{1}{\sqrt{6}} \vec{\overline{\xi}}_{\mathbf{2}} \\
\vec{\overline{x}}_{2} = & - \frac{1}{\sqrt{2}} \vec{\overline{\xi}}_{\mathbf{1}} + \frac{1}{\sqrt{6}} \vec{\overline{\xi}}_{\mathbf{2}} \\
\vec{\overline{x}}_{3} = & - \frac{2}{\sqrt{6}} \vec{\overline{\xi}}_{\mathbf{2}}
\end{array} \tag{108}$$

The relations in eq. 108 elucidate the different meaning of the suffix labels in cursive mode and boldface mode .

3-1) The *linear* oscillator basis and mode excitation numbers n_1, n_2 for 1 and 3 spatial dimensions
In order to recall the meaning of the originally adopted labels 1,2 we refer back to subsection 2-4-1 comprising eqs 17 - 29 . First we adapt from these equations eq. 20 from the barycentric coordinates z_1, z_2, defined in ref.[8] , to the dimensionless ones $\overline{\xi}_1$, $\overline{\xi}_2$ as used here for 1 spatial dimension , with the identification as in eq. 106 , together with their relative canonical momenta , denoted $\overline{\pi}_1$, $\overline{\pi}_2$ (as displayed in eqs. 21 - 24)

$$\begin{array}{c}
\overline{\xi}_{1} , \overline{\xi}_{2} \longleftrightarrow \overline{\pi}_{1} , \overline{\pi}_{2} \\
\overline{\pi}_{\mu} = \frac{1}{i} \partial_{\overline{\xi}_{\mu}} \ ; \ \mu = 1,2 \\
\hline
\overline{\xi}_{1} = \frac{1}{\sqrt{2}} \left(\overline{x}_{1} - \overline{x}_{2} \right) \ , \ \overline{\xi}_{2} = \frac{1}{\sqrt{6}} \left(\overline{x}_{1} + \overline{x}_{2} - 2 \overline{x}_{3} \right)
\end{array} \tag{109}$$

The associated (universal) oscillator vector-variables derive from the relation on the last line in eq. 25

$$\begin{array}{l}
a_{\mu k} = \frac{1}{\sqrt{2}} \left(i \overline{\pi}_{\mu k} + \overline{\xi}_{\mu k} \right) \ ; \ a_{\mu k}^{\dagger} = \frac{1}{\sqrt{2}} \left(- i \overline{\pi}_{\mu k} + \overline{\xi}_{\mu k} \right) \\
\mu = 1,2 \ ; \ k = 1,2,3 \ ; \ i \overline{\pi}_{\mu k} = \partial_{\overline{\xi}_{\mu k}}
\end{array} \tag{110}$$

which in turn yields the relativistic structure of the oscillatory $\mathcal{M}^{\,2}$ operator
substituting in eqs. 21 , 23 and 25 yield

$$\mathcal{M}^{\,2} = (2 \Lambda) \sum_{\nu=1,\, k=1}^{\nu=2\,,k=3} \left[\overline{a}^{\,\dagger}_{\,\nu\, k}\, \overline{a}_{\,\nu\, k} + \tfrac{1}{2} \right] \; ; \; 2 \Lambda = 1 / \alpha^{\,\prime} \tag{111}$$

The contribution of the zero mode oscillations , $\frac{1}{2}$ for each dimension of configuration space
(= 3) is inherent to the classical limiting form of oscillatory motion , and is accompanied in the sense of a long range approximation (especially given finite quark masses) by a constant correction .The latter remains non zero also in the limit of vanishing quark masses , as discussed in ref.[8] . In eq. 111 $1 / \alpha^{\,\prime}$ denotes the inverse of the Regge slope.
The last relation in eq. 110 determines the oscillator basis corresponding to the decomposition into *linear* oscillatory modes , which straightforwardly extend from 1 to 3 spatial dimensions

$$\begin{array}{l} a_{\,\mu\, k} = \frac{1}{\sqrt{2}} \left(\partial_{\,\overline{\xi}_{\,\mu\, k}} + \overline{\xi}_{\,\mu\, k} \right) \; ; \; a^{\,\dagger}_{\,\mu\, k} = \frac{1}{\sqrt{2}} \left(- \partial_{\,\overline{\xi}_{\,\mu\, k}} + \overline{\xi}_{\,\mu\, k} \right) \\ \mu = 1, 2 \; ; \; k = 1, 2, 3 \end{array} \tag{112}$$

Eq. 112 refers to 3 spatial dimensions while the case d = 1 would correspond to limit the suffix k in eqs. 110 - 112 to k = 1 .

3-3) Extending to 3 spatial dimensions and circular oscillatory pair-modes and wave function basis

Pair-modes have been discussed for 1 spatial dimension in subsection

"Modes of a pair of onedimensional oscillators – pairmodes and the complex plane"

which comprizes eqs. 51 - 72 .
For 1 spatial dimension we have 2 independent oscillators , which exhibit , whence the flavor & spin degrees of freedom are decoupled according to the decomposition in eq. 32 , an intrinsic SU2 symmetry of the 2 barycentric oscillatory modes . This symmetry allows to choose – always for 1 spacial dimension – a basis , henceforth called circular mode basis , in which the two oscillators take the form given in eq. 51 repeated in adapted notation below

$$\begin{array}{ll} \zeta = \frac{1}{\sqrt{2}} \left(\overline{\xi}_{\,2} + i\, \overline{\xi}_{\,1} \right) & ; x = \overline{\xi}_{\,2} \,,\; y = \overline{\xi}_{\,1} \\ a_{\,1} = \frac{1}{\sqrt{2}} \left(\partial_{\,\zeta} + \overline{\zeta} \right) & ; a_{\,2} = \frac{1}{\sqrt{2}} \left(\partial_{\,\overline{\zeta}} + \zeta \right) \\ a^{\,\dagger}_{\,1} = \frac{1}{\sqrt{2}} \left(- \partial_{\,\overline{\zeta}} + \zeta \right) & ; a^{\,\dagger}_{\,2} = \frac{1}{\sqrt{2}} \left(- \partial_{\,\zeta} + \overline{\zeta} \right) \\ \hline \left[a_{\,1} \,,\, a_{\,2} \right] = \left[a_{\,1} \,,\, a^{\,\dagger}_{\,2} \right] = \left[a_{\,2} \,,\, a^{\,\dagger}_{\,1} \right] = 0 & \end{array} \tag{113}$$

From here (eq. 113) extension from 1 to 3 space dimensions follows 'stroke by stroke' .
To demonstrate this we repeat eq. 102 in adapted form below

$$\zeta \rightarrow \zeta \longrightarrow \vec{\zeta} = \left(\zeta^{(X)} , \zeta^{(Y)} , \zeta^{(Z)} \right) \tag{114}$$

which entails the extension of the circular mode oscillators in eq. 113

$$\begin{array}{l} a_{\mu} \rightarrow \vec{a}_{\mu} \; ; \; \mu = 1,2 \rightarrow \mu = 1,2 \\ \vec{a}_{\mu} = \left(a_{\mu}^{(X)}, a_{\mu}^{(Y)}, a_{\mu}^{(Z)} \right) \; ; \; \mu = 1,2 \end{array} \tag{115}$$

Eq. 113 extended to 3 space dimensions becomes

$$\begin{array}{l} \zeta_{\mathrm{j}} = \frac{1}{\sqrt{2}} \left(\overline{\xi}_{2\,\mathrm{j}} + i \, \overline{\xi}_{1\,\mathrm{j}} \right) \quad ; x_{\mathrm{j}} = \overline{\xi}_{2\,\mathrm{j}} \, , \, y_{\mathrm{j}} = \overline{\xi}_{1\,\mathrm{j}} \\ a_{1\,\mathrm{j}} = \frac{1}{\sqrt{2}} \left(\partial_{\zeta_{\mathrm{j}}} + \overline{\zeta}_{\mathrm{j}} \right) \quad ; a_{2\,\mathrm{j}} = \frac{1}{\sqrt{2}} \left(\partial_{\overline{\zeta}_{\mathrm{j}}} + \zeta_{\mathrm{j}} \right) \\ a_{1\,\mathrm{j}}^{\dagger} = \frac{1}{\sqrt{2}} \left(- \partial_{\overline{\zeta}_{\mathrm{j}}} + \zeta_{\mathrm{j}} \right) ; a_{2\,\mathrm{j}}^{\dagger} = \frac{1}{\sqrt{2}} \left(- \partial_{\zeta_{\mathrm{j}}} + \overline{\zeta}_{\mathrm{j}} \right) \\ \hline \left[a_{1\,\mathrm{j}}, a_{2\,\mathrm{k}} \right] = \left[a_{1\,\mathrm{j}}, a^{\dagger}_{2\,\mathrm{k}} \right] = \left[a_{2\,\mathrm{k}}, a^{\dagger}_{1\,\mathrm{j}} \right] = 0 \; ; \; \mathrm{j} \, , \, \mathrm{k} =^{(X),(Y),(Z)} \end{array} \tag{116}$$

It is appropriate here – now for 3 space dimensions , and using the circular pair mode oscillator basis – to restate the vector nature of oscillator absorption and creation operators (as derived in eqs. 115 and 116) in vector- and component notation

$$\begin{array}{l} \vec{a}_{\mu} \leftrightarrow a_{\mu\,\mathrm{k}} = \left(\vec{a}_{\mu} \right)_{\mathrm{k}} \\ \vec{a}_{\mu}^{\dagger} \leftrightarrow a_{\mu\,\mathrm{k}}^{\dagger} = \left(\vec{a}_{\mu}^{\dagger} \right)_{\mathrm{k}} \end{array} \; ; \; \mu = 1,2 \, , \, \mathrm{k} = (X) \, , \, (Y) \, , \, (Z) \tag{117}$$

The components – 6 each for creation- and annihilation operators – $a_{\mu\,\mathrm{j}}$, $a_{\nu\,\mathrm{k}}^{\dagger}$, interpreted in the circular pair mode oscillator basis , satisfy the nontrivial *commutation* relations

$$\begin{array}{l} \left[a_{\mu\,\mathrm{j}} \, , \, a_{\nu\,\mathrm{k}}^{\dagger} \right] = \delta_{\mu\nu} \, \delta_{\mathrm{jk}} \, ¶ \; ; \; \left\{ \begin{array}{l} \mu \, , \nu \\ \mathrm{j} \, , \mathrm{k} \end{array} \right\} = \left\{ \begin{array}{l} 1,2 \\ (X), (Y), (Z) \end{array} \right\} \\ \text{all other commutators vanish} \end{array} \tag{118}$$

The commutation relations in eq. 118 as such do not depend on the oscillator basis (allowing an SU6 invariant structure) , but the operator realization is particularly adapted , whence circular oscillator basis is chosen . This is done in the next step

Extending the mode structure in the circular oscillator basis from 1 to 3 space dimensions

From here on new material , elaborated in 2013 , shapes the discussinon of counting oscillatory modes of quarks in baryons .
We repeat the form of the wave function in the circular oscillatry mode basis corresponding for 1 pair of oscillators , resulting from the associated structure of the pair of creation operators (eqs. 113 and 57) below

$$\begin{array}{ll} \zeta \ = \ \frac{1}{\sqrt{2}} \left(\overline{\xi}_{2} + i \, \overline{\xi}_{1} \right) & ; x \ = \overline{\xi}_{2} \ , \ y \ = \ \overline{\xi}_{1} \\ a_{1} \ = \ \frac{1}{\sqrt{2}} \left(\partial_{\zeta} + \overline{\zeta} \right) & ; a_{2} \ = \ \frac{1}{\sqrt{2}} \left(\partial_{\overline{\zeta}} + \zeta \right) \\ a_{1}^{\dagger} \ = \ \frac{1}{\sqrt{2}} \left(- \partial_{\overline{\zeta}} + \zeta \right) & ; a_{2}^{\dagger} \ = \ \frac{1}{\sqrt{2}} \left(- \partial_{\zeta} + \overline{\zeta} \right) \\ \hline \left[a_{1} \, , \, a_{2} \right] \ = \ \left[a_{1} \, , \, a_{2}^{\dagger} \right] \ = \ \left[a_{2} \, , \, a_{1}^{\dagger} \right] \ = \ 0 & \end{array} \tag{119}$$

$$\begin{array}{l} \sqrt{2} \, a_{1}^{\dagger} \ = \ \overline{\zeta} \ - \ \partial_{\zeta} \ = \ \exp \left(\zeta \, \overline{\zeta} \right) \left(- \partial_{\zeta} \right) \exp \left(- \zeta \, \overline{\zeta} \right) \\ \sqrt{2} \, a_{2}^{\dagger} \ = \ \zeta \ - \ \partial_{\overline{\zeta}} \ = \ \exp \left(\zeta \, \overline{\zeta} \right) \left(- \partial_{\overline{\zeta}} \right) \exp \left(- \zeta \, \overline{\zeta} \right) \end{array} \tag{120}$$

Thus , as shown in eqs. 58 and 59 for 1 space dimensions , the wave function in the circular oscillator basis – also corresponding to 1 space dimension takes the form repeated below

$$\begin{array}{l} \psi_{n_{1} \, , \, n_{2}} \left(\zeta , \overline{\zeta} \right) \ = \\ = \mathcal{N} \, 2^{-\frac{1}{2} \left(n_{1} + n_{2} \right)} \exp \left(\zeta \, \overline{\zeta} \right) \left(- \partial_{\zeta} \right)^{n_{1}} \left(- \partial_{\overline{\zeta}} \right)^{n_{2}} \exp \left(- 2 \, \zeta \, \overline{\zeta} \right) \\ = \mathcal{N} \, 2^{\frac{1}{2} \left(n_{1} + n_{2} \right)} \, \overline{\zeta}^{\, n_{1}} \, \zeta^{\, n_{2}} \exp \left(- \zeta \, \overline{\zeta} \right) \end{array} \tag{121}$$

and

$$\begin{array}{l} \psi_{n_{1} \, , \, n_{2}} \left(\zeta \, , \, \overline{\zeta} \right) \ = \\ = \left(\dfrac{2^{\left(n_{1} + n_{2} \right)}}{\pi \left(n_{1} ! \right) \left(n_{2} ! \right)} \right)^{\frac{1}{2}} \exp \left(\, i \left(n_{2} \ - \ n_{1} \right) \, \varphi \, \right) \, \varrho^{\left(n_{1} + n_{2} \right)} \, e^{- \varrho^{2}} \\ \hline \varrho \ = \ | \, \zeta \, | \ ; \ \varphi \ = \ arg \, \left(\, \zeta \, \right) \end{array} \tag{122}$$

The further extension from 1 to 3 space dimensions consists in assigning to the single integers $n_{1,2}$ determining the wave function in eqs. 121 and 122, vectors , denoted $\mathrm{n}_{1,2}$, which in components become

$$\mathrm{n}_{\mu} \ = \ \left(\mathrm{n}_{\mu}^{(X)} \ , \ \mathrm{n}_{\mu}^{(Y)} \ , \ \mathrm{n}_{\mu}^{(Z)} \right) \ ; \ \mu \ = \ 1{,}2 \tag{123}$$

The wave function for 1 space dimension , displayed in eq. 122 becomes for

3 such

$$\begin{array}{l} \psi_{\,\mathrm{n}_1,\,\mathrm{n}_2}\left(\vec{\zeta},\vec{\overline{\zeta}}\right) = \\ = \prod_{\mathrm{k}} \left(\begin{array}{l} \left(\dfrac{2^{\left(n_1^{\mathrm{k}} + n_2^{\mathrm{k}}\right)}}{\pi\left(n_1^{\mathrm{k}}!\right)\left(n_2^{\mathrm{k}}!\right)} \right)^{\frac{1}{2}} \exp\left(i\left(n_2^{\mathrm{k}} - n_1^{\mathrm{k}}\right)\varphi_{\mathrm{k}}\right) \times \\ \times \left[\varrho_{\mathrm{k}}^{\left(\mathrm{n}_1^{\mathrm{k}} + \mathrm{n}_2^{\mathrm{k}}\right)} \exp\left(-\varrho_{\mathrm{k}}^{2}\right) \right] \end{array} \right) \\ \hline \varrho_{\mathrm{j}} = \left|\zeta_{\mathrm{j}}\right| \;;\; \varphi_{\mathrm{j}} = arg\left(\zeta_{\mathrm{j}}\right) \;;\; \mathrm{j} = (X),(Y),(Z) \end{array} \tag{124}$$

The measure in the scalar product in eq. 103 repeated below

$$\begin{array}{l} \left\langle \Psi^{(2)} \,\middle|\, \Psi^{(1)} \right\rangle = \begin{array}{l} \lambda^{6} \int \Pi_{i=1}^{3}\, d^{3}\,\overline{\xi}_{i}\, \delta^{3}\left(\overline{\xi}_{3}\right) \times \\ \times\, \Psi^{*\,(2)}\left(\overline{x}_1,\overline{x}_2,\overline{x}_3\right) \Psi^{(1)}\left(\overline{x}_1,\overline{x}_2,\overline{x}_3\right) \end{array} \\ \hline \overline{\xi}_1 = \frac{1}{\sqrt{2}}\left(\overline{x}_1 - \overline{x}_2\right),\ \overline{\xi}_2 = \frac{1}{\sqrt{6}}\left(\overline{x}_1 + \overline{x}_2 - 2\,\overline{x}_3\right) \\ \overline{\xi}_3 = \frac{1}{\sqrt{3}}\left(\overline{x}_1 + \overline{x}_2 + \overline{x}_3\right) = \sqrt{3}\,\overline{X}_{c.m.} \rightarrow 0 \\ z_j = \lambda\,\overline{z}_j \ ,\ x_j = \lambda\,\overline{x}_j \ ;\ j = 1,2,3 \\ X_{c.m.} = \lambda\,\overline{X}_{c.m.} = \frac{1}{3}\sum_{i=1}^{3}\overline{x}_i \end{array} \tag{125}$$

is already adapted to 3 space dimensions.
This concludes all extensions from 1 to 3 space dimensions.

4. Poincaré-invariant harmonic oscillator modes for baryons in the configuration space of barycentric coordinates for 3 valence quarks u , d , s

Seen from the time line of the first years of the 70-th it took about 9 years until an explicit construction of oscillator modes of baryons exhibiting a density function

$$\varrho_n\left(m^2\right) = \frac{\partial N\left(m^2\right)}{\partial m^2} \;;\; N = m^2 \cdot \alpha^{'} \tag{126}$$

where $\alpha^{'}$ is the slope of Regge trajectories, and

$$\varrho_n\left(m^2\right) \sim \left(\frac{m^2}{m_0^2}\right)^{u} \quad \text{for } m \rightarrow \infty \tag{127}$$

$m_0\,,\ u$: characteristic parameters and u = positive integer

for valence quark (antiquark) configurations in mesons and baryons – ref.[8] – compatible with a finite number of degrees of freedom .
Figure 1 shows the logarithm of the density of states of baryons defined in formula 127, together with the density of states of mesons from ref.,[17] as a function of the main quantum number N plus one. This calculation (shown here for N+1 until 10) includes only oscillatory modes for u,d,s quarks and antiquarks for qqq baryons and $q\overline{q}$ for mesons, i.e. not including exotics. The shown density of states is valid in the limit of large N. Furthermore the here derived formulae do not include the intercept and assume a universal slope of Regge trajectories. This does not affect the counting of the number of these states. We observe that the baryon density of states grows faster than the meson density and is always larger than the latter. The data used in figure 1 come from table 1 below.

N	z(Meson)	z(Baryon)
0	36	56
1	108	210
2	216	1044
3	360	2780
4	540	5766
5	756	12510
6	1008	22366
7	1296	38052
8	1620	62826
9	1980	97540

Table 1: Density of states for each main quantum number N until N=9, for mesons and baryons with u,d,s quarks and antiquarks.

We envisage baryons within QCD , allowing *temporarily* the local color gauge group to be generalized from 3 colors to $1 \ < \ N_{\,c} \ \rightarrow \ N$ to simplify notation , whenever no confusion arises . Baryons are : fermions for N odd , bosons for N even . A concise sketch of barycentric canonically conjugate momentum and spatial coordinates – pairs of three vectors

$$(\ \vec{\pi}_{\,\nu} \ , \ \vec{z}_{\,\nu} \ ) \ ; \ \nu \ = \ 1 \ , \ 2 \ , \ \cdots \ , \ N \tag{128}$$

follows .

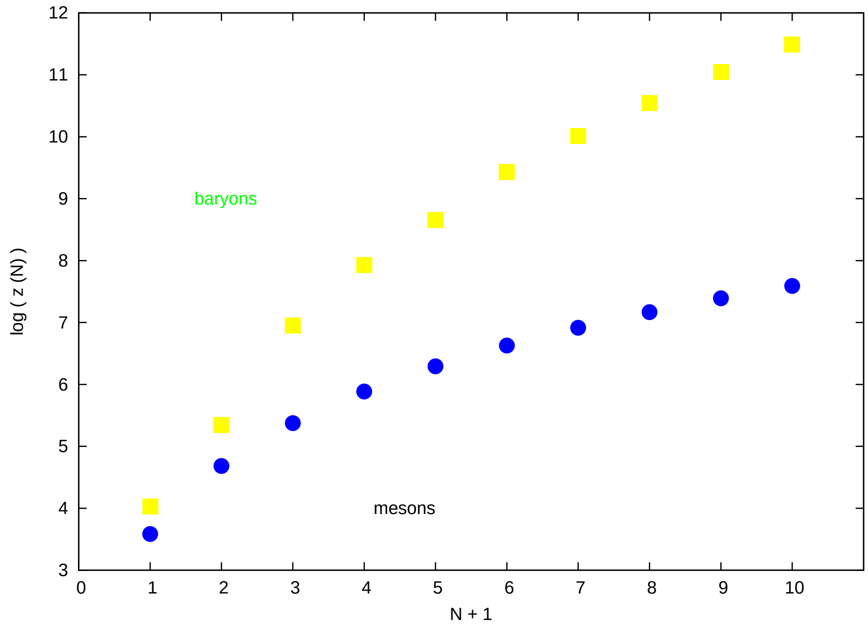

Fig. 4 : Logarithm of the density of states of baryons defined in eq. 127, together with the density of states of mesons from ,[17] as a function of the main quantum number N plus one.

Barycentric coordinates in the overall c.m. system

$$\begin{array}{ccc}
\vec{\pi}_{1} = \frac{1}{\sqrt{2}} (\vec{p}_{1} - \vec{p}_{2}) & , & \vec{z}_{1} = \frac{1}{\sqrt{2}} (\vec{x}_{1} - \vec{x}_{2}) \\
\vec{\pi}_{2} = \frac{1}{\sqrt{6}} (\vec{p}_{1} + \vec{p}_{2} - 2 \vec{p}_{3}) & , & \vec{z}_{2} = \frac{1}{\sqrt{6}} (\vec{x}_{1} + \vec{x}_{2} - 2 \vec{x}_{3}) \\
\cdot & & \cdot \\
\cdot & & \cdot \\
\vec{\pi}_{\nu} = (\nu(\nu+1))^{-1/2} \begin{pmatrix} \sum_{\alpha=1}^{\nu} \vec{p}_{\alpha} \\ -\nu \, \vec{p}_{\nu+1} \end{pmatrix} & , & \vec{z}_{\nu} = (\nu(\nu+1))^{-1/2} \begin{pmatrix} \sum_{\alpha=1}^{\nu} \vec{x}_{\alpha} \\ -\nu \, \vec{x}_{\nu+1} \end{pmatrix} \\
\cdot & & \cdot \\
\cdot & & \cdot \\
\vec{\pi}_{N-1} = \cdots & , & \vec{z}_{N-1} = \cdots \\
\hline
\vec{\pi}_{N} = N^{-1/2} \sum_{\alpha=1}^{N} \vec{p}_{\alpha} \rightarrow 0 & , & \vec{z}_{N} = N^{-1/2} \sum_{\alpha=1}^{N} \vec{x}_{\alpha} \rightarrow 0
\end{array} \tag{129}$$

The last line in eq. 129 refers to c.m. momentum and position , both fixed to vanish they do not represent genuine degrees of freedom .

4.1. Completing canonically conjugate variables in the pairs in $\mathcal{L}_N$, t

In the local Lagrangean density setting of QCD the external quark masses, denoted M_q, appropriate multipliers of the scalar densities $\overline{q}q$ composing the mass term, enter in the form, also discussed recently in refs. 16 –[27] – and 17 –[28] –

$$- \mathcal{L}_{q-mass} = \sum_{flavors} \frac{z_q}{z_M} M_q \, \overline{q}^{\,c} q^{\,c} \tag{130}$$

The quark masses M_q are independent of global c.m. frame time t and of configuration space variables $\vec{z}_\nu$; $\nu = 1, 2, \cdots, N-1$; the *mass functions* m_q do depend on the entire spacial configurarion of the (resonance-) state considered – always reduced to the overall c.m. system – and *therein* of time t .
For the dynamical meaning of c.m. time the implication of the last relation in eq. 129 repeated below

$$\vec{\pi}_N = N^{-1/2} \sum_{\alpha=1}^N \vec{p}_\alpha \rightarrow 0 \, , \, \vec{z}_N = N^{-1/2} \sum_{\alpha=1}^N \vec{x}_\alpha \rightarrow 0 \tag{131}$$

is central : the 'prima facie' canonical pair $\vec{\pi}_N$, $\vec{z}_N$ in eq. 131 is *removed* from the set of dynamical canonical (3-vector-)variable pairs

$$(\vec{\pi}_1 , \vec{z}_1) , \cdots , (\vec{\pi}_{N-1} , \vec{z}_{N-1}) \; ; \; \vec{\pi}_N = 0 \, , \, \vec{z}_N = 0 \tag{132}$$

maintaining *kinematic* simultaneous vanishing displayed in eq. 132 .
Configuration space variables $\vec{x}_\nu$; $\nu = 1, \cdots N$ inherit from the relations displayed in eqs. 131 and 132 the following c.m.-equivalence holds

$$\vec{x}_\nu \underset{c.m.}{\sim} \vec{x}_\nu - \vec{X} \; ; \; \nu = 1 , \cdots , N \; ; \; \vec{X} = \frac{1}{N} \left(\sum_{\varrho=1}^N \vec{x}_\varrho \right) \tag{133}$$

which implies that each $\vec{x}_\nu$ is equivalent to a homogeneous linear combination of the barycentric coordinates $\vec{z}_\kappa$; $1 \leq \kappa \leq N-1$. c.m. time t is defined as *common* time coordinate of four vectors

$$\begin{array}{l} \vec{x}_\nu \rightarrow \{ x^\mu \}_\nu = (t , \vec{x}_\nu) \; ; \; \begin{array}{l} \mu = 0,1,2,3 \\ \nu = 1, \cdots , N \end{array} \\ \text{consistent with the constraint : } \sum_{\nu=1}^N \vec{x}_\nu = 0 \, ; \\ \text{remembering } N = N_c \end{array} \tag{134}$$

4.2. Orbits in configuration space and velocities

The configuration space *variables* obeying c.m.-constraints and c.m. equivalence are together with c.m. time, as displayed in eq. 134

$$(t , \vec{x}_1) , (t , \vec{x}_2) , \cdots , (t , \vec{x}_N) \; ; \; \sum_{\varrho=1}^N \vec{x}_\varrho = 0 \tag{135}$$

with $\vec{x}_{\nu}\ ;\ \nu\ =\ 1,\cdots,N$ conceived *independent* of t – whereas an *orbit* is formed by binding the space-variables $\vec{x}_{\nu}\ =\ f_{\nu}\ (\ t\ )\ \equiv\ \vec{x}_{\nu}\ (t)$, as expressed in the next equation

$$\begin{array}{l} (\ t,\vec{x}_{1}\ )\qquad,(\ t,\vec{x}_{2}\ )\qquad,\cdots,(\ t,\vec{x}_{N}\ )\qquad;\sum_{\varrho=1}^{N}\ \vec{x}_{\varrho}\qquad=0 \\ (\ t,\vec{x}_{1}\ (\ t\ )\ ),(\ t,\vec{x}_{2}\ (\ t)\ ),\cdots,(\ t,\vec{x}_{N}\ (\ t)\ )\ ;\sum_{\varrho=1}^{N}\ \vec{x}_{\varrho}\ (\ t\ )=0 \end{array} \tag{136}$$

The definition of *orbit* in the second relation in eq. 136 leads to the definition of *velocities* maintaining c.m.-constraints and c.m.-equivalence

$$\begin{array}{l} \vec{v}_{\nu}\ (\ t\ )\ =\ \dfrac{d}{d\,t}\ \vec{x}_{\nu}\ (\ t\ )\ \equiv\ \overset{\bullet}{\vec{x}}_{\nu}\ \ ;\ \ \nu\ =\ 1\ ,\ \cdots\ ,\ N\ ; \\ \text{with : } \sum_{\varrho=1}^{N}\ \vec{v}_{\varrho}\ (\ t\ )\ =\ 0 \end{array} \tag{137}$$

4.3. The main quantities $\mathcal{L}_{N}$ depends on

These quantities are compiled in the list of 3 items below

1) a universal scale parameter of dimension mass-square $\Lambda\ =\ \left(\ 2\,\alpha\ '\ \right)^{-1}$ related to the inverse slope of Regge trajectories , as well as quark masses, minimally $M_{q}\ ;\ q\ =\ u\ ,\ d\ ,\ s$, composing the mass term in the local QCD Lagrangean given in eq. 130 , where the finite renormalization constants $z_{q}\ ,\ z_{m}$ are to be chosen to implement exact Ward identities , which in turn specify M_{q} – refs. [27] , [28] .
2) quark mass functions $m_{\varrho}\ (\ M_{q}\ ;\ \{\ \vec{z}_{\nu}\ \}\ )\ ;\ \nu\ =\ 1\ ,\ \cdots\ ,\ N-1\ ;\ \varrho\ =\ 1,\cdots,N$
3) velocities $\vec{v}_{\nu}\ (\ t\ )\ \equiv\ \overset{\bullet}{\vec{x}}_{\nu}\ \ ;\ \ \nu\ =\ 1\ ,\ \cdots\ ,\ N\ ;$ with : $\sum_{\varrho=1}^{N}\ \vec{v}_{\varrho}\ (\ t\ )\ =\ 0$

This leads to the Ansatz – ref. [8] – for light and heavy quark flavors remembering that we use rational units with $\hbar\ =\ c\ =\ 1$.

$$\begin{array}{l} \mathcal{L}_{N}\ =\ -\ \sum_{\alpha=1}^{N}\ \left[\ m_{\alpha}^{2}\ -\ \sum_{\beta=1}^{N}\ \sum_{\gamma=1}^{N}\ Q_{\beta\gamma}^{\alpha}\ \vec{v}_{\beta}\ \vec{v}_{\gamma}\ \right]^{1/2} \\ m_{\alpha}\ =\ m_{\alpha}\ (\ M_{\delta}\ ,\ \Lambda\ ;\ \underline{z}\ )\ \ ;\ \ Q_{\beta\gamma}^{\alpha}\ =\ Q_{\beta\gamma}^{\alpha}\ (\ M_{\delta}\ ,\ \Lambda\ ;\ \underline{z}\ ) \\ \underline{z}\ =\ \{\ \vec{z}_{1}\ ,\ \cdots\ ,\ \vec{z}_{N-1}\ \} \end{array} \tag{138}$$

The limiting quantities in the *chiral* limit obtain as follows

$$\begin{array}{l} m_{\alpha}\ (\ M_{\delta}\ ,\ \Lambda\ ;\ \underline{z}\ )\ \sim\overset{\circ}{m}_{\alpha}\ \ (\ \Lambda\ ;\ \underline{z}\ )\ +M_{\alpha}\qquad\qquad+o(M_{\delta}) \\ Q_{\beta\gamma}^{\alpha}\ (\ M_{\delta}\ ,\ \Lambda\ ;\ \underline{z}\ )\sim\overset{\circ}{Q}{}_{\beta\gamma}^{\alpha}\ \ (\ \Lambda\ ;\ \underline{z}\ )+M_{\varrho}\ A_{\beta\gamma}^{\varrho\,|\,\alpha}\ \Lambda^{\frac{1}{2}}+o\Big(M_{\delta}\ \Lambda^{\frac{1}{2}}\Big) \\ \text{for } M_{\delta}\ \rightarrow\ 0 \end{array} \tag{139}$$

In ref. [8] the quantities‘

$$\begin{array}{ccc} m_{\alpha}\ (\ M_{\delta}\ ,\ \Lambda\ ;\ \underline{z}\ ) & \longrightarrow & \overset{\circ}{m}_{\alpha}\ (\ \Lambda\ ;\ \underline{z}\ ) \\ & \text{chiral limit} & \\ Q_{\beta\gamma}^{\alpha}\ (\ M_{\delta}\ ,\ \Lambda\ ;\ \underline{z}\ ) & \longrightarrow & \overset{\circ}{Q}{}_{\beta\gamma}^{\alpha}\ (\ \Lambda\ ;\ \underline{z}\ ) \end{array} \tag{140}$$

defined in eq. 139 were identified but shall be kept distinct here , for clarity of logic.

4.4. Remarks with respect to the chiral limit

χ1 The discussion of oscillatory modes of baryons in the chiral limit is simplified but the phenomenological application is limited to the light flavors u,d,s.

χ2 In the zero width approximation the appearance of 8 massless pseudoscalar goldstone bosons

$$\overset{\circ}{\pi}_{\sigma} = \left(\pi^{+} , \pi^{0} , \pi^{-} ; \overline{K}^{0} , \overline{K}^{-} ; K^{+} , K^{0} ; \eta \right) ; \\ \sigma = 1, \cdots ,8 ; m_{\overset{\circ}{\pi}_{\sigma}} = 0 \tag{141}$$

does not enter directly , but the modes with large masses will decay to lower ones and one ore more $\overset{\circ}{\pi}$ - s – also in cascades . This is after removing infrared divergencies not significantly different for the realistic case of finite π_{σ} - masses . As a consequence such cascades of excited baryon resonance decays generate a pronounced feeding problem , aggravated if heavy ion collisions are performed at todays energy frontiers at RHIC and LHC , less so at FAIR .

χ3 In addition at high energy also hevay flavors c , b , t are or will be copiously produced aggravating the study of light flavored (anti-)baryons .

On the phenomenological side the Δ Regge trajectory from ref. 19 – ref.[29] – is shown in Fig. 5 , in which the additional trajectories were erased .

4.5. $\mathcal{L}_{N} \rightarrow \overset{\circ}{\mathcal{L}}_{N}$ in the chiral limit

For logical consistency we can consider the chiral limit of arbitrary $N = N_{c}$ and N_{fl} light flavors of quark , as long as $N > 1$ and $N > \frac{2}{11} N_{fl}$, ensuring asymptotic (ultraviolet) freedom .

The Ansatz for $\overset{\circ}{\mathcal{L}}_{N}$ becomes

$$\overset{\circ}{Q}{}^{\alpha}_{\beta\gamma} = \overset{\circ}{m}{}^{2}_{\alpha} \frac{1}{K_{N}} \delta_{\beta\gamma} ; \ K_{N} \ \text{: dimensionless , positive constant} \tag{142}$$

Eq. 138 reduces to

$$\overset{\circ}{\mathcal{L}}_{N} = - \left(\sum_{\alpha=1}^{N} \overset{\circ}{m}_{\alpha} \right) \left[1 - \frac{1}{K_{N}} v^{2} \right]^{\frac{1}{2}} ; v^{2} = \sum_{\beta=1}^{N} \vec{v}_{\beta} \vec{v}_{\beta} \\ \overline{} \\ \longrightarrow \overset{\circ}{m} = \sum_{\alpha=1}^{N} \overset{\circ}{m}_{\alpha} ; \overset{\circ}{m}_{\alpha} = \overset{\circ}{m}_{\beta} = \overset{\circ}{m} / N \ \forall \ \alpha , \beta = 1, \cdots , N \tag{143}$$

In eq. 143 $v = \sqrt{v^{2}}$ defines a Euclidean distance in a space over real numbers $R^{\widehat{d}}$ of dimension $\widehat{d} = d_{space} \times N$, with $d_{space} \rightarrow 3$, yielding the associated scalar product

$$\underline{w} = \{ \vec{w}_{1} , \cdots , \vec{w}_{N} \} \longrightarrow (\underline{v}_{2} \mid \underline{v}_{1}) = \sum_{\beta=1}^{N} \vec{v}_{2\,\beta} \vec{v}_{1\,\beta} \tag{144}$$

In $R^{\widehat{d}}$ the transformation to barycentric coordinates defined in eq. 129

$$\begin{array}{l} \underline{x} = \{ \vec{x}_1 , \cdots , \vec{x}_N \} \longrightarrow \underline{z} = \underline{z}(\underline{x}) = \{ \vec{z}_1 , \cdots , \vec{z}_N \} \\ \hline \vec{z}_1 = \frac{1}{\sqrt{2}} (\vec{x}_1 - \vec{x}_2) \\ \vec{z}_2 = \frac{1}{\sqrt{6}} (\vec{x}_1 + \vec{x}_2 - 2 \vec{x}_3) \\ \cdots \\ \vec{z}_\nu = (\nu(\nu+1))^{-1/2} \left(\sum_{\alpha=1}^{\nu} (\vec{x}_\alpha - \vec{x}_{\nu+1}) \right) \\ \cdots \\ \vec{z}_{N-1} = \cdots ; (\nu = N - 1) \\ \vec{z}_N = N^{-1/2} \sum_{\alpha=1}^{N} \vec{x}_\alpha = N^{1/2} \vec{X}_{c.m.} \end{array} \tag{145}$$

is an orthogonal $O\widehat{d}$ mapping, i.e. preserves the scalar product defined in eq. 144. [b]

The reduction from the $O\widetilde{d}$ transformation , to be proven orthogonal , to its form in $R^{\widetilde{d}} = R^N \otimes R^{d_{space}}$

$$\begin{array}{l} O^{(\widehat{d}\,)} = O^{(N)} \otimes \P_{d_{space} \times d_{space}} ; \ O^{(N)} \longleftrightarrow O_{\mu\nu} ; \\ \mu , \nu = 1, \cdots , N \ \text{with} \ \vec{z}_\mu = O_{\mu\nu} \vec{w}_\nu \end{array} \tag{146}$$

is not presented here , additional details can be found in ref. [21] and references cited therein. The proof of orthogonality of $O^{(\widehat{d}\,)}$ reduces to establish orthogonality of the matrix $O^{(N)}$ displayed in eq. 146 .

Two relations follow independently from the c.m. constraint in configuration space $\vec{X}_{c.m.} \sim 0$ from eqs. 142 - 146

$$\begin{array}{l} \sum_{\alpha=1}^{N} \vec{x}_\alpha^{\,2} = \sum_{\beta=1}^{N-1} \vec{z}_\beta^{\,2} + N \vec{X}_{c.m.}^{\,2} \\ \qquad = \sum_{\alpha=1}^{N} \left(\vec{x}_\alpha - \vec{X}_{c.m.} \right)^2 + N \vec{X}_{c.m.}^{\,2} \\ \hline \longrightarrow \sum_{\beta=1}^{N-1} \vec{z}_\beta^{\,2} = \sum_{\alpha=1}^{N} \left(\vec{x}_\alpha - \vec{X}_{c.m.} \right)^2 \end{array} \tag{147}$$

4.6. $\overset{\circ}{\mathcal{L}}_N \longrightarrow$ *constructing canonical variables of Hamiltonian quantum mechanics*

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[b] in the *approximately* realistic case we have $N \equiv N_c = d_{space} = N_{fl} = 3$ *exactly* .

$$\begin{array}{l} \vec{p}_{\beta} = \overset{\circ}{\mathcal{L}}_{N\,,\,\vec{v}_{\beta}} = \dfrac{\overset{\circ}{m}}{K_{N}} \left[1 - \omega^{2} \right]^{-1/2} \frac{1}{2} v^{2}_{\,,\vec{v}_{\beta}} \\ \qquad\qquad\qquad = \dfrac{\overset{\circ}{m}}{K_{N}} \left[1 - \omega^{2} \right]^{-1/2} \vec{v}_{\beta} \end{array} ; \\ \beta = 1, \cdots .N \\ \omega^{2} = \frac{1}{K_{N}} \sum_{\gamma=1}^{N} v_{\gamma}^{2} \ ; \ K_{N} = \frac{N}{N-1} \ ; \\ \overset{\circ}{m} = \sum_{\alpha=1}^{N} \overset{\circ}{m}_{\alpha} \tag{148}$$

Eq. 148 establishes the N canonically conjugate pairs of $d_{space}-$vectors

$$\{ \vec{p}_{\beta} , \vec{x}_{\beta} \} \ ; \ \beta = 1, \cdots , N \tag{149}$$

generated through $\overset{\circ}{\mathcal{L}}_{N}$ with general properties outlined in eq. 143. These variables do *not* satisfy the c.m. equivalence $\longrightarrow$ constraints defined in eqs. 131 - 133 collected in eq. 150 below

$$\begin{array}{l} \vec{\pi}_{N} = N^{-1/2} \sum_{\alpha=1}^{N} \vec{p}_{\alpha} \rightarrow 0 \, , \vec{z}_{N} = N^{-1/2} \sum_{\alpha=1}^{N} \vec{x}_{\alpha} \rightarrow 0 \\ (\vec{\pi}_{1} , \vec{z}_{1}) , \cdots , (\vec{\pi}_{N-1} , \vec{z}_{N-1}) \ ; \ \vec{\pi}_{N} = 0 \, , \vec{z}_{N} = 0 \\ \vec{x}_{\nu} \underset{c.m.}{\sim} \vec{x}_{\nu} - \vec{X} \ ; \ \nu = 1 , \cdots , N \ ; \ \vec{X} = \dfrac{1}{N} \left(\sum_{\varrho=1}^{N} \vec{x}_{\varrho} \right) \end{array} \tag{150}$$

Omitting many details , we proceed to construct *the* main conserved quantity, following the original ref. 4 – ref. [8]

$$\begin{array}{l} \mathcal{H}_{N} = \sum_{\alpha=1}^{N} \vec{v}_{\alpha} \vec{p}_{\alpha} - \overset{\circ}{\mathcal{L}}_{N} \ ; \ \sum_{\alpha=1}^{N} \vec{v}_{\alpha} \vec{p}_{\alpha} = \omega^{2} \left[1 - \omega^{2} \right]^{-1/2} \\ \overset{\circ}{\mathcal{L}}_{N} = - \overset{\circ}{m} \left[1 - \omega^{2} \right]^{1/2} \longrightarrow \mathcal{H}_{N} = \overset{\circ}{m} \left[1 - \omega^{2} \right]^{-1/2} \end{array} \tag{151}$$

From eqs. 148 and 151 we infer

$$K_{N} \vec{p}_{\beta}^{\,2} = \mathcal{H}_{N}^{2} \frac{1}{K_{N}} \vec{v}_{\beta}^{\,2} \longrightarrow \begin{array}{l} K_{N} \sum_{\gamma=1}^{N} \vec{p}_{\gamma}^{\,2} = \mathcal{H}_{N}^{2} \omega^{2} \\ \qquad = \mathcal{H}_{N}^{2} - \overset{\circ}{m}^{2} \end{array} \tag{152}$$

The last relation on the lower right side of eq. 152 follows from the decomposition $\omega^{2} = 1 - \left(1 - \omega^{2} \right)$ and eqs. 148 and 151 . The structure of a *genuine* harmonic oscillator is brought a step nearer rearranging eq. 152

$$K_{N} \sum_{\gamma=1}^{N} \vec{p}_{\gamma}^{\,2} + \overset{\circ}{m}^{2} = \mathcal{H}_{N}^{2} \tag{153}$$

Since eq. 153 derives from a variational principle and yields consistent equations of motion both in the classical as well as the quantized interpretation the Ansatz for $\overset{\circ}{m}{}^{2}$ realizing a system of $s = (N-1)\, d_{space}$ linear harmonic oscillators [c] implies the choice

$$\begin{array}{l} \overset{\circ}{m}{}^{2} \propto \sum_{\gamma=1}^{N} \left(\vec{x}_{\beta} - \vec{X}_{c.m.} \right)^{2} + \delta_{m^{2}} \ ; \\ \delta_{m^{2}} \ : \ \text{configuration space independent} \end{array} \tag{154}$$

The constant transforming the proportionality in eq. 154 into an equality has to be determined in agreement with canonically conjugate momenta and positions . The result is, using also the relations in eq. 147

$$\begin{array}{l} \mathcal{H}_{N}^{2} = \left[K_{N} \sum_{\alpha=1}^{N} (\vec{p}_{\alpha})^{2} \Big|_{\sum_{\beta=1}^{N} \vec{p}_{\beta} = 0} + \overset{\circ}{m}{}^{2} (x_{\gamma} - X) \right] \\ = \left[\begin{array}{l} K_{N} \sum_{\alpha=1}^{N} (\vec{p}_{\alpha})^{2} \Big|_{\sum_{\beta=1}^{N} \vec{p}_{\beta} = 0} + \\ + \dfrac{\Lambda^{2}}{K_{N}} \sum_{\alpha=1}^{N} (\vec{x}_{\alpha})^{2} \Big|_{\sum_{\beta=1}^{N} \vec{x}_{\beta} = 0} + \delta_{m^{2}} \end{array} \right] \end{array} \tag{155}$$

Eq. 155 reveals that $\mathcal{H}_{N}^{2}$ so defined *does* satisfy the c.m.-constraints , and thus by Poincaré invariance is the mass-square operator , henceforth denoted $\mathcal{M}^{2} \equiv \overset{\circ}{\mathcal{M}}{}^{2}$. For simplicity of notation the superfix $\circ$, which stands for the chiral limit , in $\mathcal{M}^{2}$, is suppressed .

Next we transform momenta and positions to barycentric coordinates and cast the relations in eqs. 153 - 155 into the form

$$\overset{\circ}{m}{}^{2} (x_{\gamma} - X) = \frac{\Lambda^{2}}{K_{N}} \sum_{\alpha=1}^{N-1} \vec{z}_{\alpha}^{\,2} + \delta_{m^{2}} \tag{156}$$

In eq. 156 Λ , of dimension mass-square , is the spring-tension [d] .

Λ^{2} is related to the gauge boson condensate of QCD ; a detailed discussion can be found in ref. 20 – ref. [30]

We arrive at the relation for $\mathcal{M}^{2}$ expressed through

$\widehat{d}_{c.m} = (N-1)\, d_{space}$ unconstrained , canonically conjugate pairs

$(\vec{\pi}_{1}, \vec{z}_{1}), \cdots, (\vec{\pi}_{N-1}, \vec{z}_{N-1})$ defined in eq. 129

$$\begin{array}{l} \mathcal{M}^{2} = \left[K_{N} \sum_{\alpha=1}^{N-1} (\vec{\pi}_{\alpha})^{2} + \dfrac{\Lambda^{2}}{K_{N}} \sum_{\alpha=1}^{N-1} (\vec{z}_{\alpha})^{2} + \delta_{m^{2}} \right] \\ \vec{\pi}_{\alpha} = \frac{1}{i} \partial_{\vec{z}_{\alpha}} \ ; \ \alpha = 1, \cdots, N-1 \ ; \ [\vec{\pi}_{\beta\, k}, \vec{z}_{\gamma\, l}] = \frac{1}{i} \delta_{\beta\gamma} \delta_{km} \ \P \\ k, l = 1, \cdots, d_{space} \end{array} \tag{157}$$

[c] For the realistic case $N = d_{space} = 3$, s is = 6 .

[d] Spring is not string .

The structure of the canonical variables in eq. 157 is intrinsically related to the trace anomaly in QCD, as derived in ref. [31]

5. Partial countings of oscillatory modes of quarks in baryons

5.1. Characteristic transformation properties of the basis functions $\psi_{\,n_1\,,\,n_2}\left(\vec{\zeta}\,,\,\vec{\overline{\zeta}}\right)$ under the permutation group S_3

It is enough to select a partial subset (not a subgroup of S_3 for the sought transformation propewrties namely the elements

$$\text{setmin} \;=\; \{T_{12}, T_{13}, T_{23} \text{ and } Z_3 : \text{the abelian subgroup of cyclic permutations}\} \tag{158}$$

The notation of the elements of S_3 ; Z_3 as well as the associated (2 one-dimensional and
1 two-dimensional unitary irreducible representations) are discussed in subsection 3-rec and for the *circular pair -mode basis* in subsectionis 3-res , 3-res-2 and 3-res-3 and eq. 101 .
First we invert the decomposition of the complex numbers $\zeta_{\,\mathrm{j}}\,,\,\overline{\zeta}_{\,\mathrm{j}}\,;\,\mathrm{j} \;=\; (X),(Y),(Z)$
into real and imaginary parts , as displayed in eq. 116

$$\begin{array}{l}
\zeta_{\,\mathrm{j}} \;=\; \frac{1}{\sqrt{2}}\left(\overline{\xi}_{\,2\,\mathrm{j}} + i\,\overline{\xi}_{\,1\,\mathrm{j}}\right) \quad ; x_{\,\mathrm{j}} = \overline{\xi}_{\,2\,\mathrm{j}}\,,\, y_{\,\mathrm{j}} \;=\; \overline{\xi}_{\,1\,\mathrm{j}} \\
a_{\,1\,\mathrm{j}} \;=\; \frac{1}{\sqrt{2}}\left(\partial_{\,\zeta_{\,\mathrm{j}}} + \overline{\zeta}_{\,\mathrm{j}}\right) \quad ; a_{\,2\,\mathrm{j}} \;=\; \frac{1}{\sqrt{2}}\left(\partial_{\,\overline{\zeta}_{\,\mathrm{j}}} + \zeta_{\,\mathrm{j}}\right) \\
a^{\,\dagger}_{\,1\,\mathrm{j}} \;=\; \frac{1}{\sqrt{2}}\left(-\,\partial_{\,\overline{\zeta}_{\,\mathrm{j}}} + \zeta_{\,\mathrm{j}}\right) ; a^{\,\dagger}_{\,2\,\mathrm{j}} \;=\; \frac{1}{\sqrt{2}}\left(-\,\partial_{\,\zeta_{\,\mathrm{j}}} + \overline{\zeta}_{\,\mathrm{j}}\right) \\
\hline
\left[\,a_{\,1\,\mathrm{j}}\,,\,a_{\,2\,\mathrm{k}}\,\right] \;=\; \left[\,a_{\,1\,\mathrm{j}}\,,\,a^{\,\dagger}_{\,2\,\mathrm{k}}\,\right] \;=\; \left[\,a_{\,2\,\mathrm{k}}\,,\,a^{\,\dagger}_{\,1\,\mathrm{j}}\,\right] \;=\; 0 \;;\; \mathrm{j}\,,\,\mathrm{k} \;=\; {}^{(X),(Y),(Z)}
\end{array} \tag{159}$$

Next we adapt the form of the induced representation with argument the cyclic permutation

$$U\left[\,\pi\begin{pmatrix}1\,2\,3\\3\,1\,2\end{pmatrix}\right] \;\longleftrightarrow\; Z \;=\; e^{\,i\,\frac{2\,\pi}{3}} \tag{160}$$

associated with the third root of 1 : $Z \;=\; e^{\,i\,\frac{2\,\pi}{3}}$ acting on the basis functions $\psi_{\,\mathrm{n}_{\,1}\,,\,\mathrm{n}_{\,2}}\left(\vec{\zeta}\,,\,\vec{\overline{\zeta}}\right)$

from 1 space time dimension as displayed in eq. 64 , to 3

$$\left(U \left[\pi \begin{pmatrix} 1\,2\,3 \\ 3\,1\,2 \end{pmatrix} \right] \psi_{\,n_1 , n_2} \right) \left(\zeta_j , \overline{\zeta}_j \right) =$$
$$= D_{n_1 n_2} (Z) \, \psi_{\,n_1 , n_2} \left(\zeta_j , \overline{\zeta}_j \right) = \psi_{\,n_1 , n_2} \left(Z^{-1} \zeta_j , Z \overline{\zeta}_j \right)$$
$$D_{n_1 n_2} (Z) = Z^{\left(\Sigma m \left(n_1^m - n_2^m \right) \right)} \; ; \; j , k , m = (X), (Y), (Z) \tag{161}$$

5.1. *Transformation properties of the basis functions $\psi_{\,n_1 , n_2} \left(\vec{\zeta} , \vec{\overline{\zeta}} \right)$ under the 3 transpositions T_{12} , T_{23} , T_{13}*

Here we need the properties of the abstract permutaion group S_3 worked out in subsection 3-res and displayed in eq. 93 partially reproduced below regarding the odd permutations in S_3

$$\begin{array}{lll} 4) \; \pi \begin{pmatrix} 1\,2\,3 \\ 2\,1\,3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 - & 1 \end{pmatrix} & - \rightarrow & T_{12} \\ 5) \; \pi \begin{pmatrix} 1\,2\,3 \\ 1\,3\,2 \end{pmatrix} \longrightarrow \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} & - \rightarrow & T_{23} \\ 6) \; \pi \begin{pmatrix} 1\,2\,3 \\ 3\,2\,1 \end{pmatrix} \longrightarrow \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} & - \rightarrow & T_{13} \end{array} \tag{162}$$

Next we recall the multiplication table of S_3 using to the numbering of elements introduced in eq. 93

$$\begin{array}{lll} 1) \; \pi \begin{pmatrix} 1\,2\,3 \\ 1\,2\,3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1\,0 \\ 0\,1 \end{pmatrix} & + \rightarrow & \P \\ 2) \; \pi \begin{pmatrix} 1\,2\,3 \\ 2\,3\,1 \end{pmatrix} \longrightarrow \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} & + \rightarrow & Z \\ 3) \; \pi \begin{pmatrix} 1\,2\,3 \\ 3\,1\,2 \end{pmatrix} \longrightarrow \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} & + \rightarrow & Z^2 \equiv Z^{-1} \end{array} \tag{163}$$

5.2. Consolidation of transformation properties of S_3 as elaborated in subsection 3-rec

The minimal discussion of products of elements of S_3 in subsection 3-rec , eqs. 75 - 92 prove insufficient . Thus we construct the partial multiplication table

$$\begin{array}{|c|cc|ccc|} \hline & 2) & 3) & 4) = T_{12} & 5) = T_{23} & 6) = T_{13} \\ \hline 2) & 3) & \P & 5) & 6) & 4) \\ 3) & \P & 2) & 6) & 4) & 5) \\ \hline 4) & 6) & 5) & \P & 3) & 2) \\ 5) & 4) & 6) & 2) & \P & 3) \\ 6) & 5) & 4) & 3) & 2) & \P \\ \hline \end{array} \tag{164}$$

We also recall the association of permutations 2) and 3) from eq. 101

$$\begin{array}{l} 2)\ d_{\pi=2}\ \pi \begin{pmatrix} 1\,2\,3 \\ 2\,3\,1 \end{pmatrix} \longmapsto \begin{pmatrix} e^{+i(2\pi/3)} & 0 \\ 0 & e^{-i(2\pi/3)} \end{pmatrix} + \rightarrow Z \\ 3)\ d_{\pi=3}\ \pi \begin{pmatrix} 1\,2\,3 \\ 3\,1\,2 \end{pmatrix} \longmapsto \begin{pmatrix} e^{-i(2\pi/3)} & 0 \\ 0 & e^{+i(2\pi/3)} \end{pmatrix} + \rightarrow Z^{2} \equiv Z^{-1} \end{array} \tag{165}$$

We note here the structure of the similarity transormations relating the three transpositions

$$\begin{array}{l} 2) \circ 4) \circ \{ 2) \}^{-1} = 6) \ ; \ 3) \circ 4) \circ \{ 3) \}^{-1} = 5) \\ 2) \circ 5) \circ \{ 2) \}^{-1} = 4) \ ; \ 3) \circ 5) \circ \{ 3) \}^{-1} = 6) \\ 2) \circ 6) \circ \{ 2) \}^{-1} = 5) \ ; \ 3) \circ 6) \circ \{ 3) \}^{-1} = 4) \end{array} \tag{166}$$

In conjunction with the conjugacy classes of a finit group $\mathcal{G}$ it os a good place to remember the orignal notions due to Frobenius[32] of conjugacy classes and their invariants , the traces over arbitrary finite unitary representations $\mathcal{D}\ (\ \mathcal{G}\ )$

$$\begin{array}{l} \text{Cjclass}\ (\ g\ ;\ \mathcal{G}\ ) \ = \ g\ \mathcal{G}\ g^{-1}\ ;\ g\ \in\ \mathcal{G} \\ \hline \mathcal{D}\ (\ \mathcal{G}\ ) \ = \ \left\{ \bigcup_g\ D\ (\ g\ )) \ \middle|\ D\ (\ g_2\ )\ D\ (\ g_1\ ) \ = \ D\ (\ g_2 \circ g_1\ ) \right\} \\ g\ ,\ g_1\ ,\ g_2\ \in\ \mathcal{G} \end{array} \tag{167}$$

5.3. *Schur's lemma* [33]

We follow here the proof of Schur's lemma laid out in ref. [26] with slight modifications to adjust to the notation used in this outline .

We consider two arbitrary *irreducible* unitary representations of $\mathcal{G}$, denoted $\mathcal{D}^{(1)}$ and $\mathcal{D}^{(2)}$, and form the quantity

$$\begin{pmatrix} Y^{(1)\ (2)}_{m\ n} \end{pmatrix}_{M\ N} = \sum_g \left(\mathcal{D}^{(1)}\ (\ g\ ) \right)_{M\ m} \left(\mathcal{D}^{(2)}\ (\ g\ ) \right)_{N\ n} \tag{168}$$

The group structure then implies that $Y^{(1)\,(2)}$ is a group invariant

$$\begin{array}{l}\left(\mathcal{D}^{(1)}\left(\widehat{h}\right)_{M\,M'}\left(\mathcal{D}^{(2)}\left(\widehat{h}\right)_{N\,N'}\begin{pmatrix}Y^{(1)\,(2)}\\ m\,n\end{pmatrix}_{M'\,N'} = \begin{pmatrix}Y^{(1)\,(2)}\\ m\,n\end{pmatrix}_{M\,N} = \\ = \sum_{g}\left(\mathcal{D}^{(1)}\left(\widehat{h}\,g\right)\right)_{M\,m}\left(\mathcal{D}^{(2)}\left(\widehat{h}\,g\right)\right)_{N\,n} \;;\; \forall\,\widehat{h}\,\in\,\mathcal{G}\longrightarrow \\ k\,=\,\widehat{h}\,g\;\;;\,g\,=\,\widehat{h}^{\,-1}\,k\,;\;\sum_{g}\,\equiv\,\sum_{k}\end{array} \tag{169}$$

In eq. 169 we have chosen as group transformation , justifying the hat symbol for $\widehat{h}$, left-multiplication

$$\widehat{h}\,:\,g\,\rightarrow\,\widehat{h}\,\circ\,g \tag{170}$$

The relation in the first line of eq. 169, using matrix notation with respect to the indices $M\,M'$, N , N' for fixed m , n becomes

$$\begin{array}{l}\mathcal{M}_{\,M\,N}\,|_{\,m\,n\text{ fixed}}\;\rightarrow\;\mathcal{M}_{\,M\,N}\;\rightarrow\;\mathcal{M}\\ \mathcal{D}^{(1)}\left(\widehat{h}\right)\,\mathcal{M}\left(\mathcal{D}^{(2)}\left(\widehat{h}\right)\right)^{T}\,=\,\mathcal{D}^{(1)}\left(\widehat{h}\right)\,\mathcal{M}\left(\mathcal{D}^{(2)}\left(\widehat{h}\right)\right)^{-1}\,=\,\mathcal{M}\\ \hline M\,,\,M'\,=\,1,\cdots,\dim\,\mathcal{D}^{(1)}\;;\;N\,,\,N'\,=\,1,\cdots,\dim\,\mathcal{D}^{(2)}\end{array} \tag{171}$$

and multiplying the relation on the second line in eq. 171 with $\mathcal{D}^{(2)}\left(\widehat{h}\right)$ from the right it follows

$$\begin{array}{l}\mathcal{D}^{(1)}\left(\widehat{h}\right)\,\mathcal{M}\,=\,\mathcal{M}\,\mathcal{D}^{(2)}\left(\widehat{h}\right)\;;\;\forall\,\widehat{h}\,\{in\,\mathcal{G}\\ \hline \mathcal{M}\,:\,dim\,\mathcal{D}^{(1)}\,\times\,dim\,\mathcal{D}^{(2)}\text{ matrix}\end{array} \tag{172}$$

Schur's lemma states that the *only* solution of eq. 172 compatible with the representations $\mathcal{D}^{(1)}$ *and* $\mathcal{D}^{(2)}$ beeing irreducible *and* unitary , *except* $\mathcal{M}\,\rightarrow M\,=\,0$, demands

$$\begin{array}{l}dim\,\mathcal{D}^{(1)}\,=\,dim\,\mathcal{D}^{(2)}\text{ and }det\,\mathcal{M}\,\rightarrow\,det\,M\,\neq\,0\text{ and}\\ \mathcal{D}^{(2)}\,=\,M^{-1}\,\mathcal{D}^{(1)}\,M\;\leftrightarrow\;\mathcal{D}^{(1)}\,=\,M\,\mathcal{D}^{(1)}\,M^{-1}\end{array} \tag{173}$$

The proof makes use of 'elementary' properties of linear algebra .
For the original paper(s) see ref.[33] .
The multiplication table of S_3 in eq. 164 shall be checked *here* , since it provides the bridge its regalar representation based on the group algebra over the group elements

$$\begin{array}{l}\mathcal{A}\,=\,\left\{a\,|\,a\,=\,\sum_{i=1}^{6}\,a^{\,i}\,g_{\,i}\text{ with }g_{\,i}\,\in\,S_{3}\;;\;a^{\,i}\text{ real}\right\}\\ a\,=\,\left(a^{\,1}\,,\,a_{\,2}\,,\,\cdots\,,\,a^{\,6}\right)\,\in\,\mathbb{C}^{\,6}\end{array} \tag{174}$$

The regular representation consists of two representations , acting on the left and on the right in a commuting way

$$\begin{array}{l} L \; : \; g_{\,i} \;\longrightarrow\; a \;\rightarrow\; g_{\,i} \,\circ\, a \;=\; \sum_{k=1}^{6} \; a^{\,k} \, g_{\,i} \,\circ\, g_{\,k} \\ R \; : \; g_{\,i} \;\longrightarrow\; a \;\rightarrow\; a \,\circ\, (\, g_{\,i}\,)^{\,-1} \;=\; \sum_{k=1}^{6} \; a^{\,k} \, g_{\,k} \,\circ\, (\, g_{\,i}\,)^{\,-1} \end{array} \tag{175}$$

From eq. 175 and the multiplication table of S_3 here , the two regular repreentations are defined

$$\begin{array}{ll} L \; : (\, g_{\,i} \,\circ\, a \,)^{\,j} & = \Big((\, D_{\,L} \,)^{\,i}_{\;j} \; (\, g_{\,i}\,) \Big) \; a^{\,j} \;=\; a^{\,k} \, g_{\,ik \,\rightarrow\, j} \\ g_{\,ik \,\rightarrow\, j} \;=\; g_{\,i} \,\circ\, g_{\,k} & \\ R \; : \Big(a \;\circ\; (\, g_{\,i}\,)^{\,-1} \Big)^{\,j} & = \Big((\, D_{\,R} \,)^{\,i}_{\;j} \; (\, g_{\,i}\,) \Big) \; a^{\,j} \;=\; a^{\,k} \, \widetilde{g}_{\,ki \,\rightarrow\, j} \\ \widetilde{g}_{\,ki \,\rightarrow\, j} \;=\; g_{\,k} \,\circ\, (\, g_{\,i}\,)^{\,-1} & \end{array} \tag{176}$$

The L , R representations derived in eqs. 174 - 176 are classical cornerstones of the theory of finite groups, attributed to Ferdinand Georg Frobenius[e] .
Eqs. 175 and 176 shall be expressed using matrix notation

$$\begin{array}{l} \widehat{g_{\,i\,L}} \, a = D_{\,L} \;(\, g_{\,i}\,)\; a = g_{\,i} \circ\, a \\ \widehat{g_{\,j\,R}} \, a = D_{\,R} \;(\, g_{\,j}\,)\; a = a \;\circ\; (\, g_{\,j}\,)^{\,-1} \end{array} \tag{177}$$

In eq. 177 the symbols $\widehat{g_{\,i\,L}}$, $\widehat{g_{\,i\,R}}$ denote the operator nature of the substitutions associated with the group multiplications on its uttermost right hand side .
From the relations in eq. 177a the bilateral representation structure following the pattern

$$\mathcal{D} \;(\, \mathcal{G} \;\times\; \mathcal{G}\,) \;=\; \mathcal{D}_{\,L} \;(\, \mathcal{G}\,) \,\otimes\, \mathcal{D}_{\,R} \;(\, \mathcal{G}\,) \tag{178}$$

It follows

$$\begin{array}{l} D_{\,L} \;(\, g_{\,k}\,)\; D_{\,L} \;(\, g_{\,i}\,) \;=\; D_{\,L} \;(\, g_{\,k} \,\circ\, g_{\,i}\,) \\ D_{\,R} \;(\, g_{\,k}\,)\; D_{\,R} \;(\, g_{\,i}\,) \;=\; D_{\,R} \;(\, g_{\,k} \,\circ\, g_{\,i}\,) \end{array} \tag{179}$$

and using $(\,6 \;\times\; 6\,)$ matrix notation

$$D_{\,R} \;(\, g_{\,k}\,)\; D_{\,L} \;(\, g_{\,i}\,) \;=\; D_{\,L} \;(\, g_{\,i}\,)\; D_{\,R} \;(\, g_{\,k}\,) \;\;\forall\; g_{\,k} \,,\, g_{\,i} \;;\;\; k \,,\, j \;=\; 1, \cdots ,6 \tag{180}$$

With this description of the regular representation all basic group- and oscillatory mode- properties are assembled , which allow to perform all necessary checks straightforwardly.

---

[e] Ferdinand Frobenius + 26 October 1849 in Berlin-Charlottenburg , Prussia , Germany
† 3 August 1917 in Berlin , Prussia , Germany

cited from ref.[32] .

We proceed to the direct count of these modes , using the circular oscillatory wave function basis and the symmetry properties of associated permutation group representations in the next subsection .

6. Counting oscillatory modes using the circular oscillatory wave function basis enforcing overall Bose symmetry under the combined permutations of $SU6$ (*fl* $\times$ *spin*) $\times$ barycentric coordinates

Traces of irreducible representations of S_3

Not beeing able to give a clearcut reference to the original publication(s) yielding knowledge with respect to what became known as 'Schur's lemma , as formulated here for unitary irreducible representations in eq. 173 the present subsection is added here.

It follows directly within the hypothesis discussed in the neughbourhood of eq. 173 that the only nontrivial relation between two unitary irrecucible linear representations of S_3 , $\mathcal{D}^{(1)}$ and $\mathcal{D}^{(2)}$ is the unitary equivalence

$$\mathcal{D}^{(2)} = M^{-1} \mathcal{D}^{(1)} M \leftrightarrow \mathcal{D}^{(1)} = M \mathcal{D}^{(1)} M^{-1} \ ; \ M : \text{unitary} \tag{181}$$

The matrix M – establishing unitary equivalence of the already assumed linear *unitary* irrecudible representations $\mathcal{D}^{1}$ and $\mathcal{D}^{2}$ – as it appears in eq. 181 is unitary but not unique and not to be identified with the quantity $\mathcal{M}$ as it is defined in eqs. 168 and 171 , with its additional indices $_{m\ n}$. See additional derivations *in extenso* in ref.[26] .

It is a general procedure within wide fields of mathematics to attempt a fully reducible construction of invariants , here with respect to the equivalence class of unitary similarity transformations as given in eq. 181 , to consider invariants for this class .

Such invariants are the traces of any given unitary irreducible representation $\mathcal{D} (\mathcal{G})$ as functions over a finite group $\mathcal{G}$, here S_3

$$\chi (h , \mathcal{D}) = tr (\mathcal{D} (h)) \ ; \ h \in \mathcal{G} \tag{182}$$

The sought invariance follows from the relation for the trace of a product of $dim_{\mathcal{D}} \times dim_{\mathcal{D}}$ matrices

$$tr (A \circ B) = tr (B \circ A) \longrightarrow tr \left(M \circ \mathcal{D} (h) \circ M^{-1} \right) = tr (\mathcal{D} (h)) \tag{183}$$

for any *given* unitary $dim\ \mathcal{D} \times dim\ \mathcal{D}$ matrix M *and* unitary irreducible representation $\mathcal{D}$.

It follows for characters , which are the same for equivalent representations (eq. 183)

$$\sum_{g} \overline{\chi} \; (g \; , \; \mathcal{D}\;') \; \chi \; (g \; , \; \mathcal{D}) \; = \; \Pi \; (\; \mathcal{G} \;) \; \delta_{\; \mathcal{D}\;' \; \mathcal{D}} \; ; \; \delta_{\; \mathcal{D}\;' \; \mathcal{D}} = \begin{cases} 1 \text{ for } \; \mathcal{D}\;' \simeq \mathcal{D} \\ 0 \text{ else} \end{cases}$$
$$\forall \text{irreducible } \mathcal{D}\;'(\mathcal{G}) \; , \; \mathcal{D} \; (\mathcal{G}) \tag{184}$$

In eq. 184 $\overline{\chi}$ is the complex conjugate of χ and $\Pi \; (\; \mathcal{G} \;)$ the number of elements of the finite group $\mathcal{G}$.
This ends my account of the pertinant messages from Issai Schur (1875-1941).
The characters form a complete orthonormal – modulo the factor $\Pi \; (\mathcal{G} \;)$ – set of functions on $\mathcal{G}$. This had been proven by Peter and Weyl[34] for compact Lie groups, but clearly applies also to the simpler case of finite groups .

6.2. Aligning statistics between the u, d , s $SU6$ (fl $\times$ spin) group and oscillator modes in 6 barycentric configuration space variables

We resume our main theme recalling the three Young tableaux induced by the statistics of 3 valence quarks u, d, s
from the group $SU6 \; (\; fl \; \times \; spin \;)$ group in subsection 3-1-3a , eq. 32 repeated below for coherence of presentation

$$\begin{aligned} 1 \; &: dim \; (\square\square\square) = \frac{4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}{5 \; !} &&= 56 \\ 2 \; &: dim \left(\begin{smallmatrix} \square\square \\ \square \end{smallmatrix} \right) = \frac{2 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{4 \; ! \; 5 \; !} &&= 70 \\ 3 \; &: dim \left(\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix} \right) = \frac{2 \cdot 4 \cdot 5 \cdot 6 \cdot 3 \cdot 4 \cdot 5 \cdot 2 \cdot 3 \cdot 4}{3 \; ! \; 4 \; ! \; 5 \; !} &&= 20 \end{aligned} \tag{185}$$

6.3. Associating symmetric and antisymmetric representations of $SU6$ (fl $\times$ spin) to oscillator mode wave functions in the circular pair-mode basis

This corresponds to the Young tableaus 1 and 3 in eq. 185 . The associated (inequivalent) irreducible representations of $S_{\;3}$ are both one dimensional : the identity for case 1 and the antisymmetric representation , assigning $+1$ for the even - and -1 for the odd permutations .

We verify eq. 184 for the two (1-dim) inequivalent representations of S_3 , denoted $\boxed{1^+}$ and $\boxed{1^-}$

g	1)	2)	3)	4)	5)	6)
1)	1)	2	3)	4)	5)	6)
2)	2)	3)	¶	5)	6)	4)
3)	3)	¶	2)	6)	4)	5)
4)	4)	6)	5)	¶	3)	2)
5)	5)	4)	6)	2)	¶	3)
6)	6)	5)	4)	3)	2)	¶

g	$\chi\left(g\,,\boxed{1^+}\right)$	$\chi\left(g\,,\boxed{1^-}\right)$
1)	$+1$	$+1$
2)	$+1$	$+1$
3)	$+1$	$+1$
4)	$+1$	-1
5)	$+1$	-1
6)	$+1$	-1
$\sum_g \chi^2$	6	6

(186)

The left panel in eq. 186 gives the full multiplication table for S_3 recalling eq. 164 from subsection 4-C .
From the right panel in eq. 186 we also veryfy tho nondiagnal orthogonality relation given in eq. 184

$$\sum_g \overline{\chi}\left(g\,,\boxed{1^-}\right)\chi\left(g\,,\boxed{1^+}\right) = 0 \tag{187}$$

We turn to the remaining irreducible (2-dim) representation of S_3 , which shall be denoted $\boxed{2}$ and is displayed in the circular pair-mode associated basis in eq. 101, shown in eq. 189 below
From the $\mathcal{D}_{\boxed{2}}$ traces we find

$$\begin{array}{l} \sum_g \overline{\chi}\left(g\,,\boxed{2}\right)\chi\left(g\,,\boxed{2}\right) = 6 \\ \sum_g \overline{\chi}\left(g\,,\boxed{2}\right)\chi\left(g\,,\boxed{1^-}\right) = \sum_g \overline{\chi}\left(g\,,\boxed{2}\right)\chi\left(g\,,\boxed{1^+}\right) = 0 \end{array} \tag{188}$$

veryfying all of eq. 184 .
$\mathcal{D}_{\boxed{2}}$ corresponds *uniquely* to Young tableau 2 in eq. 185 .
This exhausts – modulo similarity transformation – all conjugation invariant func-

tions over S_3 .

g	$\mathcal{D}_{\boxed{2}}$	$\chi\left(g, \boxed{2}\right)$
1) $d_{\pi=1} = \pi \begin{pmatrix} 1\,2\,3 \\ 1\,2\,3 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	2
2) $d_{\pi=2} = \pi \begin{pmatrix} 1\,2\,3 \\ 2\,3\,1 \end{pmatrix}$	$\begin{pmatrix} e^{+i(2\pi/3)} & 0 \\ 0 & e^{-i(2\pi/3)} \end{pmatrix}$	-1
3) $d_{\pi=3} = \pi \begin{pmatrix} 1\,2\,3 \\ 3\,1\,2 \end{pmatrix}$	$\begin{pmatrix} e^{-i(2\pi/3)} & 0 \\ 0 & e^{+i(2\pi/3)} \end{pmatrix}$	-1
4) $d_{\pi=4} = \pi \begin{pmatrix} 1\,2\,3 \\ 2\,1\,3 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	0
5) $d_{\pi=5} = \pi \begin{pmatrix} 1\,2\,3 \\ 1\,3\,2 \end{pmatrix}$	$\begin{pmatrix} 0 & e^{+i(2\pi/3)} \\ e^{-i(2\pi/3)} & 0 \end{pmatrix}$	0
6) $d_{\pi=6} = \pi \begin{pmatrix} 1\,2\,3 \\ 3\,2\,1 \end{pmatrix}$	$\begin{pmatrix} 0 & e^{-i(2\pi/3)} \\ e^{+i(2\pi/3)} & 0 \end{pmatrix}$	0

(189)

After this detour to include the basic conjugation invariant functions over S_3 , assembled in eqs. 186 and 189 , we return to the main theme of this subsection. This necessitates the insertion regarding reducible traces , from direct product representations of S_3 in the next subsection.

6.4. Reducible traces from direct product representations of S_3

[f]

The definition of traces in eq. 182 does not imply to restrict a given representation $\mathcal{D}(\mathcal{G})$ to an irreducible one . In the environment of eqs. 184 - 189 however , only the 3 inequivalent *irreducible* unitary representations of S_3 were considered .
The counting of oscillatory modes (of u, d , s quarks in baryons) involves intrinsically multiple direct product representations . Thus we are led to extend the catalog of basic traces over the direct (simple) products of these irreducible representations .

The general reduction of the trace of the direct product $\mathcal{D}$ of unitary but otherwise

[f] It is an obligation of one of us (P.M.) to refer to and thank here Kurt Schütte[35] (+ 14. October 1909 in Salzwedel ; † 18. August 1998 in Munich , Germany) . It was probably during the WS 1961/62 , when one of us (P.M.) followed as an undegraduate student his lectures on 'Algebra' , including a thorough discussion of properties of finite groups . He was then guest professor at the ETH in Zurich .

still arbitrary factor representations $\mathcal{D}^{(\alpha)}$ and $\mathcal{D}^{(\beta)}$ is

$$\mathcal{D} = \mathcal{D}^{(\alpha)} \otimes \mathcal{D}^{(\beta)} \longrightarrow \chi\,(g\,,\,\mathcal{D}\,) = \chi\,\left(g\,,\,\mathcal{D}^{(\alpha)}\right) \times \chi\,\left(g\,,\,\mathcal{D}^{(\beta)}\right) \tag{190}$$

and hence immaterial of the order of the direct product factors . As a consequence (inequivalent) products of the base , irreducible representations $\boxed{1^{+}}$ and $\boxed{1^{-}}$ of S_3 , amount to

$$\begin{array}{l} \chi\left(g\,,\,\boxed{1^{+}} \otimes \boxed{1^{+}}\right) = \left(\chi\left(g\,,\,\boxed{1^{+}}\right)\right)^{2} = \chi\left(g\,,\,\boxed{1^{+}}\right) \\ \chi\left(g\,,\,\boxed{1^{+}} \otimes \boxed{1^{-}}\right) = \chi\left(g\,,\,\boxed{1^{+}}\right)\chi\left(g\,,\,\boxed{1^{-}}\right) = \chi\left(g\,,\,\boxed{1^{-}}\right) \end{array} \tag{191}$$

The single nontrivial direct product is $\boxed{2} \otimes \boxed{2}$

$$\chi\left(g\,,\,\boxed{2} \otimes \boxed{2}\right) = \left(\chi\left(g\,,\,\boxed{2}\right)\right)^{2} = \begin{array}{c|c} \chi\ \boxed{2} \otimes \boxed{2} & g \\ \hline 4 & 1) \\ 1 & 2) \\ 1 & 3) \\ 0 & 4) \\ 0 & 5) \\ 0 & 6) \end{array} \tag{192}$$

From eq. 192 using the primitive traces of the irreducible representations $\boxed{1^{+}}$, $\boxed{1^{-}}$ in eq. 186 and $\boxed{2}$ in eq. 189 we find the decomposition

$$\left[\begin{array}{c|c} g & \chi\ \boxed{2} \otimes \boxed{2} \\ \hline 1) & 4 \\ 2) & 1 \\ 3) & 1 \\ 4) & 0 \\ 5) & 0 \\ 6) & 0 \end{array}\right] = \left[\begin{array}{c} \chi\ \boxed{2} \\ \hline \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \end{array}\right] + \left[\begin{array}{c} \chi\ \boxed{1^{+}} \\ \hline 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{array}\right] + \left[\begin{array}{c} \chi\ \boxed{1^{-}} \\ \hline 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \end{array}\right]$$
$$\longrightarrow\ \boxed{2} \otimes \boxed{2} = \boxed{2} \oplus \boxed{1^{+}} \oplus \boxed{1^{-}} \tag{193}$$

The direct product decomposition constructed here in its basic form in eqs. 191 - 193 can systematically be extended to arbitrary numbers of direct product factors. It is logically similar to the decomposition of natural numbers into products of prime ones.

6.5. Associating symmetric and antisymmetric representations of SU6 (fl × spin) to oscillator mode wave functions in the circular pair-mode basis

Having completed the two insertions – in subsections 5-1 and 5-ins – dealing with the properties of traces of general and irreducible unitary representations of a finite group $\mathcal{G}$ and their direct products , adapting to the case relevant here $\mathcal{G} \rightarrow S_3$, we resume the 'counting' of oscillatory modes of u, d , s light flavored valence quarks in baryons .
The symmetric and antisymmetric representations of S_3 correspond to simple conditions on the wave functions formed from the basis in eq. 124 repeated below

$$\psi_{n_1, n_2} \left(\vec{\zeta}, \vec{\overline{\zeta}} \right) = \\ = \prod_k \left(\begin{array}{l} \left(\dfrac{2^{\left(n_1^k + n_2^k \right)}}{\pi \left(n_1^k ! \right) \left(n_2^k ! \right)} \right)^{\frac{1}{2}} \exp \left(i \left(n_2^k - n_1^k \right) \varphi_k \right) \times \\ \times \left[\varrho_k^{\left(n_1^k + n_2^k \right)} \exp \left(- \varrho_k^2 \right) \right] \end{array} \right) \tag{194}$$

$\varrho_j = \left| \zeta_j \right| \; ; \; \varphi_j = arg \left(\zeta_j \right) \; ; \; j = (X), (Y), (Z)$
These conditions are twofold , expressed in the quantum numbers $n_1 \; ; \; n_2$ as defined in eqs. 124 and equivalently 194 . It is convenient to define the nonnegative integer quantities $N_{1,2}$ associated with $n_{1,2}$ respectively as well as their signed difference

$$\begin{array}{l} N_1 = \sum_k n_1^k \; ; \; N_2 = \sum_k n_2^k \text{ with } N = N_1 + N_2 \\ \Delta = N_1 - N_2 \end{array} \tag{195}$$

The first condition is common to both cases 1 and 3 induced by the statistics of 3 valence quarks u, d, s from the group $SU6 \; (fl \times spin)$ in subsection 3-1-3a , eqs. 32 and 32 , repeated below

$$\begin{array}{lll} 1 \; : dim \; (\square\square\square) = \dfrac{4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}{5\,!} & & = 56 \\ 2 \; : dim \left(\begin{smallmatrix} \square\square \\ \square \end{smallmatrix} \right) = \dfrac{2 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{4\,!\,5\,!} & & = 70 \\ 3 \; : dim \left(\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix} \right) = \dfrac{2 \cdot 4 \cdot 5 \cdot 6 \cdot 3 \cdot 4 \cdot 5 \cdot 2 \cdot 3 \cdot 4}{3\,!\,4\,!\,5\,!} & & = 20 \end{array} \tag{196}$$

The two conditions for cases 1 and 3 in eq. 196 are

Condition 1

$$\Delta = 0 \quad (\text{mod } 3) \;\; \text{for cases } \; 1 \text{ and } 3 \tag{197}$$

Condition 2

$$\psi_{\mathrm{n}_1,\mathrm{n}_2} \ \left(\vec{\zeta}, \vec{\overline{\zeta}}\right) \ \longrightarrow \begin{cases} \psi^{(+)}_{\mathrm{n}_1,\mathrm{n}_2} = \mathcal{N}^{+}\frac{1}{2} \ \{\psi \ \mathrm{n}_1,\mathrm{n}_2 + \psi_{\mathrm{n}_2,\mathrm{n}_1}\} \text{ for case } 1 \\ \psi^{(-)}_{\mathrm{n}_1,\mathrm{n}_2} = \mathcal{N}^{-}\frac{1}{2} \ \{ \ \psi_{\mathrm{n}_1,\mathrm{n}_2} - \psi_{\mathrm{n}_2,\mathrm{n}_1}\} \text{ for case } 3 \end{cases} \tag{198}$$

In eq. 198 $\mathcal{N}^{\pm}$ denote normalization constants .

The two conditions defined in eqs. 197 and 198 determine the counting in accordance with overall Bose symmetry for the complete bosonic wave function respecting $SU6 \ (\ fl \ \times \ spin \ )$ and oscillatory modes in 6 barycentric coordinates – for the two cases considered in this subsection .

6.6 Associating the mixed 70-representation of ***SU6 (fl × spin)*** *to oscillator mode wave functions in the circular pair-mode basis*

It turns out that the association of the symmetric and antisymmetric representations of $SU6 \ (\ fl \ \times \ spin \ )$ show the way to determine the Bose statistics association of the mixed 70-representation of $SU6 \ (\ fl \ \times \ spin \ )$ to the oscillatory modes in the circular pair-mode basis , i.e. in the remaining case 2 in eq. 196 .
In fact there is only one condition $\left(\text{modulo a restriction by a factor of } \frac{1}{2} \right)$ determining the sought association , complementing cases 1 and 3 in eqs. 197 and 198
Condition for case 2

$$\Delta \ = \ 1 \ \& \ 2 \quad (\mathrm{mod}\ 3) \ \text{ for case } \ 2 \tag{199}$$

For the counting the powers of the two sets

$$\Delta \ = \ 1 \quad (\mathrm{mod}\ 3) \ ; \ \Delta \ = \ 2 \quad (\mathrm{mod}\ 3) \tag{200}$$

are the same , allowing the required (multiple) realization of the $\boxed{2}$ representation of $S_{\ 3}$ on the wave functions in the circular pair-mode basis – intervening in case 2 . Given this realization the number of oscillator modes satisfying the condition in eq. 199 are then paired with the corresponding $\boxed{2}$ in the 70-plet of $SU6 \ (\ fl \ \times \ spin \ )$ in eq. 196 as shown in the direct product decomposition in eq. 193 . Hence overall Bose symmetry restricts the number count of wave functions satisfying the condition in eq. 199 by a facor of $\frac{1}{2}$.
This ends the *theoretical embedding* of (counting) oscillatory modes of valence u, d, s quarks in baryons and in thesr subsections: 6.3. and 6.4 and in section: 6.

What remains to be done is the actual counting , according to the conditions formulated in eqs. 197 and 198 for cases 1 and 3 and eqs. 199 and 200 for case 2 . This is best done for a finite number of main oscillator quantum numbers N in a dedicated computer program , left to be worked out soon .

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7. Appendix 1 : Some checks on results in subsection 3-1-3b The binomial reduction formula $I\ (\ K\ ,\ p+1\ )\ \rightarrow\ I\ (\ K\ ,\ p\ )$

We perform simple checks on the validity of the binomial reduction formula , derived in susection 3-1-3b . To this end we repeat eq. refeq:3-25 as starting point in eq. 201 below

$$\begin{array}{l} I\ (\ K\ ,\ p\ +\ 1\ )\ = \\ = \left[\begin{array}{c} (\ p\ +\ 2\ )^{\ -1}\ (\ K + 1\ )^{\ p+2}\ - \\ -\ (\ p\ +\ 2\ )^{\ -1}\ \sum_{\ q=1}^{\ p+1} \left(\begin{array}{c} p\ +\ 2 \\ p\ +\ 1\ -\ q \end{array} \right) I\ (\ K\ ,\ p\ +\ 1\ -q\ ) \end{array} \right] \\ \hline I\ (\ K\ ,\ 0\ )\ =\ K\ +\ 1\ ;\ \left(\begin{array}{c} p\ +\ 1 \\ 0 \end{array} \right)\ =\ 1\ \text{for}\ p \geq\ 0 \end{array} \tag{201}$$

We first set p = 0 in eq. 201

$$\begin{array}{l} I\ (\ K\ ,\ 1\ ) = \frac{1}{2}\ (\ K + 1\ )^{\ 2}\ -\ \frac{1}{2} \left(\begin{array}{c} 2 \\ 0 \end{array} \right) I\ (\ K\ ,\ 0\ ) \\ \qquad = \frac{1}{2}\ K^{\ 2}\ +\ K\ + \frac{1}{2}\ -\ \frac{1}{2}\ K\ -\ \frac{1}{2}\ =\ \frac{1}{2}\ K\ (\ K\ +\ 1\ ) \qquad (\surd) \end{array} \tag{202}$$

Eq. 202 anchors the recursion from p = 0 to p = 1 .
We proceed to calculate $I\ (\ K\ ,\ 2\ )$ setting p = 1 in eq. 201 , which yields

$$\begin{array}{l} I\ (\ K\ ,\ 2\ ) = \left[\begin{array}{c} \frac{1}{3}\ (\ K + 1\ )^{\ 3}\ - \\ -\ \frac{1}{3}\ \sum_{\ q=1}^{\ 2} \left(\begin{array}{c} 3 \\ 2\ -\ q \end{array} \right) I\ (\ K\ ,\ 2\ -\ q\ ) \end{array} \right] \\ \qquad = \frac{1}{3}\ (\ K + 1\ )^{\ 3}\ -\ I\ (\ K\ ,\ 1\ )\ -\ \frac{1}{3}\ I\ (\ K\ ,\ 0\ ) \\ \qquad = \frac{1}{3}\ (\ K + 1\ )^{\ 3}\ -\ \frac{1}{2}\ (\ K\ +\ 1\ )\ K\ - \frac{1}{3}\ (\ K\ +\ 1\ ) \\ \qquad = \frac{1}{3}\ (\ K + 1\ )\ \left[\ (\ K + 1\ )^{\ 2}\ -\ \frac{3}{2}\ K\ -\ 1\ \right] \end{array} \tag{203}$$

We check this for K = 2 , 3

$$\begin{array}{l} I\ (\ 2\ ,\ 2\ ) =\ 5\ =\ (\ 9\ -\ 3\ - 1\ ) \qquad (\surd) \\ I\ (\ 3\ ,\ 2\ ) = 14\ =\ \frac{4}{3}\ \left(\ 16\ -\ \frac{9}{2}\ - 1\ \right) \\ \qquad\qquad =\ \frac{2}{3}\ (\ 32\ -\ 11\ )\ =\ 2\ \times\ 7\ =\ 14 \qquad (\surd) \end{array} \tag{204}$$

8. Appendix 2 : The 2 x 2 representation of $S_{\ 3}$ in the basis denoted by the collection $d_{\ \pi}$ in eq. 100

First we repeat eq. 100 below , defining the similarity equation

$$\begin{array}{l} d_{\pi} \ = \ \mathcal{M} \ D_{\pi} \ \mathcal{M}^{-1} \ = \ u \ D_{\pi} \ u^{-1} \\ \hline u \ = \ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \ , \ u^{-1} \ = \ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \end{array} \tag{205}$$

Next we substitute a general (real) 2 x 2 matrix for D_{π} , yielding the transformation

$$u \ \rightarrow \ A \ \longrightarrow \ E \ = \ u \ A \ u^{\dagger} \ ; \ A \ = \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ ; \ E \ = \ \begin{pmatrix} e & f \\ g & h \end{pmatrix} \tag{206}$$

The substitution in eq. 206 becomes

$$E \ = \ \frac{1}{2} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \tag{207}$$

and proceeding step by step

$$\begin{array}{rl} E = & \frac{1}{2} \begin{pmatrix} a \ + \ i \ c & b \ + \ i \ d \\ a \ - \ i \ c & b \ - \ i \ d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \\ = & \frac{1}{2} \begin{pmatrix} a \ + \ d \ - \ i \ (\ b \ - \ c \ ) & a \ - \ d \ + \ i \ (\ b \ + \ c \ ) \\ a \ - \ d \ - \ i \ (\ b \ + \ c \ ) & a \ + \ d \ + \ i \ (\ b \ - \ c \ ) \end{pmatrix} \\ A \ = & \text{real matrix , by inspection of the collection } D_{q} \text{ in eq. 93} \end{array} \tag{208}$$

The matrix E (in eqs. 206 - 208) has complex matrix elements , which can be parametrized as follows

$$\begin{array}{l} E \ = \ \begin{pmatrix} e & f \\ g & h \end{pmatrix} \ = \ \begin{pmatrix} e & f \\ f^{*} & e^{*} \end{pmatrix} \ ; \ h \ = \ e^{*} \ , \ g \ = \ f^{*} \\ \hline e \ = \ \frac{1}{2} \ [\ a \ + \ d \ - \ i \ (\ b \ - \ c \ ) \ ] \ , \ f \ = \ \frac{1}{2} \ [\ a \ - \ d \ + \ i \ (\ b \ + \ c \ ) \ ] \end{array} \tag{209}$$

In eq. 209 e.g. e^{*} denotes the complex conjugate number relative to e .
We note here that $tr \ E \ = \ tr \ A$ is a unitary invariant , independent of the transformation matrix u .

8.1. Associating the collections D_{π} in eq. 93 and d_{π} in eq. 100 for elements beyond unity of the 2 x 2 representation of S_3

We repeat the first 2 entries of the D_{π} collection in eq. 93 below in eq. 210

$$\begin{array}{l} 1)\ \pi \begin{pmatrix} 1\,2\,3 \\ 1\,2\,3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1\,0 \\ 0\,1 \end{pmatrix} \qquad\qquad + \rightarrow \ \P \\ 2)\ \pi \begin{pmatrix} 1\,2\,3 \\ 2\,3\,1 \end{pmatrix} \longrightarrow \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} + \rightarrow \ Z \end{array} \tag{210}$$

Thus we fill the $A\ (\ a\ ,\ b\ ,\ c\ ,\ d\ )$ pattern belonging to $A\ =\ A2\ =\ D_{\ \pi=2}$

$$\begin{array}{c} \begin{pmatrix} a2\ b2 \\ c2\ \ d2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \\ \hline \frac{1}{2}\ (\ a\ +\ d\ ) = -\frac{1}{2}\ ,\ \frac{1}{2}\ (\ b\ -\ c\ ) = -\frac{\sqrt{3}}{2} \\ \frac{1}{2}\ (\ a\ -\ d\ ) = \quad 0\ ,\ \frac{1}{2}\ (\ b\ +\ c\ ) = \quad 0 \end{array} \tag{211}$$

We shall not repeat the index 2 . No confusion arises before the next index (3) is addressed .
Also the symbol e , identified with a matrix element is not to be confused with Eulers constant , even though the symbols are the same.
It follows from eqs. 209 and 211 (for $d_{\ \pi\ =2}$)

$$\begin{array}{lcl} e & = & \frac{1}{2}\ [\ a\ +\ d\ -\ i\ (\ b\ -\ c\ )\ ]\ =\ -\frac{1}{2} +\ i\ \frac{\sqrt{3}}{2} = e^{\ i\ (\ 2\pi\ /\ 3\ )} \\ f & = & \frac{1}{2}\ [\ a\ -\ d\ +\ i\ (\ b\ +\ c\ )\ ] \qquad\qquad\quad = \qquad 0 \end{array} \tag{212}$$

Finally we obtain for $d_{\ \pi=2}$

$$2)\ d_{\ \pi=2}\ \pi \begin{pmatrix} 1\,2\,3 \\ 2\,3\,1 \end{pmatrix} \longrightarrow \begin{pmatrix} e^{\ +i(2\pi/3)} & 0 \\ 0 & e^{\ -i(2\pi/3)} \end{pmatrix} + \rightarrow \ Z \tag{213}$$

From $d_{\ \pi=2}$ to $d_{\ \pi=3}$

The algebraic as well as complex conjugation properties of the $d_{\ \pi}$ collection, restricted to even permutations , allows to derive the (diagonal form) of the representation matrix $d_{\ \pi=3}$ directly from the element $d_{\ \pi=2}$

$$3)\ d_{\ \pi=3}\ \pi \begin{pmatrix} 1\,2\,3 \\ 3\,1\,2 \end{pmatrix} \longrightarrow \begin{pmatrix} e^{\ -i(2\pi/3)} & 0 \\ 0 & e^{\ +i(2\pi/3)} \end{pmatrix} + \rightarrow \ Z^{\ 2}\ \equiv\ Z^{\ -1} \tag{214}$$

8.2. The representation matrix for the transposition $d_{\pi=4}$

We repeat the fourth entry of the D_{π} collection in eq. 93 below in eq. 215

$$4)\ \pi \begin{pmatrix} 1\,2\,3 \\ 2\,1\,3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0\,- & 1 \end{pmatrix} - \rightarrow T_{12} \tag{215}$$

Here we follow the same procedure as in the subsection on $d_{\pi=2}$ in eqs. 211 - 212

$$\frac{\begin{pmatrix} a4\ b4 \\ c4\ \ d4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}{\begin{array}{l} \frac{1}{2}\,(\,a\,+\,d\,) = 0\,,\,\frac{1}{2}\,(\,b\,-\,c\,) = 0 \\ \frac{1}{2}\,(\,a\,-\,d\,) = 1\,,\,\frac{1}{2}\,(\,b\,+\,c\,) = 0 \end{array}} \tag{216}$$

The analog to eq. 212 becomes

$$\begin{aligned} e &= \tfrac{1}{2}\ [\,a\,+\,d\,-\,i\,(\,b\,-\,c\,)\,] = 0 \\ f &= \tfrac{1}{2}\ [\,a\,-\,d\,+\,i\,(\,b\,+\,c\,)\,] = 1 \end{aligned} \tag{217}$$

So we arrive with the help of eq. 209 at the result

$$4)\ d_{\pi=4}\ \pi \begin{pmatrix} 1\,2\,3 \\ 2\,1\,3 \end{pmatrix} \longrightarrow \begin{pmatrix} 0\,1 \\ 1\,0 \end{pmatrix} - \rightarrow T_{12} \tag{218}$$

8.3. The representation matrices for transpositions $d_{\pi=5}$ and $d_{\pi=6}$

We repeat entries 5) and 6) of the D_{π} collection in eq. 93 below in eq. 219

$$\begin{array}{l} 5)\ \pi \begin{pmatrix} 1\,2\,3 \\ 1\,3\,2 \end{pmatrix} \longrightarrow \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} - \rightarrow T_{23} \\ 6)\ \pi \begin{pmatrix} 1\,2\,3 \\ 3\,2\,1 \end{pmatrix} \longrightarrow \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} - \rightarrow T_{13} \end{array} \tag{219}$$

For $D_{\pi=5}$ the equation analogous to eq. 216 reads

$$\frac{\begin{pmatrix} a5\ b5 \\ c5\ \ d5 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}}{\begin{array}{l} \frac{1}{2}\,(\,a\,+\,d\,) = \ \ 0\,,\,\frac{1}{2}\,(\,b\,-\,c\,) = \ \ 0 \\ \frac{1}{2}\,(\,a\,-\,d\,) = -\frac{1}{2}\,,\,\frac{1}{2}\,(\,b\,+\,c\,) = \frac{\sqrt{3}}{2} \end{array}} \tag{220}$$

The analog to eq. 217 becomes

$$\begin{aligned} e &= \tfrac{1}{2} \left[a + d - i (b - c) \right] = 0 \\ f &= \tfrac{1}{2} \left[a - d + i (b + c) \right] = -\tfrac{1}{2} + i \tfrac{\sqrt{3}}{2} = e^{+i(2\pi/3)} \end{aligned} \tag{221}$$

The entry for $d_{\pi=5}$ becomes

$$5)\ d_{\pi=5}\ \pi \begin{pmatrix} 1\,2\,3 \\ 1\,3\,2 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & e^{+i(2\pi/3)} \\ e^{-i(2\pi/3)} & 0 \end{pmatrix} - \rightarrow T_{23} \tag{222}$$

The transition from $d_{\pi=5}$ to $d_{\pi=6}$, corresponding to the substitution $T_{23} \rightarrow T_{13}$ is achieved by complex conjugation of the matrix elements of $d_{\pi=5}$

$$6)\ d_{\pi=6}\ \pi \begin{pmatrix} 1\,2\,3 \\ 3\,2\,1 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & e^{-i(2\pi/3)} \\ e^{+i(2\pi/3)} & 0 \end{pmatrix} - \rightarrow T_{13} \tag{223}$$

9. Conclusions and outlook

In the present paper we calculated the density of states for baryons and compared it to our previous calculation for mesons.[17] The counting of baryons is done here for the first time in a complete fashion and for the mesons it has been published in a previous paper.[17] This calculation is valid in the limit of large N. It includes only oscillatory modes for u,d,s quarks and antiquarks for qqq baryons and $q\overline{q}$, i.e. is not including exotics and heavy quarks. Furthermore the here derived formulae do not include the intercept and assume a universal slope of the Regge trajectory. Within this approximation all Regge trajectory intercepts are neglected. This however does not affect the counting of the number of these states. For N=0 we find 36 states of mesons and 56 states of baryons as expected from more general Quark Model considerations for u,d,s quarks and antiquarks, and the 56 states of the baryon octet with spin=1/2 and the baryon decuplet with spin=3/2.

The density of states of baryons and mesons rise as a power law following the construction of oscillatory modes of quarks. We observe that the baryon density of states grows faster than the meson density and is always larger than the latter. The mass square density for qqq baryons and $q\overline{q}$ mesons within the oscillatory modes approach goes like a power determined by the number of independent oscillators and the leading powers are 5 for baryons and 2 for mesons.

The calculation predicts the density of baryonic states including the missing states and does not grow exponentially as in Hagedorn's model, with respect to N (or mass-square). The fact that there is no limiting temperature in the oscillatory modes approach does not mean that there is no critical temperature for the QCD phase transition .

The estimate of missing hadron states is important to take into account in thermal models that fit the hadron yields and ratios in heavy ion collisions to extract the

temperature and chemical potentials of the particle source. Such fits have been performed by many groups for example in references[36], [37], [38], [39] [40]
Furthermore, not only there are missing states of *qqbar* mesons and *qqq* baryons and their antiparticles, but probably also missing states of glueballs, tetraquarks, pentaquarks etc., which also influence the feeding and the thermal fits, while there are candidates for non excited glueball states, and strong candidates for tetraquarks and pentaquarks.
An aim for the future would be therefore to obtain main consequences from including the missing hadronic states for the description of thermal equilibrium as applying to the hadronic phase of QCD . A development of the current model would be to proceed along the path of counting oscillatory modes of heavy flavored qqq and $q\overline{q}$ hadrons as well as for multiplets for exotic states like tetraquarks, pentaquarks and glueballs.

10. Acknowledgments

We would like to warmly thank the members of the Theory Division of CERN, and especially Dr. Gian Giudice, Dr Wolfgang Lerche, Prof. Ignatios Antoniades and Dr Luis Alvarez-Gaume, as Theory group leaders, and Ms. Elena Gianolio, Ms. Anne-Marie Perrin, Ms. Michelle Connor and Ms. Jeanne Rostant for their support. We also thank Prof. Moshe Moshe from the Technion University in Haifa, Israel for the information on the paper [18] .

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