The two phase nature of QCD and its eventual root in gauge boson pair-condensation

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Abstract

**Gauge boson pair condensation is a key feature of the QCD ground state, characterized by vanishing flavor charges. Here QCD (QCD-QED) shall be restricted to the three light u,d,s flavors of quarks with realistic masses $m_{\ u,d,s}$,
and for the hadron resonance gas small $O\ (1\ \%)$ contributions from charm .
Critical thermodynamic properties are discussed for vanishing chemical potentials and related to coherence properties of QCD gauge bosons .**

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1 - Introduction

The analytical proof of a universal two phase thermal structure of QCD independent within an ample range of quark masses including their physical values and at vanishing chemical potentials is beyond our possibilitis at present. In this note we wish to discuss the consequences arising in this case whence the associated phase transition is of essentially second order with respect to the energy density.

We wish to consolidate in a constructive manner the results reported in ref. [1-2010] and to assess in detail the theoretical disagreement with lattice QCD in its present form , whence thermal properties under the same conditions are derived.

Material elaborated beyond ref. [1-2010] comprises

1) an extension of hadron states included in the evaluation of thermodynamic functions of a free hadron gas (-hg) and a careful check of the accurateness of our underlying programs in the region of temperature 150 MeV $\leq \; T \; \leq \; 700$ MeV .

2) a check of the stability of the critical temperature T_{cr} upon variation of the set of hadrons forming the free hadron gas, not changing the orders of the transition.

3) a theoretical derivation of periodicity boundary conditions leaving invariant the entire union over space-time (x) of local gauge groups $\bigcup_{\,x} \; (\, SU3 \,_c \,) \,_x$.

4) clarifications of the dynamical consequences of breaking of conformal symmetry both through the trace anomaly and spontaneously through the vacuum expected value of the local gauge boson bilinear $\langle \, \Omega \, | \;\; \frac{1}{4} (:) F^{a}_{\mu\nu} \; F^{\;\mu\nu\;a} (:) \; (x) \;\; | \, \Omega \, \rangle$ [2-1979] .

2 - The selected set of hadron resonances forming a noninteracting hadron gas Ntype 65 $\supset$ Ntype 26

The intensive thermodynamic state functions in QCD – pressure (p) , energy- , Gibbs- and entropy-density (ϱ_e , g , s) respectively, derive, for vanishing chemical potentials, from a single function of temperature or its inverse, including a priori general phases and interactions

(1) $\varrho_e (T) = - \partial_\beta g (T) \, , \; g (T) = \beta p (T) \; ; \; s (T) = \partial_T p (T) \; ; \; \beta = T^{-1}$

Ratios of hadron multiplicities have been successfully compared in heavy ion collisions at c.m. energies starting with Pb Pb collisions at the SPS $\sqrt{s_{NN}} \sim 20$ GeV per nucleon pair, [3]-[3-ctd.] , and at $\sqrt{s_{NN}} = 0.2$ TeV at RHIC, [4-2005] , with the chemical freeze out pertaining to a thermodynamic ensemble of noninteracting hadron resonances, depending on chemical potentials . We refer to the recent review by J. Cleymans, [5-2010] . Analyses of Pb Pb collisions at $\sqrt{s_{NN}} = 2.76$ TeV at the LHC, [6-2010] , are soon to follow.
Here we only discuss the extension of the input resonance spectrum in the programs originally developed for ref. [7-2001] , where the extrapolation to vanishing chemical potentials was systematically investigated up to temperatures of $150 - 170$ MeV . While chemical- and mechanical decoupling regimes indeed show energy densities justifying the neglct of interactions, this ceizes to be the case in the transition region , i.e. for temperatures above $T > \sim 170 - 220$ GeV . $\longrightarrow$

The enlarged set denoted Ntype 65 is formed by 175 meson and 148 baryon and antibaryon states , together 323 , counting all spin and charge states. It is described in tables 1-8 below and contains the smaller set Ntype 26 with 36 meson and 112 baryon and antibaryon states, used in ref. [1-2010] . The set Ntype 65 is exclusively used in the following.
More extensive lists including all resonance in the Particle Data Tables up to a mass of 2 or 2.5 GeV are currently used. In Ntype 65 we did not include higher L waves among 3 q for baryons and 3 $\overline{q}$ for antibaryons. This is justified by the increasing systematic errors resulting from the lack of taking interactions into account , whence the thermal properties of the noninteracting hadron resonance gas are extrapolated to temperatures beyond 200 MeV, always for vanishing chemical potentials.

The quantities energy density ϱ_e , pressure p divided by T^4 and the trace anomaly $(\varrho_e - 3p)/T^4$ are discussed in Apendix 1 . Comparison of the sets Ntype 65 and 26 shows that these thermal quantities depend very sensitively on the resonance sets, increasingly so whence the temperature reaches and exceeds 200 MeV , inducing additional relative errors of up to 50 % .

In Fig 9 in Appendix 1 we also compare pressure divided by T^4 for the sets Ntype 65 and 26 with the same quantity calculated on the lattice by the Wuppertal-Budapest collaboration, as reported in ref. [8-2010] . Despite the above reservations relative to the HRG extrapolation the agreement with the set Ntype 65 is quite satisfactory for 100 MeV $\leq T \leq 225 - 250$ MeV .

1-5 table 1 Ntype 1-10

nNtype(j)	bosons name & q.n.-s	#	fermions name & q.n.-s	#	total #
1 (1)	$\pi^{+,0,-}$, $I = 1$	3			
2 (2)	$K^{+,0}$, $I = \frac{1}{2}, S = +1$	2			
3 (3)	$\overline{K}^{-,0}$, $I = \frac{1}{2}, S = -1$	2			
4 (4)	η	1			
5 (26)	$\eta^{\prime}$	1			9 / 9
6 (15)	$\varrho^{+,0,-}$, $I = 1, J = 1$	9			
7 (16)	$K^{*\,+,0}$, $I = 1/2, S = +1, J = 1$	6			
8 (17)	$\overline{K}^{*\,-,0}$, $I = 1/2, S = -1, J = 1$	6			
9 (19)	ω , $J = 1$	3			
10 (18)	φ , $J = 1$	3			27 / 36

Table 1 : Pseudoscalar and vector meson $SU3_{u,d,s}$ nonets

nNtype(j)	bosons name & q.n.-s	#	fermions name & q.n.-s	#	total #
11 (5)			$\mathcal{N}^{\,+,0}\,,\ I=\frac{1}{2}, J=\frac{1}{2}, B=+1$	4	
12 (6)			$\overline{\mathcal{N}}^{\,-,0}\,,\ I=\frac{1}{2}, J=\frac{1}{2}, B=-1$	4	
13 (7)			$\Lambda\,,\ J=\frac{1}{2}, S=-1, B=+1$	2	
14 (8)			$\overline{\Lambda}\,,\ J=\frac{1}{2}, S=+1, B=-1$	2	
15 (9)			$\Sigma^{\,+,0,-}\,,\ I=1, J=\frac{1}{2}, S=-1, B=+1$	6	
16 (10)			$\overline{\Sigma}^{\,-,0,+}\,,\ I=1, J=\frac{1}{2}, S=+1, B=-1$	6	
17 (11)			$\Xi^{\,0,-}\,,\ I=\frac{1}{2}, J=\frac{1}{2}, S=-2, B=+1$	4	
18 (12)			$\overline{\Xi}^{\,0,+}\,,\ I=\frac{1}{2}, J=\frac{1}{2}, S=+2, B=-1$	4	32/68

Table 2 : Baryon and Antibaryon $SU3_{\,u,d,s}$ octets

1-7 table 3 Ntype 19-26

nNtype(j)	bosons name & q.n.-s	#	fermions name & q.n.-s	#	total #
19 (20)			$\Delta^{++,+,0,-}$, $I = \frac{3}{2}, J = \frac{3}{2}, B = +1$	16	
20 (21)			$\overline{\Delta}^{--,-,0,+}$, $I = \frac{3}{2}, J = \frac{3}{2}, B = -1$	16	
21 (22)			$\Sigma^{*+,0,-}$, $I = 1, J = \frac{3}{2}, S = -1, B = +1$	12	
22 (23)			$\overline{\Sigma}^{*-,0,+}$, $I = 1, J = \frac{3}{2}, S = +1, B = -1$	12	
23 (24)			$\Xi^{*0,-}$, $I = \frac{1}{2}, J = \frac{3}{2}, S = -2, B = +1$	8	
24 (25)			$\overline{\Xi}^{*0,+}$, $I = \frac{1}{2}, J = \frac{3}{2}, S = +2, B = -1$	8	
25 (13)			Ω^{-} , $J = \frac{3}{2}, S = -3, B = +1$	4	
26 (14)			$\overline{\Omega}^{+}$, $J = \frac{3}{2}, S = +1, B = -1$	4	80/148

Table 3 : Baryon and Antibaryon $SU3_{u,d,s}$ decuplets

The resonances listed in tables 1-3 form the set denoted Ntype=26 .

1-8 table 4 Ntype 27-30

nNtype	bosons name & q.n.-s	#	total #
27	$D^{+,0}$, $I = \frac{1}{2}, C = +1, c\overline{d}, c\overline{u}$	2	
28	$\overline{D}^{-,0}$, $I = \frac{1}{2}, C = -1, \overline{c}d, \overline{c}u$	2	
29	D_s^{+} , $S = +1, C = -1, c\overline{s}$	1	
30	$\overline{D}_s^{-}$, $S = -1, C = -1, \overline{c}s$	1	6/154

Table 4 : Charmed pseudoscalar mesons C=1 $SU3_{\,u,d,s}$ antitriplet and C=-1 $SU3_{\,u,d,s}$ triplet

1-9 table 5 Ntype 31-40

nNtype	fermions name & q.n.-s	#	total #
31	Λ_c^{+} , $B=+1, C=+1, J=\frac{1}{2}, udc$	2	
32	$\overline{\Lambda}_c^{\,-}$, $B=-1, C=-1, J=\frac{1}{2}, \overline{udc}$	2	
33	$\Xi_c^{+,0}$, $B=+1, C=+1, S=-1, I=\frac{1}{2}, J=\frac{1}{2}, usc, dsc$	4	
34	$\overline{\Xi}_c^{\,-,0}$, $B=-1, C=-1, S=+1, I=\frac{1}{2}, J=\frac{1}{2}, \overline{usc}, \overline{dsc}$	4	
35	$\Sigma_c^{++,+,0}$, $B=+1, C=+1, I=1, J=\frac{1}{2}, uuc, udc, ddc$	6	
36	$\overline{\Sigma}_c^{\,--,-,0}$, $B=-1, C=-1, I=1, J=\frac{1}{2}, \overline{uuc}, \overline{udc}, \overline{ddc}$	6	
37	$\Xi_c^{\prime\;+,0}$, $B=+1, C=+1, S=-1, I=\frac{1}{2}, J=\frac{1}{2}, usc, dsc$	4	
38	$\overline{\Xi}_c^{\,-,0}$, $B=-1, C=-1, S=+1, I=\frac{1}{2}, J=\frac{1}{2}, \overline{usc}, \overline{dsc}$	4	
39	Ω_c^{0} , $B=+1, C=+1, S=-2, J=\frac{1}{2}, ssc$	2	
40	$\overline{\Omega}_c^{\,0}$, $B=-1, C=-1, S=+2, J=\frac{1}{2}, \overline{ssc}$	2	36/190

Table 5 : S-wave $q\ q'\ c$ baryons and $\overline{q}\ \overline{q}'\ \overline{c}$ antibaryons with q , q' = u,d,s

1-10 table 6 Ntype 41-50

nNtype	bosons name & q.n.-s	#	total #
41	$f0(980)\ ,\ 0^{++},\ singlet$	1	
42	$f0(1500)\ ,\ 0^{++},\ octet$	1	
43	$a0(980)\ ^{+,0,-}\ ,\ 0^{++}, u\overline{d}, \frac{1}{\sqrt{2}}(u\overline{u} - d\overline{d}), d\overline{u}$	3	
44	$K_0{}^*(1430)\ ^{+,0}\ ,\ 0^{++}, u\overline{s}, d\overline{s}$	2	
45	$\overline{K}_0{}^*(1430)\ ^{-,0}\ ,\ 0^{++}, s\overline{u}, s\overline{d}$	2	
46	$f1(1285)\ ,\ 1^{++}$	3	
47	$f1(1420)\ ,\ 1^{++}$	3	
48	$a1(1260)\ ^{+,0,-}\ ,\ 1^{++}, u\overline{d}, \frac{1}{\sqrt{2}}(u\overline{u} - d\overline{d}), d\overline{u}$	9	
49	$K_1{}^*(1400)\ ^{+,0}\ ,\ 1^{++}, u\overline{s}, d\overline{s}$	6	
50	$\overline{K}_1{}^*(1400)\ ^{-,0}\ ,\ 1^{++}, u\overline{s}, d\overline{s}$	6	36/226

Table 6 : P-wave $q\overline{q}'$, $q = u, d, s$ **mesons ,** $SU3\ _{u,d,s}$ **nonets with** $J^{PC_n} = 0^{++}, 1^{++}$

1-11 table 7 Ntype 51-60

nNtype	bosons name & q.n.-s	#	total #
51	$f2(1270)\ ,\ 2^{++},\ nonstrange$	5	
52	$f2(1525)\ ,\ 2^{++},\ strange$	6	
53	$a2(1320)\ ^{+,0,-}\ ,\ 2^{++}, u\overline{d}, \frac{1}{\sqrt{2}}(u\overline{u}-d\overline{d}), d\overline{u}$	15	
54	$K_2^*(1430)\ ^{+,0}\ ,\ 2^{++}, u\overline{s}, d\overline{s}$	10	
55	$\overline{K}_2^*(1430)\ ^{-,0}\ ,\ 2^{++}, s\overline{u}, s\overline{d}$	10	
56	$h1(1170)\ ,\ 1^{+-}$	3	
57	$h1(1380)\ ,\ 1^{+-}$	3	
58	$b1(1235)\ ^{+,0,-}\ ,\ 1^{+-}, u\overline{d}, \frac{1}{\sqrt{2}}(u\overline{u}-d\overline{d}), d\overline{u}$	9	
59	$K_1^*(1270)\ ^{+,0}\ ,\ 1^{+-}, u\overline{s}, d\overline{s}$	6	
60	$\overline{K}_1^*(1270)\ ^{-,0}\ ,\ 1^{+-}, u\overline{s}, d\overline{s}$	6	72/298

Table 7 : P-wave $q\overline{q}'$, $q = u, d, s$ **mesons ,** $SU3\ _{u,d,s}$ **nonets with** $J^{PC_n} = 2^{++}, 1^{+-}$

1-12 table 8 Ntype 61-65

nNtype	bosons name & q.n.-s	#	total #
61	$gb\ ,\ 0^{++}$	1	
62	$gb\ ,\ 0^{+-}$	1	
63	$gb\ ,\ 2^{++}$	5	
64	$qg\overline{q}\ ,\ 1^{-+}, I = 1$	9	
65	$qg\overline{q}'\ ,\ 1^{-+}, I = 1$	9	25/323

Table 8 : glueballs and nonstrange $I = 1\ ,\ J^{PC_n} = 1^{-+}$ hybrids

The resonances listed in tables 1-8 form the set denoted Ntype=65 .

3 - Conditions of enveloping local gauge invariance, integrability of field strengths from connections and boundary values in eventual conflict with lattice QCD

Connections and associated field strengths pertaining to a local, compact and semi-simple gauge group $\mathcal{G}$ $(\rightarrow \ SU3_c$ for QCD $)$ shall be called complete, if extendable to the full ring of representations of potential matter fields , irrespective of the actual presence of such fields in the full gauge theory . The requirement of continuity with respect to space time derivatives shall apply to classical configurations, as substrate of path integrals.

Let a general irreducible unitary representation of $\mathcal{G}$ of dimension $dim\,(\,\mathcal{D}\,)$ be denoted $\mathcal{D}$ with

(2)
$$\begin{array}{l} (\,D\,(\,g\,)\,)_{\,\alpha\beta} \in \mathcal{D} \,:\, \alpha\,,\,\beta\,=\,1\,,\cdots,\,dim\,(\,\mathcal{D}\,) \quad ; \quad g\,\in\,\mathcal{G} \\ (\,D\,(\,g\,)\,)_{\,\alpha\beta} \,\rightarrow\, D\,(\,g\,) \end{array}$$

Then the $Lie\,(\,\mathcal{D}\,)$ associated connection is represented by the connection one form

(3)
$$\begin{array}{l} \left(\,W^{\,(1)}\,(\,\mathcal{D}\,)\,\right)_{\,\alpha\beta} \,=\, W_{\,\mu}^{\,r}\,(\,x\,)\,\left(\,d_{\,r}\,(\,\mathcal{D}\,)\,\right)_{\,\alpha\beta}\,d\,x^{\,\mu} \\ W_{\,\mu}^{\,r}\,(\,x\,) \,:\, \textbf{real} \quad ; \quad r\,=\,1,\,\cdots,\,dim\,(\,\mathcal{G}\,) \\ \left(\,W^{\,(1)}\,(\,\mathcal{D}\,)\,\right)_{\,\alpha\beta} \;\rightarrow\; W^{\,(1)}\,(\,\mathcal{D}\,) \;\rightarrow\; W^{\,(1)}\,(\,|_{\,\mathcal{D}}\,) \\ (\,d_{\,r}\,(\,\mathcal{D}\,)\,)_{\,\alpha\beta} \;\rightarrow\; d_{\,r}\,(\,\mathcal{D}\,) \;\rightarrow\; d_{\,r} \,=\, -\,d_{\,r}^{\,\dagger} \quad ; \quad r\,=\,1\,,\cdots,\,dim\,\mathcal{G} \end{array}$$

The antihermitian matrices $d_{\,r}$ in eq. 3 form a basis of $Lie\,(\,\mathcal{D}\,)$. $\rightarrow$

$Lie\,(\,\mathcal{D}\,)$ **is aligned with the adjoint representation** $Lie\,(\,\mathcal{G}\,)$ **as explained below, but is conceived in an apparently simpler context all by itself through the exponential mapping and its inverse**

(4)
$$D\,(\,g\,)\,D\,(\,h\,)\;=\;D\,(\,g\,.\,h\,)\;\;;\;\;D\,(\,g\,)\,D^{\,\dagger}\,(\,g\,)\;=\;\P\,|_{\,dim\,(\,\mathcal{D}\,)\,\times\,dim\,(\,\mathcal{D}\,)}$$
$$D\,(\,g\,)\;\rightarrow\;D\;\;;\;\;Det\,(\,D\,)\;=\;1$$

The unimodularity of the matrices D follows from the requirement that $\mathcal{G}$ **be semi-simple , i.e. be a direct product of simple factor groups, none of which contain any continuous normal subgroups .**
The exponential mapping associates the linear space of antihermitian matrices $\equiv\;Lie\,(\,\mathcal{D}\,)$ **with the set of representation matrices** $\{\,D\,\}\;\equiv\;\mathcal{D}$

(5)
$$(\,\omega\,)_{\,\alpha\beta}\;\rightarrow\;\widehat{\omega}\;\in\;Lie\,(\,\mathcal{D}\,)\;\;;\;\;\widehat{\omega}^{\,\dagger}\;=\;-\,\widehat{\omega}$$
$$D\;=\;\exp\;\widehat{\omega}\;\;;\;\;\widehat{\omega}\;=\;\omega^{\,r}\,d_{\,r}\;\;,\;\;\omega^{\,r}\;:\;\textbf{real}\;\;;\;\;r\;=\;1\,,\cdots,\,dim\,\mathcal{G}$$

The precise definition of the matrix valued quantity $\widehat{\omega}$ **introduced in eq. 5 is given in eq. 7 below.**
The exponential mapping $Lie\,(\,\mathcal{D}\,)\;\rightarrow\;\mathcal{D}$ **as defined in eq. 5 is embedded into the one parameter abelian subgroup of** $\mathcal{D}$ **as represented restricting to** $\mathcal{D}$ **by associating first the adjoint representation and then also** $\mathcal{D}$ **with the 'notion of motion'**

(6)
$$\omega\;\rightarrow\;\tau\,\omega\;\;;\;\;\tau\;:\;\textbf{real}\;\;\rightarrow\;\;D\,(\,\tau\,;\,\omega\,)\;=\;exp\,(\,\tau\,\widehat{\omega}\,)$$
$$D\,(\,\tau_{\,1}+\,\tau_{\,2}\,;\,\omega\,)\;=\;D\,(\,\tau_{\,1}\,;\,\omega)\,D\,(\,\tau_{\,2}\,;\,\omega\,)\;=\;D\,(\,\tau_{\,2}\,;\,\omega)\,D\,(\,\tau_{\,1}\,;\,\omega\,)\qquad\rightarrow$$

Interpreting the variable τ as representing the time development of the group element $g \in \mathcal{G}$ a differential eq. follows from eq. 6

$$\frac{d}{d\,\tau}\; D\,(\,\tau\,;\,\omega\,)\;=\;\frac{D\,(\,d\,\tau\,;\,\omega)\;-\;\P}{d\,\tau}\Bigg|_{d\tau\to 0}\qquad D\,(\,\tau\,;\,\omega\,)\;\equiv\;\widehat{\omega}\,D\,(\,\tau\,;\,\omega\,)$$

$$(7)\qquad \frac{D\,(\,d\,t\,;\,\omega)\;-\;\P}{d\,t}\;\equiv\;\frac{d}{d\,t}\,D\,(\,t\,;\,\omega\,)\Bigg|_{t=0}\qquad\equiv\;\widehat{\omega}\;=\;\omega^{\,r}\,d_{\,r}\;\in\;Lie\,(\,\mathcal{D}\,)$$

$$\widehat{\omega}\;=\;\widehat{\omega}\,(\,\mathcal{D}\,)\;=\;\dot{D}\,(\,0\,;\,\omega\,)\quad;\quad{}^{\cdot}\;=\;\frac{d}{d\,\tau\ \text{or}\ d\,t}$$

The relations in eqs. 5 - 7 define the real coordinates $\omega^{\,r}$ and antihermitian base matrices $d_{\,r}$ – both independent of τ – the latter forming the matrix valued structure $\widehat{\omega}\,(\,\mathcal{D}\,)$, identified for shortness of notation with $Lie\,(\,\mathcal{D}\,)$. The solution to the differential equation in eq. 7 with initial condition $D\,(\,0\,;\,\omega\,)\;=\;\P$ is given in eq. 6 .

3 a - Derivations from continuous coordinate transformation groups representing $\mathcal{G}$, fibre manifolds and irreducible submanifolds at the origin of conditions on complete connections

This section shall contain a most concise résumé of those notions inherent to the mathematics underlying compact semi-simple Lie groups as is necessary to infer conditions on field theoretical connections as announced in points 3) and 4) of the introduction, whereby most derivations are omitted. A minimum of historical and textbook references shall be given here [19-1951-1961] - [22-1963] . The results presented below are based on my treatment in ref. [23-2010] .

3 a-1 - general fibres $\longrightarrow$ irreducible ones $\simeq$ homogenous spaces [21-1962]

A general fibre manifold $\mathcal{F}$ (called $\mathcal{B}$ in ref. [23-2010]) with structure group $\mathcal{G}$ is required to allow a continuous representation of $\mathcal{G}$ by coordinate transformations of $\mathcal{F} \longrightarrow \mathcal{F}$. The set of these coordinate transformations shall be denoted $\mathcal{T}_{\mathcal{F}} = \left\{ \bigcup_{a} T_{a} \, ; \, a \in \mathcal{G} \right\}_{\mathcal{F}}$

$$\phi = \left(\phi^{1} , \cdots , \phi^{F} \right) \; ; \; F = dim \, (\, \mathcal{F} \,) \; : \; \text{coordinates on } \mathcal{F}$$

$$T_{a} \, \phi = \psi \, (\, \phi \, ; \, a \,) \; ; \; \psi^{j} = \psi^{j} \left(\phi^{1} , \cdots \phi^{F} ; a^{1} , \cdots , a^{G} \right) \tag{8}$$

$$j = 1 , \cdots , F \; ; \; G = dim \, \mathcal{G}$$

with suitable continuity / differentiability requirements for the functions $\psi \, (\, \phi \, ; \, a \,)$ defined in eq, 8 . $\longrightarrow$

The group property of $\mathcal{T}_{\mathcal{F}}$ then translates to, by the assciative property of coordinate transformations

(9) $$T_b\,(T_a\,\phi) = (T_b T_a)\,\phi\,;\; T_b T_a = T_{b\,a} \text{ with } b\,a = b\,.\,a \;:\; \text{group multiplication} \in \mathcal{G}$$

in coordinates : $\psi\,(\psi\,(\phi\,;\,a)\,;\,b) = \psi\,(\phi\,;\,b\,.\,a)\;;\;\psi\,,\,\phi \in \mathcal{F}\,;\,b\,,\,a \in \mathcal{G}$

The group transformation properties on $\mathcal{G}$ enter implicitely into the $\mathcal{T}_{\mathcal{F}}$ ones, as shown in eq. 9

(10) $$b\,.\,a = @\,(b\,,a)\;;\quad @^{\,\nu} = @^{\,\nu}\,(b^{\,1}\,,\cdots b^{\,G}\,;\,a^{\,1}\,,\cdots\,,\,a^{\,G})$$
$$\nu = 1\,,\cdots\,,\,G$$

We use the symbol @ (instead of c) to denote the G functions $@^{\,\nu}\,,\,\nu = 1\,,\cdots\,,\,G$ determining multiplication on $\mathcal{G}$ in order to freely use the symbols $a\,,\,b\,,\,c\,,\,\cdots$ for group elements $\in \mathcal{G}$.
We display here the eligibility of $\mathcal{G}$ as a special fiber manifold using left multiplication first, renaming the fiber variable h and the variables $a\,,\,b\,,\,c\,\cdots$ for $T \in \mathcal{T}_{\mathcal{G}\,L}$

(11) $$T_L \to T \in \mathcal{T}_{\mathcal{G}\,L} \to T_a\,h = @\,(a\,;\,h)$$
$$T_b\,(T_a\,h) = (T_b T_a)\,h = @\,(b\,;\,@\,(a\,;\,h))$$
$$@\,(b\,;\,@\,(a\,;\,h)) = b\,.\,@\,(a\,;\,h) = b\,.\,(a\,.\,h) = (b\,.\,a)\,.\,h$$

$\longrightarrow$

Clearly the choice $\mathcal{F} = \mathcal{G}$ is singled out, since there exists in this case, independently of the left multiplication transfomation group – $\mathcal{T}_{\mathcal{G}\,L}$ – also the right multiplication one – $\mathcal{T}_{\mathcal{G}\,R}$ with the associations

(12)
$$\begin{aligned}
& T_R \rightarrow T \in \mathcal{T}_{\mathcal{G}\,R} \rightarrow T_a\, h = @\left(h\,;\,a^{-1}\right) \\
& T_b\,(T_a\,h) = (T_b\,T_a)\,h = @\left(@\left(h\,;\,a^{-1}\right)\,;\,b^{-1}\right) \\
& @\left(@\left(h\,;\,a^{-1}\right)\,;\,b^{-1}\right) = @\left(h\,;\,a^{-1}\right).\,b^{-1} \\
& \qquad = \left(h\,.\,a^{-1}\right).\,b^{-1} = h\,.\,(b\,.\,a)^{-1}
\end{aligned}$$

Requiring an inversion symmetry on the tangent spaces of $\mathcal{F}$ – following ref. [21-1962] – allows to identify the irreducible parts of the fibre manifolds on which $\mathcal{T}_{\mathcal{F}}$ acts transitively to the (right or left -) coset spaces

(13)
$$\mathcal{F} \rightarrow \mathcal{F}_{irr} \rightarrow \mathcal{G}\,/\,\mathcal{H} \;\;;\;\; \mathcal{H} \;:\; \textbf{Lie subgroup of } \mathcal{G}$$

With the identification $\mathcal{F} \rightarrow \mathcal{F}_{irr}$ in eq. 13 and choosing right cosets for definitensess $\mathcal{T}_{\mathcal{F}}$ becomes

(14)
$$\begin{aligned}
& \mathcal{F} \ni \phi = h \sim h\,.\,h_R \;\forall\, h_R \in \mathcal{H} \subset \mathcal{G} \\
& \mathcal{T}_{\mathcal{F}} \ni T_a \;\;,\;\; \mathcal{T}_{\mathcal{G}\,L} \ni T'_a \;:\; T_a\,\phi = \left(T'_a\,h\right) \sim a\,.\,h\,.\,h_R \;\forall\, h_R \in \mathcal{H}
\end{aligned}$$

$\rightarrow$

The classification of fibre manifolds according to cosets $\mathcal{G} \,/\, \mathcal{H}$ as specified in eq. 13 allows a graduation of fibres $\mathcal{F}$:
regularity conditions for connections as described in eq. 3 compatible with $\mathcal{F}_{\,1} \;=\; \mathcal{G} \,/\, \mathcal{H}_{\,1}$ are not less restrictive than relative to $\mathcal{F}_{\,2} \;=\; \mathcal{G} \,/\, \mathcal{H}_{\,2}$ for $\mathcal{H}_{\,1} \;\subseteq\; \mathcal{H}_{\,2}$. We denote this graduation as

(15) $$W^{\,(1)}_{\,\mathcal{F}_{\,1}} \,(\,\mathcal{D}\,) \;\succeq\; W^{\,(1)}_{\,\mathcal{F}_{\,2}} \,(\,\mathcal{D}\,) \;\;\text{for}\;\; \mathcal{F}_{\,1} \,\succeq\, \mathcal{F}_{\,2} \;\;\equiv\;\; \mathcal{H}_{\,1} \subseteq \mathcal{H}_{\,2}$$

At this stage the connection 1-forms are still defined for a given matrix representation $Lie\,(\,\mathcal{D}\,)$. The direct anchoring of connections to the fibre manifolds $\mathcal{F}$ as considered in this subsection will be defined after its conclusion .
While $\mathcal{H}_{\,0} \;=\; \{\,¶\,\}$ – i.e. consisting only of the unit element of $\mathcal{G}$ – is strictly speaking not a Lie (sub)group of $\mathcal{G}$, we adjoin the group $\mathcal{G}$ as the maximal fibre manifold, with both tranformation groups $\mathcal{T}_{\,\mathcal{G}\;L}$ and $\mathcal{T}_{\,\mathcal{G}\;R}$, as defined in eqs. 10 - 12 . Accordingly we adjoin as unique discrete subgroup $\mathcal{H}_{\,0} \;=\; \{\,¶\,\}$ to the set of genuine Lie subgroups $\bigcup\,(\,\mathcal{H}\,)$. Then eq. 15 takes the form

(16) $$\mathcal{F}_{\,max} \;\rightarrow\; \mathcal{G} \;\;;\;\; \mathcal{F} \;=\; \mathcal{G}\,/\,\mathcal{H} \;\;\longrightarrow\;\; W^{\,(1)}_{\,\mathcal{G}} \,(\,\mathcal{D}\,) \;\succeq\; W^{\,(1)}_{\,\mathcal{F}} \,(\,\mathcal{D}\,) \;\;\forall\;\; \mathcal{F}$$

This subsection $(\,$3 a-1$\,)$ serves to define the selection of fibre manifolds and among them the maximal one according to eqs. 13 - 16 characterizing complete connection as introduced in eqs. 2 - 3 .

3 b - Ordered differentials : Killing fields on the group fibre manifold $\mathcal{G}$ with transformation group(s) $\mathcal{T}_{\mathcal{G}\ L(R)}$ and as derivations on associated induced representations

3 b-1 - adjoint representation from infinitesimal group coordinates and the Lie algebra

Coordinates of group elements of $\mathcal{G}$ in the sense of a classical manifold shall be denoted with the same symbol as the group elements as such.

(17) $$\mathcal{G} \rightarrow \mathcal{M}\,|_{\mathcal{G}} \;\sim\; h \in \mathcal{G} \rightarrow h = (h^{\nu}) = \left(h^{1}, \cdots, h^{G} \right) \Big|_{\mathcal{M}}$$

The suffix $\big|_{\mathcal{M}}$ in eq. 17 signals that h as coordinates on $\mathcal{M}$ are not unique. It shall be suppressed for simplicity with exceptions granted to avoid confusion .
The unit elements $e \rightarrow T_e \; (\doteq ¶)$ – with $T \in \mathcal{T}$ arbitrary – have the property

(18) $$e \,.\, a = a \,.\, e = a \;\leftrightarrow\; T_a T_e = T_e T_a = T_a \; (\forall\, a)$$

It is no loss of generality to assign the neutral element e the coordinates in $\mathcal{G}$

(19) $$e = \left(e^{1}, \cdots, e^{G} \right) \;;\; e^{\nu} = 0 \;,\; \nu = 1, \cdots, G$$

The infinitesimal neighbourhood of e forms the substrate of tangent space at e . Here we anchor the exponential mapping discussed with respect to a finite dimensional linear representation $\mathcal{D} (\mathcal{G}) \rightarrow Lie (\mathcal{D})$ defined in eqs. 4 - 7 in $Lie (\mathcal{G})$ $\longrightarrow$

with the substitutions

$$\omega \;=\; (\,\omega^{\,\nu}\,) \,\in\, \mathcal{G} \;\;\longrightarrow\;\; d\,\tau\,\omega \,\in\, \mathcal{G}\,|_{\,\mathcal{M}} \;\; ; \; d\,\tau\,\omega \;\;\longrightarrow\;\; T_{\,d\,\tau\,\omega} \,\in\, \mathcal{T} \;(\,\textbf{arbitrary}\,)$$

(20) $$\left. \frac{T_{\,d\,\tau\,\omega} \;-\; \P}{d\,\tau} \right|_{d\tau \to 0} \;\;\longrightarrow\;\; \widehat{\omega} \;=\; \widehat{\omega}\;(\,\mathcal{T}_{\,(1)}\,) \;=\; \omega^{\,\nu}\,\widehat{I}_{\,\nu} \;\; ; \;\; \widehat{I}_{\,\varrho} \,\in\, \mathcal{T}_{\,(1)}$$

$$\widehat{\omega} \;\;\longrightarrow\;\; \omega \,\in\, Lie\,(\,\mathcal{G}\,)\,|_{\,\mathcal{M}^{\,(1)}\,(\,e\,)}$$

In eq. 20 we have introduced two notions

1) $\mathcal{T}_{\,(1)}$ **: the set of differential quotients of transformations** $T_{\,h} \,\in\, \mathcal{T}_{\mathcal{G}}$ **.**

2) $\mathcal{M}^{\,(1)}\,(\,h\,)\,;\; h \,\in\, \mathcal{G}\,|_{\,\mathcal{M}}$ **: the tangent spaces of** $\mathcal{M}$ **at** $h \,\in\, \mathcal{G}$ **.**

All differential operations on the ordered fibre manifolds admitted as described in subsection 3 a-1 in particular on the group manifold itself are allowed by the condition of differentiablity – once for $\mathcal{T}_{\,(1)}$ **and** $\bigcup_{\,h}\,\mathcal{M}^{\,(1)}\,(\,h\,)$ **– imposed 'eo ipso' on fibres and Lie groups .**
With the definitions given in eqs. 17 - 20 we set out next to find the adjoint representation . $\longrightarrow$

The finite adjoint matrix representation denoted $\mathcal{A}$ arises directly from the group tranformations on $\mathcal{G} = \mathcal{M}$

$$d\,c = a\,.\,d\,\tau\,\omega\,.\,a^{-1}\,|_{\mathcal{M}} \quad ; \quad d\,c = d\,\tau \Big(\omega^{\,\prime} \,(\,a\,;\,\omega\,) \Big)$$

(21) $$\omega^{\,\prime\,\nu}\,(\,a\,,\,\omega\,) = Ad\,(\,a\,)^{\,\nu}{}_{\mu}\,\omega^{\,\mu} \quad ; \quad {}^{\nu}\,,\,{}_{\mu} = 1\,,\cdots,\,G \quad \rightarrow \quad \omega^{\,\prime} = Ad\,(\,a\,)\,\omega$$

$$Ad\,(\,b\,)\,Ad\,(\,a\,) = Ad\,(\,b\,.\,a\,)\,|_{\mathcal{M}} \quad ; \quad \mathcal{A} = \Big\{ \bigcup_{\,a\,\in\,\mathcal{G}} Ad\,(\,a\,) \Big\}$$

The real matrices $Ad\,(\,h\,)$ depend on the coordinates chosen on $\mathcal{M}$ and are for general choices not unitary, i.e. not orthogonal , but depend continuously on the coordinates h . The condition that $\mathcal{G}$ be semi-simple implies

(22) $$Det\;Ad\,(\,h\,) = 1 \;\; \forall\, h \in \mathcal{G}$$

At this point we extend the order of differentials to 2 which allows to introduce the Lie algebra .
To this end we use the exponential mapping for the operator valued quantities defined on the second line of eq. 20 ordering them as a pair

(23) $$\widehat{\omega}_{\,(k)} = (\,\omega^{\,\nu}\,)_{\,(k)}\,\widehat{I}_{\,\nu}\,,\;k = 1,2 \quad ; \quad \widehat{I}_{\,\varrho} \in \mathcal{T}_{(1)}$$

We associate an initially finite 'time like' quantity $\tau_{\,1}$ with $\widehat{\omega}_{\,(1)}$ and a first order differential $d\,\tau_{\,2}$ with $\widehat{\omega}_{\,(2)}$ $\longrightarrow$

and consider the equivalence exponential mapping and the differential equation with respect to τ_1 , valid for finite τ_1

$$\widehat{\Omega}(\tau_1) = \exp(\tau_1 \widehat{\omega}_{(1)}) \left(\P + d\tau_2 \widehat{\omega}_{(2)} \right) \exp(-\tau_1 \widehat{\omega}_{(1)})$$

(24)
$$\frac{d}{d\tau_1} \widehat{\Omega}(\tau_1) = \exp(\tau_1 \widehat{\omega}_{(1)}) \, d\tau_2 \left[\widehat{\omega}_{(1)}, \widehat{\omega}_{(2)} \right] \exp(-\tau_1 \widehat{\omega}_{(1)})$$

$$\left[\widehat{\omega}_{(1)}, \widehat{\omega}_{(2)} \right] = \widehat{\omega}_{(1)} \widehat{\omega}_{(2)} - \widehat{\omega}_{(2)} \widehat{\omega}_{(1)}$$

In eq. 24 the operator valued commutator $\left[\widehat{\omega}_{(1)}, \widehat{\omega}_{(2)} \right]$ naturally appears .

Next we let also τ_1 become infinitesimal $\tau_1 \rightarrow d\tau_1$ and expand $\widehat{\Omega}(d\tau_1)$ up to second order in the differentials $d\tau_k \, ; \, k = 1, 2$

(25)
$$\widehat{\Omega}(d\tau_1) = \left\{ \begin{array}{c} \left(\P + d\tau_1 \widehat{\omega}_{(1)} + \frac{1}{2} (d\tau_1)^2 \widehat{\omega}^2_{(1)} \right) \times \\ \times \left(\P + d\tau_2 \widehat{\omega}_{(2)} \right) \times \\ \times \left(\P - d\tau_1 \widehat{\omega}_{(1)} + \frac{1}{2} (d\tau_1)^2 \widehat{\omega}^2_{(1)} \right) \end{array} \right\} = \left\{ \begin{array}{c} \P + \\ d\tau_1 \, d\tau_2 \left[\widehat{\omega}_{(1)}, \widehat{\omega}_{(2)} \right] \end{array} \right\}$$

The absence of a term $\propto (d\tau_1)^2$ on the right hand side of eq. 25 is due to the exponential mapping . $\longrightarrow$

It follows that the finite commutator $\left[\,\widehat{\omega}_{(1)}\,,\,\widehat{\omega}_{(2)}\,\right]$ upon the combined exponential mapping, defined in eq. 26 below, generates a commuting one parameter family of transformations . by the semi-simple condition on $\mathcal{G}$ a one parameter commutative subgroup of $\mathcal{G}$

(26)
$$T\,(\,\vartheta\,)\;=\;T_{\,h\,(\,\vartheta\,)}\;=\;\exp\,\left(\,\vartheta\,\left[\,\widehat{\omega}_{(1)}\,,\,\widehat{\omega}_{(2)}\,\right]\,\right)\quad;\quad h(\,\vartheta\,)\,\in\,\mathcal{G}\,\forall\,\vartheta\quad\textbf{with}$$
$$h(\,\vartheta_{\,1}\,+\,\vartheta_{\,2}\,)\;=\;h\,(\,\vartheta_{\,1}\,)\,.\,h\,(\,\vartheta_{\,2}\,)\;=\;h\,(\,\vartheta_{\,2}\,)\,.\,h\,(\,\vartheta_{\,1}\,)$$

The Lie algebra relation follows, first on the level of all transformation groups $\mathcal{T}_{\,\mathcal{F}}$ defined in section 3 a

$$\left[\,\widehat{\omega}_{(1)}\,,\,\widehat{\omega}_{(2)}\,\right]\;=\;\omega^{\,\varrho}_{\,(1)}\,\omega^{\,\sigma}_{\,(2)}\,\left[\,\widehat{I}_{\,\varrho}\,,\,\widehat{I}_{\,\sigma}\,\right]\quad;\quad\widehat{I}_{\,\varrho}\,,\,\widehat{I}_{\,\sigma}\,\in\,\mathcal{T}_{\,(1)}\;\leftrightarrow\textbf{eq. 20}$$

$$\downarrow$$

(27)
$$\left[\,\widehat{I}_{\,\varrho}\,,\,\widehat{I}_{\,\sigma}\,\right]\;=\;f^{\,\chi}{}_{\,\varrho\,\sigma}\,\widehat{I}_{\,\chi}$$

$$f^{\,\chi}{}_{\,\varrho\,\sigma}\;=\;-\,f^{\,\chi}{}_{\,\sigma\,\varrho}\quad:\quad\textbf{structure constants of }\mathcal{G}\quad;\quad\chi\,,\,\varrho\,,\,\sigma\;=\;1\,,\,\cdots\,,\,G$$

Clearly, in order to implement the operator relations on the level of $\mathcal{T}_{\,\mathcal{G}\;L\;(R)}$ to the coordinate multiplication as given by the functions $@\,(\,b\,;\,a\,)\;=\;b\,.\,a$ for $b\,,a\,\in\,\mathcal{G}$ as defined in eq. 10 at least twofold partial differentiability of $@$ is necessary . $\longrightarrow$

This further becomes threefold partial differentiability to safeguard the operator Jacobi identity a necessary condition to ensure regular convergence of nested exponential mappings of all allowed transformation groups $\mathcal{T}_{\mathcal{F}}$

$$\left[\widehat{I}_{\alpha} , \left[\widehat{I}_{\beta} , \widehat{I}_{\gamma} \right] \right] + (\alpha\beta\gamma \rightarrow \gamma\alpha\beta) + (\alpha\beta\gamma \rightarrow \beta\gamma\alpha) = 0 \;\; \forall \;\; \alpha, \beta, \gamma$$

(28) $$\longrightarrow \quad f^{\sigma}{}_{\alpha\varrho} f^{\varrho}{}_{\beta\gamma} + f^{\sigma}{}_{\gamma\varrho} f^{\varrho}{}_{\alpha\beta} + f^{\sigma}{}_{\beta\varrho} f^{\varrho}{}_{\gamma\alpha} = 0$$

$$f^{\sigma}{}_{\alpha\gamma} \doteq (\xi_{\alpha})^{\sigma}{}_{\gamma}$$

With the matrix substitutions $(\xi_{\alpha})^{\sigma}{}_{\gamma} \rightarrow \xi_{\alpha} \;\; ; \;\; {}^{\sigma} , \alpha , \gamma = 1 , \cdots , G$ **the relation on the second line of eq. 28 can be cast into the form**

(29) $$\Big([\xi_{\alpha} , \xi_{\beta}] = f^{\chi}{}_{\alpha\beta} \xi_{\chi} \Big)^{\sigma}{}_{\gamma} \;\; \sim \;\; \left[\widehat{I}_{\alpha} , \widehat{I}_{\beta} \right] = f^{\chi}{}_{\alpha\beta} \widehat{I}_{\chi} \;\; \leftrightarrow \;\; \textbf{eq. 27}$$

The equivalence relation in eq. 29 $\xi_{\alpha} \sim \widehat{I}_{\alpha}$ **allows to reconstruct the adjoint representation** $\mathcal{A}$ **defined in eqs. 21 - 22 by the finite dimensional exponential mappings**

$$(Ad (h (\tau ; \omega)) = \exp (\tau \omega^{\alpha} \xi_{\alpha}))^{\sigma}{}_{\gamma} \;\; ; \;\; h (\tau ; \omega) \in \mathcal{G}$$

(30) $$h (\tau_1 + \tau_2 ; \omega) = h (\tau_1 ; \omega) . h (\tau_2 , \omega)$$

$\omega \in$ **tangent space of** $\mathcal{G}$ **at** e

$\rightarrow$

Coordinates on $\mathcal{G}$, for compact, semi-simple $\mathcal{G}$ can be chosen such, that all finite dimensional representations $\mathcal{D} \left(\mathcal{G} \right)$ are unitary, unimodular as shown (e.g.) in refs. [19-1951-1961] , [23-2010] .
From the present (sub-) perspective we then infer from eq. 30

(31)
$$
\begin{array}{l}
h \ = \ h \left(\tau \, ; \, \omega \right) \ \rightarrow \ Ad \left(h \right) \left(Ad \left(h \right) \right)^{T} \ = \ ¶_{G \times G} \ \longrightarrow \\
\left(\xi_{\alpha} \right)^{\gamma}{}_{\beta} \ = \ - \left(\xi_{\alpha} \right)^{\beta}{}_{\gamma} \quad ; \quad \left(\xi_{\alpha} \right)^{\gamma}{}_{\beta} \ = \ f^{\gamma}{}_{\alpha \beta}
\end{array}
$$

In eq. 31 ^{T} denotes matrix transposition.
The identifications in the relations on the second line of eq. 31 justify the substitutions – after unitarity of all finite dimensional representations of $\mathcal{G}$ is achieved –

(32)
$$
\left(\xi_{\alpha} \right)^{\gamma}{}_{\beta} \rightarrow \left(\xi_{\alpha} \right)_{\gamma \beta} \quad \textbf{and} \quad f^{\gamma}{}_{\alpha \beta} \rightarrow f_{\gamma \alpha \beta}
$$

With the substitutions defined in eq. 32 we rewrite eq. 31

(33)
$$
\begin{array}{lll}
\left(\xi_{\alpha} \right)_{\gamma \beta} \ = \ - \left(\xi_{\alpha} \right)_{\beta \gamma} \quad \longleftrightarrow \quad f_{\gamma \alpha \beta} & = & - f_{\beta \alpha \gamma} \\
 & & - f_{\gamma \beta \alpha}
\end{array}
$$

The commutator definition in eq. 27 yields the second relation for the structure constants in eq. 33 .
As a consequence of the two independent antisymmetry conditions for the structure constants $f_{\alpha \beta \gamma}$ the latter are totally antisymmetric with respect to their three indices . $\longrightarrow$

Recapitulation of characteristics of the Lie algebra and adjoint representation (eqs. 31 - 33)

The orthogonal, adapted adjoint representation is rewritten in 3 equations below, using the (standard) symbols or definitions

$$(\xi_\alpha)_{\gamma\,\beta} \doteq \left(ad_{(\alpha)} \right)_{\gamma\,\beta} = f_{\gamma\alpha\beta}$$

$$ad(\omega) = \omega^{\varrho}\, ad_{(\alpha)} \;\; ; \;\; \omega = \left(\omega^{1}, \cdots, \omega^{G} \right) \in \text{ tangent space of } \mathcal{G} \text{ at } e$$

(34) $$\widehat{\omega} = \omega^{\varrho}\, \widehat{I}_{\varrho} \;\; ; \;\; \widehat{\omega} \in \mathcal{T}_{(1)} \text{ relative to general } \mathcal{T}_{\mathcal{G}} \text{ as defined in eq. 20}$$

$$\mathcal{D} : d(\omega) = \omega^{\varrho}\, d_{(\varrho)} \in Lie(\mathcal{D}) \;\; d_{\varrho} \rightarrow d_{(\varrho)} \text{ from eq. 3}$$

$$d(\omega) = \left(d(\omega) \right)_{rs} \;\; ; \;\; r, s = 1, \cdots, dim(\mathcal{D})$$

Lie algebra relations depend for all representations on the universal structure constants of $\mathcal{G}$

$$\left[ad_{(\alpha)}, ad_{(\beta)} \right] = f_{\alpha\beta\,\gamma}\, ad_{(\gamma)} \;\; , \;\; \left[d_{(\alpha)}, d_{(\beta)} \right] = f_{\alpha\beta\,\gamma}\, d_{(\gamma)}$$

(35) $$\left[\widehat{I}_{\alpha}, \widehat{I}_{\beta} \right] = f_{\alpha\beta\gamma}\, \widehat{I}_{\gamma}$$

$f_{\alpha\beta\gamma}$: **totally antisymmetric with respect to $\alpha\beta\,\gamma$**

$\longrightarrow$

Respective exponential mapping on the adjoint representation $\mathcal{A}$, on a general finite dimensional unitary representation $\mathcal{D}$ as well as on all operator valued transformation group representation $\mathcal{T}_{\mathcal{G}}$ determine on $\mathcal{G}$ a commuting one parameter family of group elements $h(\tau;\omega) \in \mathcal{G}$

$$\left\{ \begin{array}{c} \mathcal{A} \\ \mathcal{D} \\ \mathcal{T}_{\mathcal{G}} \end{array} \right\} \quad h(\tau;\omega) \leftarrow \left\{ \begin{array}{lcl} \exp(\tau\, ad(\omega)) & = & Ad(h(\tau;\omega)) \\ \exp(\tau\, d(\omega)) & = & D(h(\tau;\omega)) \\ \exp(\tau\, \widehat{\omega}) & = & T_{h(\tau;\omega)} \end{array} \right\} \tag{36}$$

$$h(\tau_1 + \tau_2;\omega) = h(\tau_1;\omega) \,.\, h(\tau_2;\omega)$$

The construction of this family – $h(\tau;\omega) \in \mathcal{G}$ – indicated as $h(\tau;\omega) \leftarrow$ in eq. 36 corresponds to a system of G first order differential equations with respect to τ, subject of the subsection 3 b-2 below .

We recall that up to this point threefold partial differentiability is required as regularity condition on the group transformation functions $(b\,.\,a)^{\nu} = @^{\nu}\left(b^1, \cdots b^G; a^1, \cdots, a^G\right)$.

This ends the present subsection $($3 b-1$)$.

3 b - Killing fields on the group fibre manifold $\mathcal{G}$ with transformation group(s) $\mathcal{T}_{\mathcal{G}\ L(R)}$ – continued

An excellent exposition can be found in ref. [18-1941-1986] .

Here we follow the thread laid out in ref. [23-2010] starting from eqs. 10 , 11 defining the action of $T_a \in \mathcal{T}_{\mathcal{G}\ L}$ on the group manifold $\mathcal{G}$, repeated below

(37)
$$\begin{aligned} & T_a \in \mathcal{T}_{\mathcal{G}\ L} \ ; \ a \, , h = (a^{\nu}) \, , (h^{\nu}) \ ; \ \nu = 1, \cdots , G \in \mathcal{G} \ \rightarrow \\ & T_a \, h = a \, . \, h = @ \, (a \, ; h) \ \longleftrightarrow \ (a \, . \, h)^{\nu} = @^{\nu} \, (a^{\chi} \, ; h^{\varrho}) \\ & T_b \, (T_a \, h) = (T_b T_a) \, h = @ \, (b \, ; @ \, (a \, ; h)) \\ & @ \, (b \, ; @ \, (a \, ; h)) = b \, . \, @ \, (a \, ; h) = b \, . \, (a \, . \, h) = (b \, . \, a) \, . \, h \end{aligned}$$

The differential quotient in eq. 20 for $\mathcal{T}_{(1)} \, (\mathcal{T}_{\mathcal{G}\ L})$ defines left multiplication Killing fields on $\mathcal{G}$

$$\omega = (\omega^{\nu}) \in \mathcal{G} \ \longrightarrow \ d \tau \, \omega \in \mathcal{G} \, |_{\mathcal{M}} \ ; \ d \tau \, \omega \ \longrightarrow \ T_{d \tau \omega} \in \mathcal{T}_{\mathcal{G}\ L}$$

(38)
$$\left. \frac{T_{d \tau \omega} - ¶}{d \tau} \right|_{d\tau \rightarrow 0} \longrightarrow \ \widehat{\omega} = \widehat{\omega} \, (\mathcal{T}_{(1)}) = \omega^{\nu} \, \widehat{I}_{\nu} \ ; \ \widehat{I}_{\varrho} \in \mathcal{T}_{(1)} \, (\mathcal{T}_{\mathcal{G}\ L})$$

$$\widehat{\omega} \ \rightarrow \ \omega \in Lie \, (\mathcal{G}) \, |_{\mathcal{M}^{(1)} \, (e)}$$

$$\left(\widehat{I}_{\varrho} \, h \right)^{\chi} = (u_{(\varrho)})^{\chi} \, (h) = \partial_{a^{\varrho}} @^{\chi} \, (a \, ; h) \, |_{a = 0}$$

→

We use up to second order differentials as given in eq. 23 in the left multiplication order

(39)
$$\widehat{\omega}_{(k)} = (\omega^{\nu})_{(k)} \widehat{I}_{\nu}, \; k = 1,2 \; ; \; \widehat{I}_{\varrho} \in \mathcal{T}_{(1)}(\mathcal{T}_{\mathcal{G}\,L})$$
$$d\tau_2\,\omega_{(2)} = d\tau_2\,(\omega^{\nu})_{(2)} \to d\tau_1\,\omega_{(1)} = d\tau_1\,(\omega^{\nu})_{(1)}$$

The last relation in eq. 37 then becomes , upon identifying the first order differentials
$b \leftrightarrow d\,b\,;\, a \leftrightarrow d\,a$

(40)
$$@(b\,;\,@(a\,;\,h)) = b\,.\,@(a\,;\,h) \longrightarrow b = d\tau_2\,\omega_{(2)} \;\; ; \;\; a = d\tau_1\,\omega_{(1)}$$
$$\begin{aligned}(b\,.\,a\,.\,h)^{\nu} &= @^{\nu}\Big(d\tau_2\,(\omega_{(2)})^{\beta}\,;\, h^{\alpha} + d\tau_1\,\omega^{\chi}_{(1)}\,(u_{(\chi)}(h))^{\alpha}\Big) \\ &= h^{\nu} + d\tau_1\,\omega^{\chi}_{(1)}\,u^{\nu}_{(\chi)}(h) \\ &\qquad + d\tau_2\,\omega^{\chi}_{(2)}\,u^{\nu}_{(\chi)}\Big(h + d\tau_1\,\omega^{\psi}_{(1)}\,u_{(\psi)}(h)\Big) \\ &= h^{\nu} + \big(d\tau_1\,\omega_{(1)} + d\tau_2\,\omega_{(2)}\big)^{\chi}\,u^{\nu}_{(\chi)}(h) \\ &\qquad + \Big(d\tau_2\,d\tau_1\,\omega^{\chi}_{(2)}\,\omega^{\psi}_{(1)}\Big)\,u^{\varrho}_{(\psi)}(h)\,\partial_{\xi^{\varrho}}\,u^{\nu}_{(\chi)}(\xi)\Big|_{\xi = h}\end{aligned}$$

$\longrightarrow$

We rewrite the last relation in eq. 40 to render the structure of sequential differential orders, denoted $[\,.\,]_{0,1,\cdots}$, up to second order more transparent

$$(b \sim db = d\tau_2\,\omega_{(2)})^{\beta} \;,\; (a \sim da = d\tau_1\,\omega_{(1)})^{\alpha} \;;\; \beta, \alpha = 1, \cdots, G$$

$$(b\,.\,a\,.\,h)^{\nu} = (b\,.\,a\,.\,h)^{\nu}_{0} + (b\,.\,a\,.\,h)^{\nu}_{1} + (b\,.\,a\,.\,h)^{\nu}_{2} + \cdots$$

---

(41)
$$\begin{aligned} (b\,.\,a\,.\,h)^{\nu}_{0} &= h^{\nu} \\ (b\,.\,a\,.\,h)^{\nu}_{1} &= \left(d\tau_1\,\omega_{(1)} + d\tau_2\,\omega_{(2)}\right)^{\chi} u^{\nu}_{(\chi)}(h) \\ (b\,.\,a\,.\,h)^{\nu}_{2} &= d\tau_2\, d\tau_1 \left(\omega_{(2)}\right)^{\beta} \left(\omega_{(1)}\right)^{\alpha} L^{\nu}{}_{\beta\,\alpha}(h) \end{aligned}$$

$$L^{\nu}{}_{\beta\,\alpha}(h) = \left(u_{(\alpha)}\right)^{\varrho} \partial_{\xi^{\varrho}} \left(u_{(\beta)}\right)^{\nu}(\xi)\Big|_{\xi = h}$$

It follows from eq. 41 that the zeroth and first orders of $(b\,.\,a\,.\,h)^{\nu}$ are symmetric under the exchange $\omega_{(2)} \leftrightarrow \omega_{(1)}$, whereas the second order is not .

We further note the relation

(42)
$$\begin{aligned} X^{\nu}{}_{\varrho\,\beta} &= \partial_{\xi^{\varrho}} \left(u_{(\beta)}\right)^{\nu}(\xi)\Big|_{\xi = h} = \partial_{\xi^{\varrho}} \partial_{\xi^{\beta}} @^{\nu}(\xi)\Big|_{\xi = h} = X^{\nu}{}_{\beta\,\varrho} \\ X^{\nu}{}_{\varrho\,\beta} &= X^{\nu}{}_{\varrho\,\beta}(h) \end{aligned}$$

$\longrightarrow$

From eq. 41 we infer the Lie algebra relation for $\mathcal{T}_{\mathcal{G}\,L}$

$$L^{\nu}{}_{\beta\,\alpha}(h) - L^{\nu}{}_{\alpha\,\beta}(h) = f^{\gamma}{}_{\beta\alpha}\left(u_{(\gamma)}(h)\right)^{\nu} \quad \longrightarrow$$

(43) $$\left(u_{(\alpha)}(\xi)\right)^{\varrho}\partial_{\xi\,\varrho}\left(u_{(\beta)}(\xi)\right)^{\nu} - (\alpha \leftrightarrow \beta) = f_{\gamma\beta\alpha}\left(u_{(\gamma)}(\xi)\right)^{\nu}$$

upon the substitutions : $h \to \xi \;\; ; \;\; f^{\gamma}{}_{\beta\alpha} \to f_{\gamma\beta\alpha}$

We assume that coordinates on $\mathcal{G}$ are adapted such that the structure constants allow the substitution $f^{\gamma}{}_{\beta\alpha} \to f_{\gamma\beta\alpha}$ in eq. 43 and become totally antisymmetric – as discussed in subsection 3b-1 previously $\left(\text{eqs. 31 - 33}\right)$.

The induced representation [20-1953] operating on a collection of functions $\varphi(\xi)\,;\,\xi \in \mathcal{G}$ relative to $\mathcal{T}_{\mathcal{G}\,L}$ shall be denoted $\blacktriangleright\,\mathcal{T}_{\mathcal{G}\,L}$ and the transformations forming the induced representation as $\blacktriangleright\,T_{a}$

$$\blacktriangleright\,T_{a} \in \blacktriangleright\,\mathcal{T}_{\mathcal{G}\,L} \quad \longleftrightarrow \quad T_{a} \in \mathcal{T}_{\mathcal{G}\,L} \;\; ; \;\; a\,,\xi \in \mathcal{G} \quad \to$$

$\blacktriangleright\,T_{a}\,\varphi(\xi) = \varphi(a^{-1}\,.\,\xi)$; with the multiplication

$$\blacktriangleright\,T_{b}\,(\blacktriangleright\,T_{a}\,\varphi)\,(\xi) = \varphi\left(a^{-1}\,.\,b^{-1}\,.\,\xi\right) = \varphi\left((b\,.\,a)^{-1}\,.\,\xi\right) = \blacktriangleright\,T_{b\,.\,a}\,\varphi(\xi)$$

---

(44) $$\longrightarrow \quad \blacktriangleright\,T_{b}\;\blacktriangleright\,T_{a} = \blacktriangleright\,T_{b\,.\,a}$$

$\rightarrow$

Then , using eq. 38 to define

(45) $$\widehat{\omega} = \widehat{\omega} \left(\blacktriangleright \mathcal{T}_{(1)} (\blacktriangleright \mathcal{T}_{\mathcal{G}\ L}) \right) = \omega^{\nu} \widehat{I}_{\nu} \ ; \ \widehat{I}_{\varrho} \in \blacktriangleright \mathcal{T}_{(1)} (\blacktriangleright \mathcal{T}_{\mathcal{G}\ L})$$

we obtain $\widehat{I}_{\varrho}$ as a derivation with respect to the Killing fields pertaining to $\mathcal{T}_{\mathcal{G}\ L}$ multiplied with -1

(46) $$\widehat{I}_{\varrho} = - \left(u_{(\varrho)} (\xi) \right)^{\nu} \partial_{\xi^{\nu}} \ ; \ \left[\widehat{I}_{\alpha} , \widehat{I}_{\beta} \right] = f_{\alpha\beta\gamma} \widehat{I}_{\gamma}$$

$$\left[\widehat{I}_{\alpha} , \widehat{I}_{\beta} \right] = \left[\ \left(u_{(\alpha)} \right)^{\varrho} \left(\partial_{\xi^{\varrho}} \left(u_{(\beta)} \right)^{\nu} \right) - (\alpha \leftrightarrow \beta) \ \right] \partial_{\nu}$$

In eq. 46 the derivation operators are implied to act from the left on a function $\varphi (\xi)$, which is suppressed for simplification of notation .

The expression in brackets in the last relation in eq. 46 agrees with the one derived with respect to $\mathcal{T}_{\mathcal{G}\ L}$ given in eq. 43 , verifying the universal nature of the structure constants .

The collection of functions $\varphi (\xi)$ is understood to form the Hilbert space over the left and right invariant Haar measure [15-1933] with respect to $\mathcal{T}_{\mathcal{G}\ L\ \&\ R}$

$$\left\{ \bigcup_{\varphi} \varphi (\xi) \right\} \sim \mathcal{H} (\mathcal{G}) = L_{2} \left[\mathcal{G} ; (d \mu)_{Haar} (\xi)) \right]$$

(47) allowing both induced representations : $\blacktriangleright \mathcal{T}_{\mathcal{G}\ L}$ and $\blacktriangleright \mathcal{T}_{\mathcal{G}\ R}$

$$L \rightarrow R : u_{(\varrho)} \rightarrow v_{(\varrho)} \text{ with } \left(v_{(\varrho)} (\xi) \right)^{\nu} = - \partial_{a^{\varrho}} @^{\nu} (\xi , a) |_{a=0}$$

In eq. 47 $v_{(\varrho)}$ denote the Killing vector fields pertaining to $\mathcal{T}_{\mathcal{G}\ R}$. $\rightarrow$

Invariance of the Haar measure [15-1933] with respect to $\mathcal{T}_{\mathcal{G}\;L\;\&\;R}$

(48) $$(\,d\,\mu\,)_{\,Haar}\,(\,\xi\,)\;=\;(\,d\,\mu\,)_{\,Haar}\,(\,a\,.\,\xi\,)\;=\;(d\,\mu\,)_{\,Haar}\,(\,\xi\,.\,b)\quad;\quad a\,,\,b\;\in\;\mathcal{G}$$

sets the stage for the operators $T_{\,a}\;\in \blacktriangleright\;\mathcal{T}_{\mathcal{G}\;L\;\&\;R}$ to become unitary ones in $\mathcal{H}$ as defined in eq. 47 and in turn through the invere exponential mapping using eq. 45 and the definitions in eq. 36

(49) $$\begin{aligned} &T_{\,h\,(\,\tau\,;\,\omega\,)}\;=\;\exp\,(\,\tau\,\widehat{\omega}\,)\;\in\;\blacktriangleright\;\mathcal{T}_{\mathcal{G}\;L\;\textbf{or}\;R}\\ &\widehat{\omega}\;=\;\omega^{\,\nu}\,\widehat{I}_{\,\nu}\quad;\quad\widehat{I}_{\,\nu}\;\in\;\blacktriangleright\;\mathcal{T}_{(1)}\,(\,\blacktriangleright\;\mathcal{T}_{\mathcal{G}\;L\;\textbf{or}\;R}\,)\;\longrightarrow\\ &T_{\,h}\,T_{\,h}^{\,\dagger}\;=\;\P\,|_{\,\mathcal{H}}\quad;\quad\widehat{I}_{\,\nu}\;=\;-\,\widehat{I}_{\,\nu}^{\,\dagger}\quad,\quad\textbf{with}\;\;^{\dagger}\;\;:\;\textbf{self adjoint conjugation in}\;\;\mathcal{H}\end{aligned}$$

In eq. 49 self adjoint conjugation refers to the hermitian scalar product in $\mathcal{H}$

(50) $$\zeta\,,\,\varphi\;\in\;\mathcal{H}\;:\;\langle\,\zeta\,,\,\varphi\,\rangle\;=\;\int(\,d\,\mu\,)_{\,Haar}\,(\,\xi\,)\;[\,\zeta^{\,*}\,(\,\xi\,)\,\varphi\,(\,\xi\,)\,]$$

Distinguishing explicitely $T_{\,a\;L}$ and $T_{\,a\;R}$, $\widehat{I}_{\,\nu\;L}$ and $\widehat{I}_{\,\nu\;R}$ as defined for the induced respective representations in $\mathcal{H}$, eq. 49 is extended below to include their Lie algebra relations

(51) $$\begin{aligned} &T_{\,h\;L}\,T_{\,h\;L}^{\,\dagger}\;=\;T_{\,h\;R}\,T_{\,h\;R}^{\,\dagger}\;=\;\P\,|_{\,\mathcal{H}}\quad;\quad\widehat{I}_{\,\nu\;L}\;=\;-\,\widehat{I}_{\,\nu\;L}^{\,\dagger}\quad;\quad\widehat{I}_{\,\nu\;R}\;=\;-\,\widehat{I}_{\,\nu\;R}^{\,\dagger}\\ &\left[\,\widehat{I}_{\,\alpha\;K}\,,\,\widehat{I}_{\,\beta\;K}\,\right]\;=\;f_{\,\alpha\beta\gamma}\,\widehat{I}_{\,\gamma\;K}\;;\;K\;=\;L,R\;\;;\;\;\left[\,\widehat{I}_{\,\alpha\;L}\,,\,\widehat{I}_{\,\beta\;R}\,\right]\;=\;0\end{aligned}$$

In eqs. 44 - 51 the properties of the induced representations, denoted $\blacktriangleright\;\mathcal{T}_{\mathcal{G}\;L\;\&\;R}$, relevant for the following are made explicit. $\longrightarrow$

The apparent digression from the derivation of the differential equations determining the abelian one parameter subgroups as shown in eq. 36 in subsection 3b-1 serves to make intrinsic use of the theorem by Peter and Weyl [14-1927] , which holds precisely for the induced representations ▶ $\mathcal{T}_{\mathcal{G}\ L\ \&\ R}$. Now these representations were constructed with the regular representation of finite groups as guideline, the latter yielding in a clear deductive way to a complete reduction involving all irreducible linear representations of the finite group $\mathcal{G}_{\ finite}$ in question with the dimension of the group beeing equal to the multiplicity of its appearances within the regular representation. So a similar construction was sought and found for compact semi-simple Lie groups, indeed analogous to the regular representation in what the theorem of Peter and Weyl asserts : the reprentations ▶ $\mathcal{T}_{\mathcal{G}\ L\ \&\ R}$, for $\mathcal{G}$ a compact semi-simple Lie group can be fully reduced with respect to finite dimensional irreducible representations, if each simple factor subgroup and in this way all such representations are recovered with each representation appearing with a multiplicity equal to its dimension , allowing both $\mathcal{T}_{\mathcal{G}\ L}$ and $\mathcal{T}_{\mathcal{G}\ R}$ to be represented – commuting according to eq. 51 .
This property is thus characteristic or self consistent for complete connections within the ordered sequence of fibre manifolds considered here, as discussed in subsection 3a-1 , to which we will return in the next section .

3 b-2 – construction of one parameter abelian subgroups on $\mathcal{G}$

We return to the construction of one parameter abelian subgroups of $\mathcal{G}$ using the transformation group $\mathcal{T}_{\mathcal{G}\ L}$ to be specific , using the notions introduced in eqs. 36 and 38 . $\longrightarrow$

The exponential mapping induced by the left multiplication operators $T_a \in \mathcal{T}_{\mathcal{G}\,L}$

(52)
$$\widehat{\omega} = \omega^{\varrho} \widehat{I}_{\varrho} \; ; \; \widehat{\omega} , \widehat{I}_{\varrho} \in \mathcal{T}_{(1)} \left(\mathcal{T}_{\mathcal{G}\,L} \right) \longrightarrow$$
$$\exp \left(\tau \widehat{\omega} \right) e = h \left(\tau ; \omega \right) \in \mathcal{G} \; \textbf{with} \; e = \textbf{unit element} \in \mathcal{G}$$

defines a set of one parameter abelian subgroups of $\mathcal{G}$ **with the initial condition**

(53)
$$h \left(\tau ; \omega \right) |_{\tau = 0} = e$$

which involves higher order differentials than third for the group multiplication functions as specified in eqs. 10 and 11

(54)
$$\left(b \, . \, a \right)^{\nu} = @^{\nu} \left(b^{1} , \cdots b^{G} ; a^{1} , \cdots , a^{G} \right)$$

Using the first order differentials (eqs. 40 - 42) we infer the system of differential equations

(55)
$$\begin{aligned} \frac{d}{d\tau} h^{\nu} \left(\tau ; \omega \right) &= \omega^{\varrho} \left[\partial_{a^{\varrho}} @^{\nu} \left(a , h \left(\tau , \omega \right) |_{a = 0} \right] \right. \\ &= \omega^{\varrho} \left[u_{(\varrho)}^{\;\nu} \left(h \left(\tau ; \omega \right) \right) \right] \end{aligned}$$

$$u_{(\varrho)}^{\;\nu} \left(0 \left(\leftrightarrow e \right) \right) = \delta_{(\varrho)}^{\;\nu} \; ; \; h^{\nu} \left(\tau = 0 ; \omega \right) = 0 \left(\leftrightarrow e \right)$$

In eq. 55 $\left[\partial_{a^{\varrho}} @^{\nu} \left(a , \xi \right) |_{a = 0} \right] = u_{(\varrho)}^{\;\nu} \left(\xi \right)$ **denote the Killing vector fields on** $\mathcal{G}$ **generated by the first order differentials** $\mathcal{T}_{(1)} \left(\mathcal{T}_{\mathcal{G}\,L} \right)$ **as defined in eqs. 37 and following .** $\longrightarrow$

We rewrite the differential equation (eq. 55) suppressing the argument ω of the one parameter family coordinates $h^{\nu}(\tau;\omega)$, which plays a parametric important role, for simplicity of notation

$$h^{\nu}(\tau;\omega) \rightarrow h^{\nu}(\tau) \longrightarrow$$

$$(56) \quad \frac{d}{d\tau} h^{\nu}(\tau) = \omega^{\varrho} \, u_{(\varrho)}{}^{\nu} (h(\tau)) \quad ; \quad \frac{d}{d\tau} h^{\nu}(0) = \omega^{\nu} \quad ; \quad h^{\nu}(0) = 0$$

Conversely let $h^{\nu}(\tau)$ satisfy the differential equations and initial conditions in eq. 56 .
Then we consider the function associated with the group product $p(\tau,\vartheta) \sim (h(\tau) . h(\vartheta))$

$$p^{\nu}(\tau,\vartheta) = @^{\nu}(h(\tau);h(\vartheta)) \longrightarrow$$

$$(57) \quad \left[\begin{array}{l} p^{\nu}(\tau + d\tau, \vartheta) \\ - p^{\nu}(\tau,\vartheta) \end{array} \right] \begin{array}{l} = @^{\nu}(d\tau\,\omega ; p(\tau,\vartheta)) \\ = d\tau\,\omega^{\varrho} \; u_{(\varrho)}{}^{\nu}(p(\tau,\vartheta)) \\ = d\tau\,\omega^{\varrho} \; u_{(\varrho)}{}^{\chi}(h(\tau)) \; \partial_{a\,\chi} @^{\nu}(a;h(\vartheta)) \,|_{\,a=h(\tau)} \end{array}$$

$\longrightarrow$

Eq. 57 gives rise to the differential equations for $p^{\nu}(\tau,\vartheta)\ \ (=\ (h(\tau).h(\vartheta))^{\nu})$

$$\text{(58)}\quad \frac{d}{d\tau}\, p^{\nu}(\tau,\vartheta) = \omega^{\varrho}\ u_{(\varrho)}{}^{\nu}\,(p(\tau,\vartheta))$$

$$= \omega^{\varrho}\ u_{(\varrho)}{}^{\chi}\,(h(\tau))\ \partial_{a\,\chi}\,@^{\nu}\,(a\,;h(\vartheta))\,|_{a=h(\tau)} \longrightarrow$$

$$\frac{d}{d\tau}\, p^{\nu}(0,\vartheta) = \omega^{\varrho}\ u_{(\varrho)}{}^{\nu}\,(h(\vartheta))\quad ;\quad p(0,\vartheta) = h(\vartheta)$$

We compare differential equations and initial conditions for $p^{\nu}(\tau,\vartheta)$ **in eq. 58 with the ones for**

$$q^{\nu}(\tau,\vartheta) = h^{\nu}(\tau+\vartheta)$$

$$\text{(59)}\quad \frac{d}{d\tau}\, q^{\nu}(\tau,\vartheta) = \omega^{\varrho}\ u_{(\varrho)}{}^{\nu}\,(q(\tau,\vartheta))$$

$$\frac{d}{d\tau}\, q^{\nu}(0,\vartheta) = \omega^{\varrho}\ u_{(\varrho)}{}^{\nu}\,(h(\vartheta))\quad ;\quad q(0,\vartheta) = h(\vartheta)$$

which are both identical . It follows $\longrightarrow$

by the uniqueness of solutions to a system of first order differential equations subject to the same initial conditions

(60)
$$p(\tau, \vartheta) = q(\tau, \vartheta) \quad \longleftrightarrow \quad h(\tau) . h(\vartheta) = h(\tau + \vartheta)$$

in the domain of validity of solutions to eqs. 56 - 59 .

From the structure of all one parameter families $h^{\nu}(\tau ; \omega)$ and the first order differentsal equations they satisfy – eqs. 56 - 60 – at least for small values of τ it follows that first in the neighbourhood of the unit element group elements can be uniquely parametrized by the tangent vectors $\omega = (\omega^{1}, \cdots, \omega^{G})$ and using these 'normal' coordinates the convolution functions $@^{\nu}(\omega_{(1)} ; \omega_{(2)}) = \left(h(\omega_{(1)}) . h(\omega_{(2)}) \right)$ once threefold differentiability is assumed – become (real -) analytic functions of the special tangent vector variables $\omega_{(1)}, \omega_{(2)}$ through the exponential mapping

(61)
$$@^{\nu}(\omega_{(1)} ; \omega_{(2)}) =$$
$$\sum_{m_1 \cdots m^G ; n_1 \cdots n^G = 0}^{\infty} \left(a_{m_1, \cdots, m_G ; n_1, \cdots, n_G} \right) \times$$
$$\times \left(\omega^{1}_{(1)}\right)^{m_1} \cdots \left(\omega^{G}_{(1)}\right)^{m_G} \left(\omega^{1}_{(2)}\right)^{n_1} \cdots \left(\omega^{G}_{(2)}\right)^{n_G}$$: convergent in a neighbourhood of e

From here $\mathcal{G}$ emerges as its universal covering group .

This ends the present subsection (3 b-2) and section (3 b) .

3 c - Complete connections : regularity conditions from the full collection of fibre manifolds as defined in subsection (3 a-1)

With the complete structure of analytic coordinates and the property of universal covering group with respect to 'path homotopy' of semisimple compact Lie groups $\mathcal{G}$ fully specified

in sections (3 a) , (3 b) and subsections (3 b-1) , (3 b-2)
and the selction of graded fibre manifolds in subsection (3 a-1)

we return to the properties of complete connections , introduced at the beginning of section 3 . The words used in modern mathematics to describe complete connections are : connections regular for the complete ring of representations of $\mathcal{G}$ [26-1968] .

In the following $\mathcal{G}$ shall be generically a compact simple Lie group, specifically $SU3_{\,c}$ and $\mathcal{D}$ a general irreducible representation of $\mathcal{G}$

(62) $$\mathcal{G} \;=\; \left\{ \begin{array}{l} \text{compact simpe Lie group} \\ \\ \quad SU3_{\,c} \text{ specifically} \end{array} \right. \qquad ; \quad \mathcal{D} \;=\; \mathcal{D}\,|_{\,\mathcal{G}} \;\in\; \mathcal{R} \;=\; \mathcal{R}|_{\,\mathcal{G}}$$

$\mathcal{R}$ = complete ring generated by finite dimenional, unitary , irreducible representations of $\mathcal{G}$

Following the notation of eqs. 2 - 7 $\longrightarrow$

we denote by $Lie\,(\,\mathcal{D}\,)$ a basis of antihermtian $dim\,(\,\mathcal{D}\,)\,\times\,dim\,(\,\mathcal{D}\,)$ matrices satisfying the Lie algebra commutation relations pertaining to $\mathcal{D}$

$$\left(\,d_{\,r}\,(\,\mathcal{D}\,)\,\right)_{\,\alpha\beta} \;\rightarrow\; d_{\,r}\,(\,\mathcal{D}\,) \;\rightarrow\; d_{\,r} \,=\, -\,d_{\,r}^{\,\dagger} \;\;;\;\; r \,=\, 1\,,\cdots,\,dim\,\mathcal{G} \,\in\, Lie\,(\,\mathcal{D}\,)$$

(63) $$\left[\,d_{\,p}\,,\,d_{\,q}\,\right] \,=\, f_{\,pqr}\,d_{\,r} \;\;:\;\; f_{\,pqr} \,=\, \begin{cases} \text{real, totally antisymmetric} \\ \text{structure constants of } \mathcal{G} \end{cases}$$

$$\alpha\,,\,\beta \,=\, 1\,,\cdots,\,dim\,(\,\mathcal{D}\,)$$

Now lets assume the situation as defined in eq. 52 , where the one parameter subgroup $h\,(\,\tau\,;\,\omega\,)\,\in\,\mathcal{G}$ was constructed through the exponential mapping

(64) $$\widehat{\omega} \,=\, \omega^{\,r}\,\widehat{I}_{\,r} \;\;;\;\; \widehat{\omega}\,,\,\widehat{I}_{\,r} \,\in\, \mathcal{T}_{(1)}\,(\,\mathcal{T}_{\mathcal{G}\,L}\,) \;\;;\;\; \left[\,\widehat{I}_{\,p}\,,\,\widehat{I}_{\,q}\,\right] \,=\, f_{\,pqr}\,\widehat{I}_{\,r}$$
$$\longrightarrow\;\; \exp\,(\,\tau\,\widehat{\omega}\,)\,e \,=\, h\,(\,\tau\,;\,\omega\,) \,\in\, \mathcal{G} \;\;\text{with}\;\; e \,=\, \text{unit element} \,\in\, \mathcal{G}$$

of which the tangent vector at e is $\omega\,=\,(\,\omega^{1}\,,\,\cdots\,,\,\omega^{\,G}\,)$, as shown again in eq. 64 , with the transformation group $\mathcal{T}_{\mathcal{G}\,L}$, i.e. left multiplication on $\mathcal{G}$. Then the finite transformation matrix $D\;(\,h\,(\,\tau\,;\,\omega\,)\,)\,\in\,\mathcal{D}$ is obtained through the exponential mapping

(65) $$D\;(\,h\,(\,\tau\,;\,\omega\,)\,) \,=\, \exp\,(\,\tau\,\omega^{\,r}\,d_{\,r}\,) \,\in\, \mathcal{D} \;\;\forall\;\; \mathcal{D} \,\in\, \mathcal{R}$$

This anchors the meaning of eqs. 4 - 7 as is necessary to analyze complete connections , next . $\rightarrow$

3 c-1 – Complete connections in detail

At this stage we come back to the connection one form as defined in eq. 3 , expanding on the regularity conditions implied by the adopted complete fibre manifolds – in subsection 3 a-1 – as they give rise to the complete ring of representations $\cup_{\mathcal{D}} = \mathcal{R}$, defined in eq. 62

$$\begin{aligned} & \left(W^{(1)} (\mathcal{D}) \right)_{\alpha\beta} = W_{\mu}^{r} (x) \left(d_{r} (\mathcal{D}) \right)_{\alpha\beta} \, d x^{\mu} \\ & W_{\mu}^{r} (x) \ : \ \textbf{real} \ ; \ r = 1 , \cdots , G = dim (\mathcal{G}) \\ & \left(W^{(1)} (\mathcal{D}) \right)_{\alpha\beta} \ \rightarrow \ \alpha , \beta = 1 , \cdots , D = dim (\mathcal{D}) \\ & W^{(1)} (\mathcal{D}) \ \rightarrow \ W^{(1)} \left(|_{\mathcal{D}} \right) \ ; \ \mathcal{D} \in \mathcal{R} \ \left(\textbf{unrestricted} \right) \\ & \left(d_{r} (\mathcal{D}) \right)_{\alpha\beta} \ \rightarrow \ d_{r} (\mathcal{D}) \ \rightarrow \ d_{r} \in Lie (\mathcal{D}) \end{aligned} \tag{66}$$

The brackets around $_{\mathcal{D}}$ specifying the chosen representation forming the connection 1-form $W^{(1)} \left(|_{\mathcal{D}} \right)$ in eqs. 3 and 66 shall indicate that this suffix may be suppressed in the following for simplicity of notation.
The argument x and the differentials $d x^{\mu}$; $^{\mu} = 0, 1, 2, 3$ refer to 1 + 3 dimensional uncurved space-time as base-space of the connection 1-form . Eventually the base space can be continued to its Euclidean version . The conditions in eq. 66 define a complete classical connection together with its regularity conditions . $\longrightarrow$

In order to maintain exact local gauge invariance , imposed here, complete connections are understood to form a collection of such, denoted $\mathcal{C}$

(67) $$\mathcal{C} = \bigcup_{W} \left(W^{(1)} \right) \text{ complete connections}$$

invariant under local gauge transformations to which we turn next.

Let $F(x)$ be a classical field quantity – with arbitrary spin, untouched by charge like gauges – transforming according to the local irreducible representation $\Omega(x) \in \mathcal{D}(\mathcal{G})$

(68) $$\begin{aligned} &\Omega(x) : F(x) \rightarrow F^{\Omega}(x) = \Omega(x) F(x) \\ &F_{\alpha} \rightarrow F^{\Omega}_{\alpha} = \Omega_{\alpha\beta} F_{\beta} \; ; \; \Omega \in \mathcal{D} \end{aligned}$$

In eq. 68 and subsequently we use matrix notation and also suppress the local argument x , whenever clarity of notation allows .

Then the $\left(\mathcal{D} \text{ adapted} \right)$ covariant derivative 1-form is determined from the relations, derivatives meant to act on F , F^{Ω} from the left

(69) $$\begin{aligned} &d x^{\mu} \delta_{\alpha\beta} \partial_{\mu} = d x^{\mu} \P_{D \times D} \partial_{\mu} = \partial \\ &\Omega : D^{(1)} = \partial + W^{(1)} \rightarrow D^{\Omega (1)} = \partial + W^{\Omega (1)} \text{ with} \\ &D^{\Omega (1)} F^{\Omega} = \Omega D^{(1)} F = \Omega D^{(1)} \left(\Omega^{-1} F^{\Omega} \right) \end{aligned}$$

In eq. 69 the usual derivative symbol d is replaced by ∂ to ditinguish the matrices d_r . $\longrightarrow$

From eq. 69 the inhomogeneous transformation law for connections follows

(70) $$W^{\,\Omega\,(1)} = \Omega \left(\partial + W^{\,(1)} \right) \Omega^{-1} = \Omega \, D^{\,(1)} \, \Omega^{-1}$$

Performing the inverse of the exponential mapping or equivalently for infinitesimal $\Omega - ¶$ we obtain

(71) $$\delta_{\,\omega} W^{\,(1)} = - \left(\partial \omega + \left[W^{\,(1)} , \omega \right] \right) \text{ in components } \rightarrow$$
$$- \delta_{\,\omega} W_{\mu}^{\,r} \, d_{\,r} = \partial_{\,\mu} \omega^{\,r} \, d_{\,r} + f_{\,pqr} W_{\mu}^{\,p} \, \omega^{\,q} \, d_{\,r} \quad \forall \mathcal{D} \in \mathcal{R}$$

Dealing with complete connections, the common factor $d_{\,r}$ can be projected out and eq. 71 becomes

(72) $$- \delta_{\,\omega} W_{\mu}^{\,r} = \partial_{\,\mu} \omega^{\,r} + f_{\,rpq} W_{\mu}^{\,p} \, \omega^{\,q} = \left(\partial_{\,\mu} \delta_{\,rq} + \left(W_{\mu}^{\,ad} \right)_{\,rq} \right) \omega^{\,q}$$
$$\left(W_{\mu}^{\,ad} \right)_{\,rq} = W_{\mu}^{\,p} \left(ad_{\,p} \right)_{\,rq} \quad ; \quad \left(ad_{\,p} \right)_{\,rq} = f_{\,rpq}$$

In turn if we identify $\omega^{\,r} \left(x \right) \sim F_{r} \in Ad \left(\mathcal{G} \right)$ as a field quantity transforming under the adjoint representation of $\mathcal{G}$ the right hand side of the first relation in eq. 72 represents its covariant derivative for a given connection $W^{\,(1)}$ with respect to this representation . The above local apparent reduction of both gauge connections and field strengths , to allow their full characterization through the adjoint representation is in particular for complete connections incorrect, as a consequence of nonlocal gauge co- and invariance emerging through finite distance parallel transport, as discussed in the next subsection (3 c-2) . Here we continue defining local field strengths . →

From a complete connection relative to the irreducible local gauge group representation $\mathcal{D}\,(\,\mathcal{G}\,)$ as defined in eqs. 66 (- 72) the $\mathcal{D}$ associated field strengths obtain through the two form

$$
\begin{array}{l}
W^{\,(2)}\,(\,\mathcal{D}\,)\;\rightarrow\;W^{\,(2)}\;=\;\partial\,W^{\,(1)}\;+\;\left(\,W^{\,(1)}\,\right)^{\,2} \\
\left(\,W^{\,(2)}\,\right)_{\,\alpha\beta}\;=\;\frac{1}{2}\,W^{\,r}_{\,\mu\nu}\;(\,d_{\,r}\,)_{\,\alpha\beta}\;d\,x^{\,\mu}\;\wedge\;d\,x^{\,\nu}\quad;\quad d_{\,r}\;\in\;Lie\,(\,\mathcal{D}\,)\quad\rightarrow \\
W^{\,(2)}_{\,\mu\nu}\;=\;\partial_{\,\mu}\,W^{\,(1)}_{\,\nu}\;-\;\partial_{\,\nu}\,W^{\,(1)}_{\,\mu}\;+\;\left[\,W^{\,(1)}_{\,\mu}\,,\,W^{\,(1)}_{\,\nu}\,\right] \\
W^{\,r}_{\,\mu\nu}\;=\;-\,W^{\,r}_{\,\nu\mu}\;=\;\partial_{\,\mu}\,W^{\,r}_{\,\nu}\;-\;\partial_{\,\nu}\,W^{\,r}_{\,\mu}\;+\;f_{rpq}\,W^{\,p}_{\,\mu}\,W^{\,q}_{\,\nu} \\
\hline
W^{\,(2)}\,(\,\mathcal{D}\,)\;\equiv\;B^{\,(2)}\,(\,\mathcal{D}\,)\quad;\quad W^{\,r}_{\,\mu\nu}\;\equiv\;B^{\,r}_{\,\mu\nu}\qquad\text{components of field strengths independent of }\mathcal{D}
\end{array}
\tag{73}
$$

In the last relation in eq. 73 we have identified the curvature two form and its $\mathcal{D}-$ independent components with the letter B for field strengths pertaining to charge like gauges. $d\,x^{\,\mu}\;\wedge\;d\,x^{\,\nu}$ denotes the antisymmetric de Rham wedge product for $(2)\;\rightarrow\;(k)-$ forms [27-1931] .
We note the local, covariant gauge transformation properties of $F^{\,(2)}$ following from eqs. 70 and 73

$$
\begin{array}{l}
W^{\,(1)}\;\rightarrow\;W^{\,\Omega\,(1)}\;=\;\Omega\;\left(\,\partial\;+\;W^{\,(1)}\,\right)\;\Omega^{\,-1}\;=\;\Omega\,D^{\,(1)}\,\Omega^{\,-1}\,\big|_{\,\mathcal{D}}\;\longrightarrow \\
B^{\,(2)}\;\rightarrow\;B^{\,\Omega\,(2)}\;=\;\Omega\,B^{\,(2)}\,\Omega^{\,-1}\,\big|_{\,\mathcal{D}}\quad;\quad\Omega\,(\,x\,)\;\in\;\mathcal{D}
\end{array}
\tag{74}
$$

$\rightarrow$

The Bianchi identity, only local consequence of a system of local connections and field strengths

In a system as specified in the (sub-) title above one identity follows, generalising the homogeneous Maxwell equations in QED

$$
\begin{aligned}
B^{(3)} &= \partial B^{(2)} + \left[W^{(1)}, B^{(2)} \right] \Big|_{\mathcal{D}} \equiv 0 \\
&= \partial \, \partial W^{(1)} + \left[\begin{array}{c} \left[\left(\partial W^{(1)} \right), W^{(1)} \right] + \left[W^{(1)}, \left(\partial W^{(1)} \right) \right] \\ + \left[W^{(1)}, \left(W^{(1)} \right)^2 \right] \end{array} \right]
\end{aligned} \tag{75}
$$

$$
\longleftrightarrow \; B^{(2)} (\mathcal{D}) \rightarrow W^{(2)} = \partial W^{(1)} + \left(W^{(1)} \right)^2
$$

The (Bianchi -) identity in eq. 75 holds independently of whether other matter fields, e.g. q , $\overline{q}$ are included with finite masses or not . However the identity only holds if $F^{(2)} (\mathcal{D})$ actually derives from connections , and furthermore if allowed connections do satisfy regularity conditions, here those appropriate for complete such .

This ends subsection 3 c-1 and we turn towards nonlocal properties of complete connections in the next subsection 3 c-2 .

3 c-2 – Parallel transport with complete connections

Here I follow the outline in ref. [28-2004] [a] .

We first establish the relation of notation between ref. [28-2004], Appendix A.3 op.cit.: hep-ph/0405032v1, pp. 42 - 52 , eqs. 93 - 127 , to the present work : subsection 3 c-1 , eqs. 66 - 75 .

In ref. [28-2004] only the adjoint representation is discussed which implies

	present work	ref. [28-2004]
	$\mathcal{D} \; = \; Ad\,(\,\mathcal{G}\,)$	$ABC \; \longrightarrow \; qpr$
		$\frac{1}{i}\,\mathcal{F}_{A=r\;B=p\;C=q} \; = \; f_{\,rpq} \; = \; (\,ad_{\,p}\,)_{\,rq}$
(76)		$(\,\mathcal{W}_{\,\mu}\,)_{\,A=r\;B=q} \; = \; \left(\,W_{\,\mu}^{\,ad}\,\right)_{\,rq} \; = \; W_{\,\mu}^{\,p}\;(\,ad_{\,p}\,)_{\,rq}$
		$V_{\,\mu}\,(\,x\,,\;D=p\,) \; = \; -\,W_{\,\mu}^{\,p}\,(\,x\,)$
		$X\,(\,x\,,\;B=r\,) \; = \; F^{\,r}\,(\,x\,)$
		$F_{\,[\,\mu\nu\,]}\,(\,x\,,\;D=r\,) \; = \; W_{\,\mu\nu}^{\,r}\,(\,x\,)$

---

[a] **In ref. [28-2004] I called instead of 'complete connections over the complete ring generate by finite dimenional, unitary , irreducible representations of $\mathcal{G}$ ' as defined here in eq. 62 : 'universal bundle' . However the notion of 'universal bundle' in the mathematical literature is reserved for another structure. I thank Martin Lüscher, for pointing this error out to me . It applies to my nomenclature , not to the derivations .** $\longrightarrow$

The identifications in eq. 76 shall suffice here, they allow to deduce all further such.

We return to general complete connections based on a given representation $\mathcal{D} \left(\mathcal{G} \right)$ as specified in subsection 3 c-1 .

Parallel transport – a priori along a general curve C – in the base space $\mathcal{B} = \left\{ x \right\}$ – leads to the path ordered exponential integral of a given connection 1-form denoted $W^{(1)} \left(x \right) \Big|_{\mathcal{D} \left(\mathcal{G} \right)}$ as defined in eq. 66 out of the collection $\mathcal{C}$ of complete connections (eq. 67) denoted $U = U_{\alpha\beta} \in \mathcal{D}$

$$U \left(x \overset{C}{\leftarrow} y \right) = P \exp \left(- \int_C W^{(1)} \left(\overline{x} \right) \right) \rightarrow \left(U \left(x , C , y \right) \right)_{\alpha\beta} \in \mathcal{D} \left(\mathcal{G} \right)$$

(77) $C = C \left\{ \overline{x} \right\} \quad : \quad \overline{x} = \overline{x} \left(s \right) \quad ; \quad \tau \geq s \geq 0 \quad ; \quad s$: **path parameter**

$$\overline{x} \left(s = \tau \right) = x \quad ; \quad \overline{x} \left(s = 0 \right) = y$$

In eq. 77 P denotes matrix ordering along the path C since $W^{(1)}$ is matrix valued

(78) $$\left(W^{(1)} \left(\mathcal{D} \right) \right)_{\alpha\beta} = W_\mu^{\,r} \left(\overline{x} \right) \left(d_r \left(\mathcal{D} \right) \right)_{\alpha\beta} \, d\overline{x}^{\,\mu}$$

However since we consider here classical configurations the quantities $W_\mu^{\,r} \left(\overline{x} \right)$ are commuting for arbitrary $\overline{x}$. We will only use straight line curves C , which would leave local field variables commuting only for space like straight line paths , except for the derivation of the differential equation for $U \left(x , C , y \right)$, which would also be true for noncommuting field variables, next . $\longrightarrow$

The differential equation derives from the parallel transport and has its roots in expanding on the meaning of eq. 71. We do not explicitely do so here . We assume that the general path C , as specified by the functions $\overline{x}^{\,\mu}(s)$ (eq. 77) , which respect all regularity conditions and are known functions also beyond the specific boundary value τ described in eq. 77

Then it follows for $\tau \rightarrow \tau + d\tau$

$$U \rightarrow U(\tau)$$

$$U(\tau) + d\tau \frac{d}{d\tau} U = \left(¶ - d\tau\, \mathbf{v}^{\mu}(\tau) W_{\mu}^{\,r}(y(\tau))\, d_r \right) U \quad \longrightarrow$$

---

(79)

$$\frac{d}{d\tau} U(\tau) = -(\mathbf{v}^{\mu} \mathcal{W}_{\mu})(\tau)\, U(\tau) \ ; \quad \left[\begin{array}{l} \mathbf{v}^{\mu} = \frac{d}{d\tau} \overline{x}^{\,\mu}(\tau) \\ \mathcal{W}_{\mu} = W_{\mu}^{\,r} d_r = \mathcal{W}_{\mu}(\tau) \end{array} \right]$$

$$U(\tau = 0) = U_y = ¶$$

The regularity conditions imposed on complete connections go over $\longrightarrow$

to the quantities $U(\tau) \to U\left(x \overset{C}{\leftarrow} y\right)$ if and only if all paths are chosen to respect these conditions forming a network, i.e. the functions $\overline{x}(s)$ are chosen accordingly, e.g. to be real analytic functions or n-fold differentiable ones of the argument s .

If two paths C_1 and C_2 can be joined without reduction of imposed regularity conditions individually to a combined path C_{2+1} then the associated unitary operators can be combined and obey the composition law

(80) $$U\left(x_2 \overset{C_2}{\leftarrow} x_1\right) U\left(x_1 \overset{C_1}{\leftarrow} y\right) = U\left(x_2 \overset{C_{2+1}}{\leftarrow} y\right)$$

in a natural way. The best known case, whereby parallel transport is called holonomy (whence applied to a field F^{β} at y) for a closed path $x \to x_{end} = y$.

But we leave the paths open and consider a local gauge transformation as defined in eq. 70

(81) $$W^{\Omega\,(1)}(x) = \Omega(x)\left(\partial_x + W^{(1)}(x)\right)\Omega^{-1}(x) = \Omega\, D^{(1)}\,\Omega^{-1}(x)$$

Then it follows

(82) $$\begin{array}{ccc} U\left(x \overset{C}{\leftarrow} y\, ;\, W^{\Omega\,(1)}\right) & = & P \exp\left(-\int_C W^{\Omega\,(1)}\right) \\ \| & & \\ \Omega(x)\, U\left(x \overset{C}{\leftarrow} y\, ;\, W^{(1)}\right)\Omega^{-1}(y) & & \end{array}$$

$\longrightarrow$

The vertical relation in eq. 82 is repeated below

$$U \left(x \overset{C}{\leftarrow} y \, ; \, W^{\,\Omega \, (1)} \right) \, = \, \Omega \, (\, x \,) \, U \, \left(x \overset{C}{\leftarrow} y \, ; \, W^{\,(1)} \right) \, \Omega^{\,-1} \, (\, y \,)$$

(83)

$$W^{\,\Omega \, (1)} \, (\, z \,) \, = \, \Omega \, (\, z \,) \, \left(\partial_{\,z} \, + \, W^{\,(1)} \, (\, z \,) \right) \, \Omega^{\,-1} \, (\, z \,) \, = \, \Omega \, D^{\,(1)} \, \Omega^{\,-1} \, (\, z \,)$$

$\Omega \, (\, x \,) \, , \, \Omega \, (\, y \,) \, , \, \Omega \, (\, x \,) \, \in \, \mathcal{D} \, (\, \mathcal{G} \,) \;\; \forall \;\; \mathcal{D}$ **and** $\forall$ **complete** $W^{\,(1)} \, \big|_{\,\mathcal{D}}$

The relation on the first line of eq. 83 demonstrates that complete connections – by splitting the arguments of $\Omega \, (\, x \,)$ from $\Omega \, (\, y \,)$ for all $\mathcal{D} \, (\, \mathcal{G} \,)$ – allow to reconstruct the full global structure of $\mathcal{G}$. Hence they do not admit singularities at finite discrete normal subgroups of $\mathcal{G}$ a (semi-) simple compact group , as e.g. the center $Z_{\,3}$ of $SU3_{\,c}$.

The eventual conflicts with regularity conditions respected in lattice formulations of QCD are based on the properties derived for complete connections, maintaining associated extended gauge invariance, in sections 3 a - 3 c and are the subject of the next section 3 d .

This concludes subsection 3 c-2 and section 3 c .

3 d - Complete connections , regularity conditions in potential conflict with lattice QCD – selected specific points

We shall enumerate specific points below

1) the Bianchi identity $\left(\right.$ here defined in eq. 75 $\left.\right)$ repeated below

$$
\begin{aligned}
B^{(3)} &= \partial B^{(2)} + \left[W^{(1)} , B^{(2)} \right] \Big|_{\mathcal{D}} \equiv 0 \\
&= \frac{1}{6} \begin{pmatrix} \partial_{\nu} \mathcal{B}_{\varrho\sigma} + \left[\mathcal{W}_{\nu} , \mathcal{B}_{\varrho\sigma} \right] \\ + \partial_{\sigma} \mathcal{B}_{\nu\varrho} + \left[\mathcal{W}_{\sigma} , \mathcal{B}_{\nu\varrho} \right] \\ + \partial_{\varrho} \mathcal{B}_{\sigma\nu} + \left[\mathcal{W}_{\sigma} , \mathcal{B}_{\nu\varrho} \right] \end{pmatrix}_{\alpha\beta} d x^{\nu} \wedge d x^{\varrho} \wedge d x^{\sigma}
\end{aligned}
$$

---

(84)

$$
W^{(1)} = \mathcal{W}_{\mu} \, d x^{\mu} \quad ; \quad B^{(2)} = \frac{1}{2} \mathcal{B}_{\mu\nu} \, d x^{\mu} \wedge d x^{\nu}
$$

$$
B^{(3)} = \frac{1}{6} \mathcal{B}_{\mu\nu\varrho} \, d x^{\mu} \wedge d x^{\nu} \wedge d x^{\varrho}
$$

$$
\begin{pmatrix} \mathcal{W}_{\mu} \\ \mathcal{B}_{\mu\nu} \\ \mathcal{B}_{\mu\nu\varrho} \end{pmatrix}_{\alpha\beta} = \begin{pmatrix} W^{\,r}_{\mu} \\ B^{\,r}_{\mu\nu} \\ B^{\,r}_{\mu\nu\varrho} \end{pmatrix} \left(d_{r} \right)_{\alpha\beta} \in Lie\left(\mathcal{D} \right)
$$

$\longrightarrow$

1) (continued)

From eq. 84 we infer

(85)
$$\frac{\frac{1}{2}\,\varepsilon^{\,\mu\nu\varrho\sigma}\,(\,\partial_{\,\nu}\,\mathcal{B}_{\,\varrho\sigma}\,+\,[\,\mathcal{W}_{\,\nu}\,,\,\mathcal{B}_{\,\varrho\sigma}\,]\,)\,\equiv\,0\quad;\quad\widetilde{\mathcal{B}}^{\,\mu\,\nu}\,\doteq\,\frac{1}{2}\,\varepsilon^{\,\mu\nu\varrho\sigma}\,\mathcal{B}_{\,\varrho\sigma}}{\longrightarrow\quad\partial_{\,\nu}\,\widetilde{\mathcal{B}}^{\,\mu\,\nu}\,+\,\left[\,\mathcal{W}_{\,\nu}\,,\,\widetilde{\mathcal{B}}^{\,\mu\,\nu}\,\right]\,\equiv\,0\quad\forall\;\mathcal{D}\,(\,\mathcal{G}\,)}$$

It is worth noting that for the time-space signatures $(+---)$ as well as $(-+++)$ and independent of the sign of $\varepsilon_{\,0123}\;\pm\;1$ we have contrary to d=4 Euclidean space

(86)
$$\varepsilon^{\,\mu\nu\varrho\sigma}\,=\,-\,\varepsilon_{\,\mu\nu\varrho\sigma}\quad\longrightarrow\quad\widetilde{\widetilde{\mathcal{B}}}_{\,\mu\nu}\,=\,-\,\mathcal{B}_{\,\mu\nu}$$

The structure of eq. 76 implies that the Bianchi identity is indeed an identity, provided the field strengths are derived from a connection , and complete connections then imply regularity conditions for all Lie algebra valued such associated with any one out of all representations $\mathcal{D}\,(\,\mathcal{G}\,)$, not to be obstructed by singular connections for which the identity may be violated minimally in one singular point . It is this regularity feature, which is at least not clearly satisfied in lattice QCD or versions thereof e.g. without quark flavors. [a]

$\longrightarrow$

---

[a] **I am indepted to Uwe-Jens Wiese for his patience and many discussions, in which he brought up the question as to fulfilment of Bianchi identities in lattice QCD .**

2) **complete connections and boundary conditions for finite time $\Delta t = \beta = 1 / T$ thermal path integrals with respect to gauge bosons**

Subsection 3 c-2 is conceived particularly to assess the boundary conditions between two spacelike parallel planes with constant time each, in the rest system of a thermal equilibrium ensemble, set apart by $\Delta t = \beta = 1 / T$ with T denoting the temperature, and at finite but asymptotically large space volume V. The Gibbs potential is then associated with the generating functional for fixed intensive variables β, $\chi_a = \mu_a \beta$; μ_a : chemical potentials

(87)
$$Z(\beta, \mu_a, V) \simeq |_{V \to \infty} \; tr \left(\exp \left[- \beta \mathbf{H} + \sum_b \chi_b \mathbf{N}_b \right] \right)$$

$$\text{with } \chi_a = \mu_a / T \; ; \; Z \sim e^{g V} \; ; \; g = g(\beta, \chi_a) = \beta p$$

In eq. 87 H , N $_a$; $a = 1, \cdots, n_{fl}$ denote the conserved operators for energy and net charges respectively , and p the pressure .

Just in order to keep most simple, precise and correct notions, the number of exactly conserved charges in the absence of all electroweak interactions and leptons and scalar elementary fields , for n_{fl} of quarks (and antiquarks) with nondegenrate and nonzero masses is equal to n_{fl} .

A formally equivalent way to calculate Z as defined in eq. 87 is to use imaginary time and perform a path integral weighted as $\exp(- S_\beta)$ using the Euclidean form of the action in the associated Euclidean field theory . An excellent exposé of thermodynamic notions in the environment

→

of local fields can be found in ref. [29-2002] .

Now we go back to subsection 3 c-2 and recall eq. 83 below

$$U \left(x \overset{C}{\leftarrow} y \, ; \, W^{\,\Omega \, (1)} \right) \, = \, \Omega \, (\, x \,) \, U \left(x \overset{C}{\leftarrow} y \, ; \, W^{\,(1)} \right) \, \Omega^{\,-1} \, (\, y \,)$$

---

(88)

$$W^{\,\Omega \, (1)} \, (\, z \,) \, = \, \Omega \, (\, z \,) \, \left(\partial_{\,z} \, + \, W^{\,(1)} \, (\, z \,) \right) \, \Omega^{\,-1} \, (\, z \,) \, = \, \Omega \, D^{\,(1)} \, \Omega^{\,-1} \, (\, z \,)$$

$\Omega \, (\, x \,) \, , \, \Omega \, (\, y \,) \, , \, \Omega \, (\, x \,) \, \in \, \mathcal{D} \, (\, \mathcal{G} \,) \;\; \forall \;\; \mathcal{D}$ **and** $\forall$ **complete** $W^{\,(1)} \, \big|_{\,\mathcal{D}}$

The relation on the first line of eqs. 83 , 88 demonstrates that complete connections – by splitting the arguments of $\Omega \, (\, x \,)$ from $\Omega \, (\, y \,)$ for all $\mathcal{D} \, (\, \mathcal{G} \,)$ – allow to reconstruct the full global structure of $\mathcal{G}$. Hence ensuing regularity conditions, based on the properties derived for complete connections, maintaining associated extended gauge invariance can only tolerate periodic connections modulo gauge transformations , from the space time points $x \, = \, (\, 0 \, , \, \vec{x} \,) \, \longrightarrow \, x \, + \, \Delta \, t \, = \, (\, \Delta \, t \, , \, \vec{x} \,)$

(89)

$$W^{\,(1)} \, (\, x \, + \, \Delta \, t \,) \, = \, W^{\,\Omega(1)} \, (\, x \,) \, = \, \Omega \, \left(\partial \, + \, W^{\,(1)} \right) \, \Omega^{\,-1} \, (\, x \,)$$

for a suitably general set of gauge transformations $\Omega \, (\, x \,) \, \in$ any $\mathcal{D} \, (\, \mathcal{G} \,)$

$W^{\,\Omega \, (1)}$ **is defined in eq. 70 . The generalized boundary conditions in eq. 89 maintain periodicity of (fully) gauge invariant local quantities .** $\longrightarrow$

Furthermore the quantity $U\left(x+\Delta t \overset{C}{\leftarrow} x\,;W^{(1)}\right)$ **transforms under a local gauge tranformation as**

$$\begin{aligned} & U\left(x+\Delta t \overset{C}{\leftarrow} x\,;W^{(1)}\right) \rightarrow \\ & U\left(x+\Delta t \overset{C}{\leftarrow} x\,;W^{\Omega\,(1)}\right) = \\ = \ & \Omega\,(x+\Delta t)\,U\left(x+\Delta t \overset{C}{\leftarrow} x\,;W^{(1)}\right)\Omega^{-1}\,(x) \end{aligned} \tag{90}$$

with $\Omega\,(x+\Delta t) \neq \Omega\,(x)$ **for general allowed gauge transformations**

As a consequence what is known as the trace of the Polyakov loop

$$\begin{aligned} & Tr\,U\left(x+\Delta t \overset{C}{\leftarrow} x\,;W^{\Omega\,(1)}\right) = \\ & = Tr\,\Omega^{-1}\,(x)\left(\Omega\,(x+\Delta t)\;U\left(x+\Delta \overset{C}{\leftarrow} x\,;W^{(1)}\right)\right) \\ & \neq Tr\,U\left(x+\Delta t \overset{C}{\leftarrow} x\,;W^{(1)}\right) \end{aligned} \tag{91}$$

is not gauge invariant within the conditions imposed by complete connections .

The consequences from eqs. 89 - 91 are in conflict with the conditions imposed on lattice QCD applied to the thermodynamical environment .

Specific non-complete fiber spaces (not all manifolds) were defined and discussed in ref. [30-1991] .

The general option therein can be adapted to complete connections , notwithstanding the alternative conjectures suggested by the authors . This ends section 3 d and all of chapter 3 .

4 - The dominantly second order phase transition for vanishing chemical potentials

In this section the hypothesis is investigated, that the thermal structure of QCD phases at and near zero chemical potentials is determined by long range coherence, inducing the gauge boson pair condensate, and its thermal extension, representing a fundamental order parameter. A consistent framework for thermal behaviour including interactions is derived in which the condensate does not produce any latent heat as it vanishes at the critical temperature inducing a second order phase transition with respect to energy density neglecting eventual numerically small critical exponents. Localization and delocalization of color fields are thus separated by a unique critical temperature.

The existence and nature of the QCD phase transition is theoretically accessible to a universal description underlying a thermal system with all its general and specific restrictions. At the moment key theoretical questions concerning the phase structure of thermal systems in QCD remain. Lattice QCD calculations lead to very clear predictions, establishing the lack of any phase transition at zero chemical potentials [8-2010, 31-2009]. The validity of this method and its results is not uncontroversial, for example see references [32-2010, 33-2010] and the material elaborated in section 3 .

The selection of resonances , called Ntype 65 , underlying the Hadron Reosnance Gas (HRG) description for $T \; < \; T_c$ is presented in section 2 and tables 1 - 8 , as well as in Appendix 1, wherein the HRG thermal relative densities for Ntype 65 and the smaller set Ntype 26 are presented . $\longrightarrow$

ϱ_e / T^4 **is given in table 9 and displayed in figure 8 ,** p / T^4 **in table 10 and figure 9 , and the trace anomaly asociated dscale =** $(\varrho_e - 3p) / T^4$ **in table 11 and figure 10 .**

There is a longstanding experimental effort to measure characteristic signatures of the phase transition in collisions of heavy ions at several facilities at AGS at BNL, SPS at CERN, RHIC at BNL, LHC at CERN and in the future FAIR at GSI. The present state of these studies yields significant results, well interpretable as a strongly interacting partonic matter formed in the earlier stages of the heavy ion collisions [4-2005, 3, 3-ctd.].

These results suggest the existence of a phase transition in the course of the collision. Assuming the correctness of this conclusion, the detailed structure including several potential orders could not be yet determined.

The experimentally measured reactions cover a large range of collision energies and chemical potentials in thermal analyses. In particular, with increasing energies the baryochemical potential in hadron-hadron or heavy ion collisions decreases. For a comparison reducing these systems to a common baryochemical potential we refer to [7-2001]. $\longrightarrow$

In this section we propose to give an ”outline in principle” of the thermal phases of QCD at vanishing chemical potentials and their relation to the gauge boson pair condensate of QCD. We proceed to show that the two phases, separated by the critical temperature T_c at $\mu = 0$ can indeed be understood as a dynamical consequence of uniquely the bosonic pair condensate at nonzero quark masses, in contrast to supersymmetric models with no spontaneous breaking of supersymmetry.
The two phases are represented *approximatively* by a collection of noninteracting hadrons for $0 < T < T_c$ and a collection of 8 gauge bosons and 4 flavours of quarks and antiquarks (u,d,s,c & anti) for $T > T_c$, in which interactions are modeled. The adopted approximation allows to illustrate the thermal phase properties in principle.

4 a - Energy momentum density tensor and gauge boson condensate

We begin with the local, symmetric and conserved energy momentum density tensor $\vartheta_{\mu\nu}$ existing and renormalized as a basic consequence of QCD .
The above properties imply exact Poincaré invariance.

(92) $$\{ \vartheta_{\mu\nu} = \vartheta_{\nu\mu} \} (x) \; ; \; \partial^{\nu} \vartheta_{\mu\nu} = 0$$

The central defining quantities, of the existence and order of the phase transition, are pressure, energy density and entropy density for zero chemical potentials. $\longrightarrow$

The *local* operator form of the trace anomaly in QCD reads:

(93)
$$\vartheta^{\mu}{}_{\mu} = \sum_f m_f \, S_{\dot{f} f}(x) + \delta_0(x)$$
$$\delta_0 = \left(-2\beta(g) / g^3\right) \left[-\tfrac{1}{4} (:) \, F^A_{\mu\nu}(x) \, F^{\mu\nu\, A}(x) (:) \right]_{s.d.}$$
$$= -2 b_0 \left[\tfrac{1}{4} (:) F^A_{\mu\nu}(x) \, F^{\mu\nu\, A}(x) (:) \right]$$

In eq. 93 the subscript $s.d.$ stands for ”renormalization scale dependent”, while $b_0 = 9/(16\pi^2)$. All quantities are defined to be renormalized and renormalization group invariant, except those with the subscript $s.d.$ $\beta(g)$ denotes the (Callan-Symanzik-) rescaling function, where g is the coupling constant. $\frac{1}{4} (:) \, F^{A}_{\mu\nu} \, F^{\mu\nu A} (:)$ is the *local* scalar density – to be called *local* gauge boson bilinear – composed bilinearly of field strength tensors $F^{A}_{\mu\nu}$ – the latter including *multiplicatively* the gauge coupling constant in their definition, relative to the perturbative field strength normalization.
A denotes the color component within the adjoint representation of $SU(3)_c$. (:)(:) stands for a *suitable* normal ordering. m_f denotes the mass of quark flavour f : $S_{\dot{f} f}$ denotes the *local* scalar density of antiquark $\dot{f}$ and quark f flavors : $S_{\dot{f} f}(x) = (:) \, \overline{q}_{\dot{f}\dot{c}s}(x) q^{s}_{f c}(x) \, (:)$, where s denotes the spin.
The focus is to consider vacuum expected values of the *local* operators in eq. 93, which by translation invariance $\longrightarrow$

are independent of the position x, suppressed in the following

$$\eta^{\mu\nu} \langle\Omega| \vartheta_{\mu\nu}(x) \ |\Omega\rangle = -2b_0 \langle\Omega| \tfrac{1}{4}(:)F^A_{\mu\nu} \, F^{\mu\nu\, A}(x)(:) \,|\Omega\rangle + m_f \ \langle\Omega\,|\, S_{\dot{f}\,f} \ (x)\,|\Omega\rangle \tag{94}$$

$\langle\, \Omega \,|\, (:) \frac{1}{4} F^A_{\mu\nu} \, F^{\mu\nu \ A} (:) \,|\, \Omega \,\rangle = \mathcal{B}^2$ **shall be called the gauge boson pair condensate , abbreviated by** $\mathcal{B}^{\,2}$ **.** $\langle\, \Omega \,|\ S_{\ \dot{f}\ f}\ |\, \Omega \,\rangle \ \neq \ 0$ **induces spontaneous chiral symmetry breaking and is generally called quark condensate.**

In connection with *normal ordering ambiguities* **it is important to admit in the precise form of the energy momentum tensor a nontrivial vacuum expected value , which as a consequence of** *exact* **Poincaré invariance must be of the form**

$$\langle\, \Omega \,|\ \vartheta_{\mu\nu} \ (x) \ |\, \Omega \,\rangle = -\eta_{\mu\nu} \, p_{vac}$$

$$\left\{ \ \eta_{\mu\nu} = diag \ (1, -1, -1, -1) \ \ ; \ p_{vac} = -\rho_{vac} \ \right\}$$

(95) **independent of x** $\ \longrightarrow$

$$\Delta\vartheta_{\mu\nu} \ (x) = \vartheta_{\mu\nu} \ (x) - \langle\Omega|\, \vartheta_{\mu\nu} \ (x) \,|\Omega\rangle \ \ |\Omega\rangle \ \langle\Omega|$$

with $\ \partial^{\ \nu} \, \Delta \, \vartheta_{\ \mu \ \nu} \ (\ x \ ) \ = \ 0 \ ; \ \langle\, \Omega \,|\ \Delta\vartheta_{\mu\nu} \ (x) \ |\, \Omega \,\rangle = 0$

In eq. 95 $\ P_{\,\Omega} \ = \ |\, \Omega \,\rangle \ \langle\, \Omega \,|$ **denotes the projector on the ground state .** $\longrightarrow$

From the two local, *conserved* **tensors in eq. 95 only $\Delta\vartheta_{\mu\nu}(x)$ with vanishing vacuum expected value is acceptable as representing the conserved 4 momentum** *operators* **and their densities yielding the integral form**

(96)
$$\widehat{P}_{\mu} \ = \ \int_t d^3\, x\, \Delta\vartheta_{\mu 0}\, (t, \vec{x})$$

We use here throughout strictly thermal , 'extension in phase space' associated potentials, depending in subtle ways on vacuum condensates. To these potentials no vacuum associated spontaneous parameters like $p_{vac} \ = \ -\rho_{vac}$, defined in eq. 95, contribute in a direct way, dominating in the limit $T \ \rightarrow \ 0\,;\ \mu_{\ \alpha} \ \equiv \ 0$.

From eqs. 94 and 95 we obtain the relation and estimates 'pour fixer les idées'

(97)
$$\begin{aligned} p_{vac} \quad &= \quad \frac{9}{32\,\pi^{\ 2}}\mathcal{B}^{\ 2} \ + \ \tfrac{1}{4}\,\Lambda = \left\{ \begin{array}{l} 0.00302 \ \textbf{GeV}^{\ 4} \\ 0.00658 \ \textbf{GeV}^{\ 4} \end{array} \right. \\ \mathcal{B}^{\ 2} \quad &= \quad \left\{ \begin{array}{l} 0.125 \ \textbf{GeV}^{\ 4} \ \textbf{[2-1979]} \\ 0.250 \ \textbf{GeV}^{\ 4} \ \textbf{[34-1998]} \end{array} \right. \\ \Lambda \quad &= \quad -\textstyle\sum_f \ m_f \ \langle\Omega| \ S_{\dot{f}\,f} \ |\Omega\rangle \\ &\sim \quad f_{\ \pi}^{\ 2}\,(\ \tfrac{1}{2}\, m_{\ \pi}^{\ 2} \ + \ m_{\ K}^{\ 2}\ ) = 0.00217 \ \textbf{GeV}^{\ 4} \end{aligned}$$

4 b - Construction of a thermal model including interactions

We follow the strategy layed out in ref. [7-2001] taking into account the modifications described above, distinguishing two *eventual* phases

1) the hadronic (h)-phase , with color localized within stable hadrons and selected hadron resonances. Thermal potentials of the (h)-phase are approximated by those of free hadrons, neglecting decay widths, as described in ref. [7-2001].

2) the quark-antiquark-gauge boson (qg)-phase, wherein thermal potentials are related but not equal to those of free quarks and antiquarks, restricted to the flavors u,d,s and c, and 8 gauge bosons pertaining to the gauge group $SU3_c$. Next we describe the modeling of interactions in the (qg)-phase, which deviates from noninteracting constituents assumed in ref. [7-2001] .

We introduce for later use for the (qg)-phase , the Gibbs density $g_{\ qg}^{(0)}$ and energy density $\varrho_{\ e\ qg}^{(0)}$ $\longrightarrow$

pertaining to noninteracting tricolored quarks , antiquarks with flavors u , d , s , c and eightfold colored gauge bosons

(98)

$$g_{qg}^{(0)}\,(T) = \sum_{\alpha_{qg}} w_{\alpha_{qg}}\ (1/(2\pi^2))\ \int_{m_{\alpha_{qg}}}^{\infty} l\, E\, p\, d\, E$$

$$w_{\alpha_{qg}} = (2spin_{\alpha_{qg}} + 1) \begin{cases} 3 \text{ for } q, \overline{q} \\ 8 \text{ for } g \end{cases} = \begin{cases} 6 \text{ for } q, \overline{q} \\ 16 \text{ for } g \end{cases}$$

$$\beta \equiv 1/T\,;\, l = \mp \log\left[1 \mp \exp\left(-\beta E\right)\right]$$

$$\varrho_{e\ qg}^{(0)}\,(T) = -\,(d/d\beta)\ g_{qg}^{(0)}\,(T)\ =\ T^{\,2}\ (d/dT)\ g_{qg}^{(0)}\,(T)$$

In eq. 98 the index α_{qg} runs over the different constituents of the (qg) phase, while $w_{\,\alpha_{qg}}$ denotes the multiplicity beyond momentum phase space associated with the constituent α_{qg} . The sign ($\mp$) in the expression for l is - for bosons and + for fermions .

We choose the following masses for quark flavors u, d, s, c

(99)

$$[\, m_u = 0.00525, m_d = 0.00875, m_s = 0.175, m_c = 1.27\,] \text{ GeV}$$

The absolute masses of the u,d,s light flavors as well as their ratios $m_{\,u}\ :\ m_{\,d}\ :\ m_{\,s}\ =\ 3\ :\ 5\ :\ 100$, [35-1979, 36-2010, 37-2010] , used here play no decisive role in the present derivations, within generous ranges of $\pm$ 20 % . $\longrightarrow$

The inclusion of the charmed quark serves the purpose to check whether it has any significant influence on the thermal parameters in the region of $T_c \sim 0.2$ GeV , wich turns out to be in the few percent range .

We proceed to modify the free quark antiquark gauge boson (qg-) parametrization of the Gibbs potential and the energy density, which for $\mu_\alpha = 0$ must obey the exact relation

(100) $$g_{qg} = g_{qg}\,(T \equiv \beta^{-1})\ ; \quad -(d/d\beta)\ g_{qg}\,(T) = \varrho_{e\,qg}\,(T)$$

and $g_{qg} \leftrightarrow g_h \ , \ \varrho_{e\,qg} \leftrightarrow \varrho_{e\,hg}$

The Gibbs- and energy-densities in the hadron phase are constructed from the expressions analogous to the ones given in eq. 98 , where the index $\alpha_{qg} \rightarrow \alpha_{hg}$ runs over a suitable choice of hadrons and hadron resonances as defined in ref. [7-2001] with real masses and neglecting interactions among these states . The ensuing parametrization of interactions is understood as representing the phase structure in principle and not in numerical detail. The modeling of interactions in the qg-phase is performed setting two parameters $k\ ,\ \Delta\, g$, independent of temperature, as approximately parametrizing the interaction in the deconfined phase in a limited region of $T \geq T_{cr}$

(101) $$\begin{aligned} \varrho_{e\,qg}(T;k) &= k\,\varrho^{(0)}_{e\,qg}\,(T) \\ g_{qg}(T;k,\Delta\, g) &= k\,g^{(0)}_{qg}(T) - \Delta\, g \end{aligned}$$

The parameter $0 < k < 1$ is taking into account the reduction of Gibbs density or pressure →

relative to the noninteracting (Stefan-Boltzmann) limit , noted in perturbative QCD calculations of thermal parameters for $T \simeq T_c$ of interest here [38-2007] , while the second parameter Δg arises as integration constant from the differential equation (eq. 100) , which is clearly satisfied for arbitrary values of $(k , \Delta g)$.

We proceed in two steps to map out the structure of the phase transition, using $T_c \equiv T_{cr}$

I : the condition determining $T_c \leftrightarrow k$

The equality of the energy densities – in the hadron phase $\varrho_{e\ had}$ for $T \leq T_c$ as outlined in ref. [7-2001] and in the qg-phase as defined in eq. 101 $\varrho_{e\ qg}$ for $T \geq T_c$ determine the critical temperature

(102) $$\varrho_{e\ had}(T) = \varrho_{e\ qg}(T;k) \quad \leftrightarrow \quad T = T_c(k)$$

The matching (eq. 102) is further restricted to yield the value

(103) $$T_c \sim 0.2 \text{ GeV} \quad \leftrightarrow \quad k \sim 0.452 \quad ; \quad \text{for Ntype = 65}$$

in accordance with the estimate of one of us [39-1988] .

II : the condition avoiding singular behaviour of pressure gradient

This condition implies

(104) $$g_{hg}(T_c) = g_{qg}(T_c; k, \Delta g) \ \leftrightarrow \ \Delta g = \Delta g(T_c(k))$$

and determines Δg

(105) $$\Delta g \sim 0.00753 \text{ GeV}^3 \simeq 0.94 \text{ fm}^{-3} \text{for Ntype = 65}$$

4 c - Results and discussion

We present figures 1 - 7 in sequence, each followed by an extended caption.

Figure 1 shows how the transition temperature is determined and its stable variation , for the two sets of resonances, Ntype 65 and Ntype 26 forming the HRG $\leftrightarrow$ hg . The two sets are described in chapter 2 , tables 1 - 8 and in Appendix 1 , tables 9 - 11 and figures 8 - 10 .

Figure 2 shows in detail the energy- and Gibbs densities for the selected choice Ntype 65 , as well as on the parameters k , Δg as defined in eqs. 100 - 101 in subsection 4b .

Figure 3 shows the second order nature of the transition – unmodified and modified on the quark-gluon-side by subleading critical exponents – of the quantity $\varrho_{e} (T) / T^{4}$, piecewise described by $\varrho_{e\, hg(-65)} / T^{4}$ for $T \leq T_{c}$ from the HRG side and $\rho_{e\, qg\, no\, mod\, (mod)} / T^{4}$ for $T \geq T_{c}$ from the quark-gluon side, as defined in chapter 2 , Appendix 1 and subsection 4b .

Figure 4 shows the third order nature of the transition under the same conditions as underlying figure 3 for the quantity p / T^{4} , piecewise described by $p_{e\, hg(-65)} / T^{4}$ for $T \leq T_{c}$ from the HRG side and $p_{qg\, no\, mod\, (mod)} / T^{4}$ for $T \geq T_{c}$ from the quark-gluon side .

Figure 5 shows the second order nature of the transition under the same conditions as figures 3 and 4 for the quantity dscale $= (\varrho_{e} - 3\, p) / T^{4}$, displaying the form of an 'indian tent' . The transition is piecewise described by dscale $_{hg}$ for $T \leq T_{c}$ from the HRG side and dscale $_{qg\, no\, mod\, (mod)}$ for $T \geq T_{c}$ from the quark-gluon side .

The modification by subleading critical exponents is given in eq. 106 below .

$\rightarrow$

- Figure 6 shows the quantities forming the 'indian tent' identical to figure 5 . The quantity dscale obtained in ref. [8-2010] from lattice simulation of QCD under the same thermal conditions is also plotted for comparison .
- Figure 7 shows the first order nature of the transition of the square of the velocity of sound , under the same conditions as for figures 3 - 6 , piecewise described by the quantities v^{2}_{hg} for $T \leq T_c$ from the HRG side and $v^{2}_{qg\ no\ mod}$ for $T \geq T_c$ from the quark-gluon side . Only the unmodified setting is used for v^{2}_{gq} .

In figures 3 - 6 the modifications of $\varrho_{e\ qg\ mod} / T^4$ and $p_{qg\ mod} / T^4$ introduce a subleading critical exponent ν in the vicinity of $T = T_c$ allowing the free quark-gluon limits to be reached for $T \to \infty$. The modified quantities are defined as

$$\left\{ \begin{array}{c} \varrho^{\nu}_{e\ qg} \\ p^{\nu}_{qg} \end{array} \right\} = \left\{ \begin{array}{l} fmod\,(\nu\,,\,T\,/\,T_c\,)\,\varrho_{e\ qg} + fmod1\,(\nu\,,\,T\,/\,T_c\,)\,p_{qg} \\ fmod\,(\nu\,,\,T\,/\,T_c\,)\,p_{qg} \end{array} \right\}$$

(106)

$$fmod\,(\nu\,,\,T\,/\,T_c\,) = 1 + |\,1 - T_c\,/\,T\,|^{2\nu}\,(\,1\,/\,k - 1\,)$$

$$fmod1\,(\nu\,,\,T\,/\,T_c) = T\,(\,d\,/\,d\,T\,)\,fmod\,(\nu\,,\,T\,/\,T_c\,)$$

The relations in eq. 106 ensure that the differential equation in eq. 1 is satisfied .

fig1

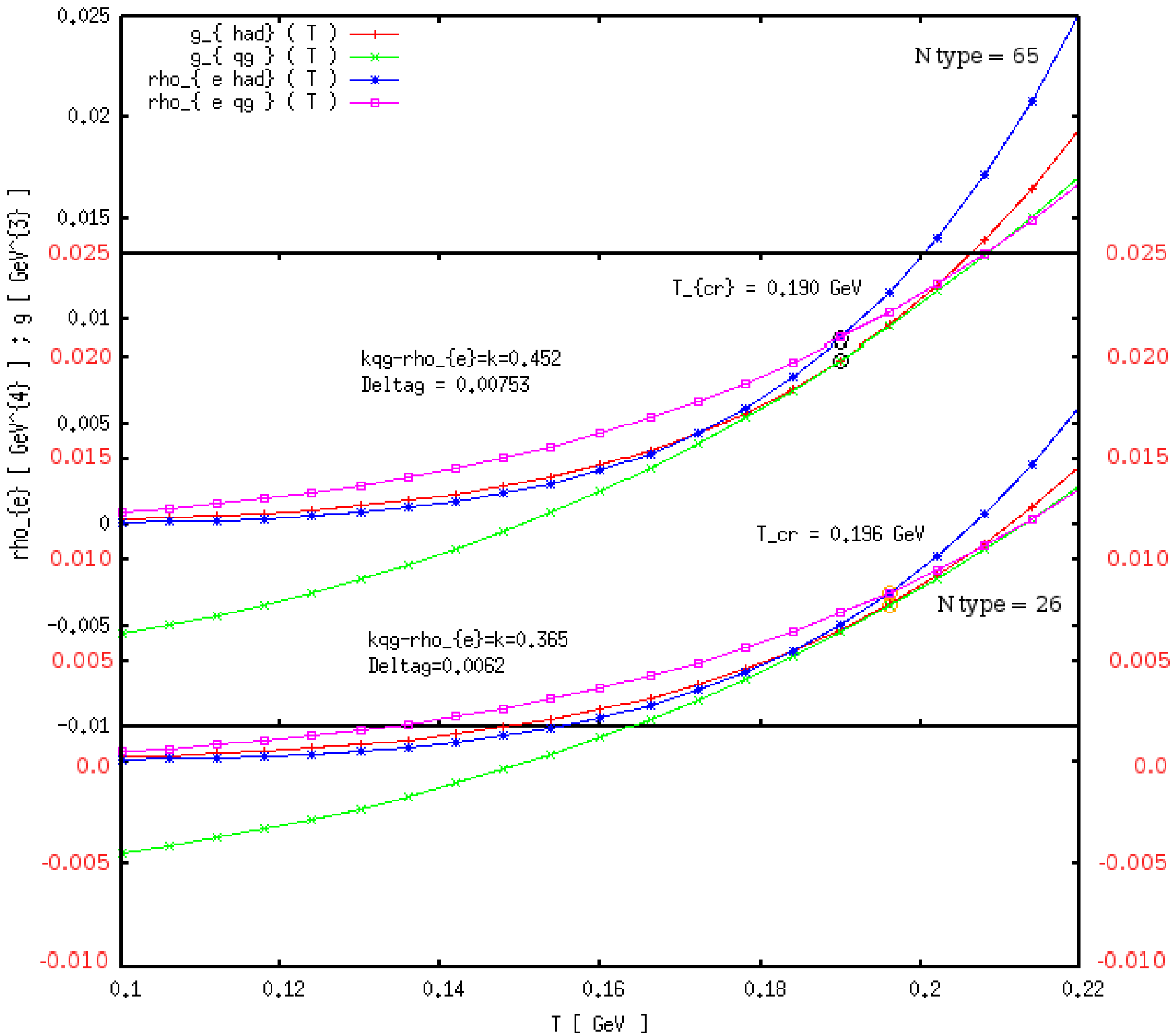


Fig 1 : Dependence of energy- and Gibbs densities $(\varrho_e\ ,\ g)_{hg}$ pertaining to a free hadron gas, for two, offset, choices of spectra : Ntype $= 65 \supset 26$, and associated $(\varrho_e\ ,\ g)_{qg\ no\ mod}$. $\longleftrightarrow$

To Fig 1 : In units of GeV ^{4} and GeV ^{3} for energy- and gibbs densities the same ordinates are used.

The upper offset ordinate corresponds to Ntype = 65 and is marked by ticmarks in red , while the lower ordinate corresponds to Ntype 26 , with ticmarks in black .

The two conditions for a second , third order transition with respect to $\varrho_e \, , \, g$

(107) $$\left\{ \begin{array}{c} \varrho_e \\ \\ g \end{array} \right\}_{hg} (T) = \left\{ \begin{array}{c} \varrho_e \\ \\ g \end{array} \right\}_{qg} (T) \quad \textbf{for} \quad T = T_c$$

are marked by two $\circ$ **symbols separately for Ntype = 65 , 26 . The two sets of parameters** $T_c \, , \, k \, (\equiv rk) \, , \, \Delta g$ **given in eqs. 104 , 105 become**

(108)

T_c	k	Δg	Ntype
0.190 GeV	0.452	0.00753 GeV ^{3}	65
0.196 GeV	0.365	0.0062 GeV ^{3}	26

The quantities with suffix $_{qg}$ **refer to unmodified ones (no mod) .**

fig2

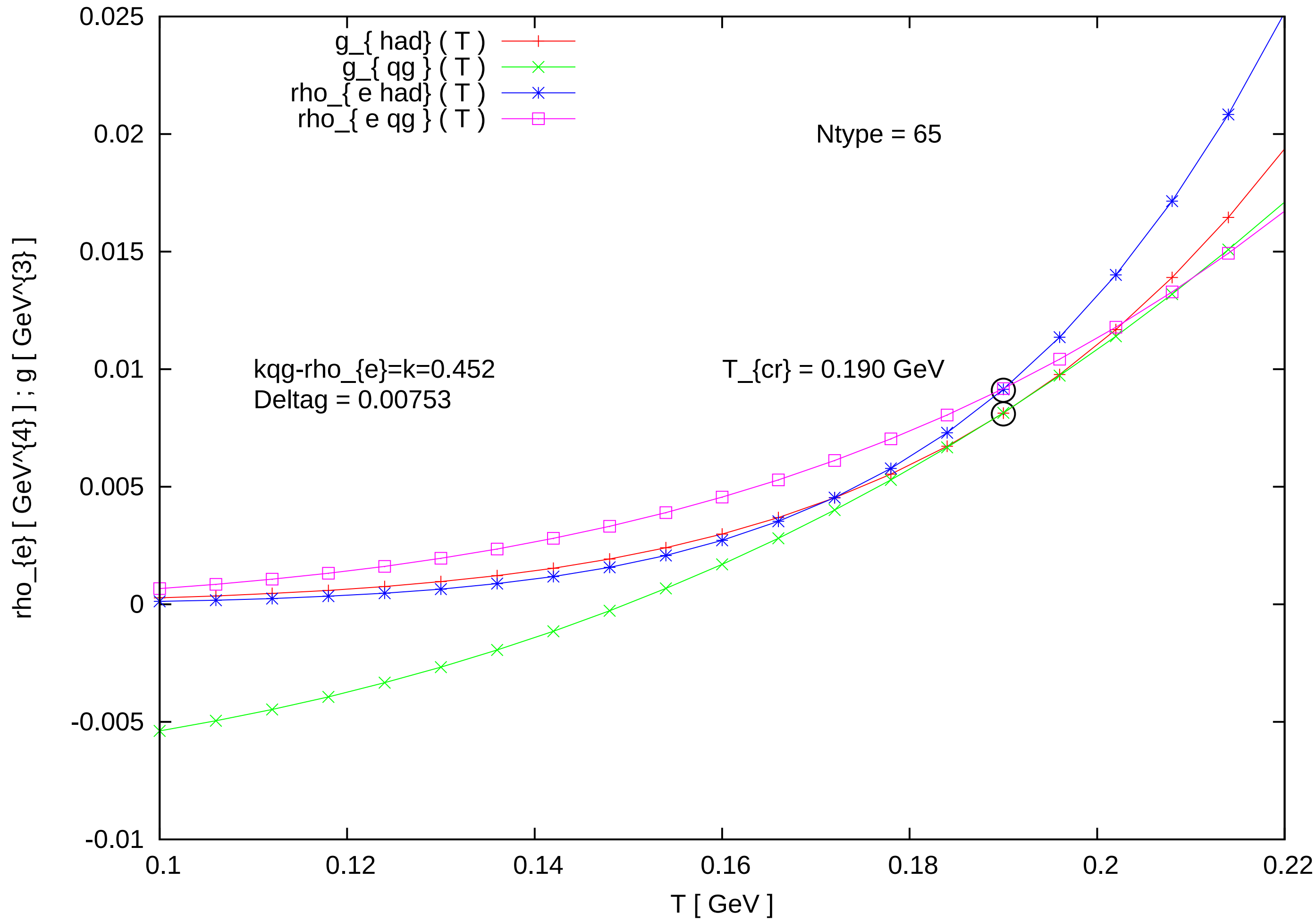


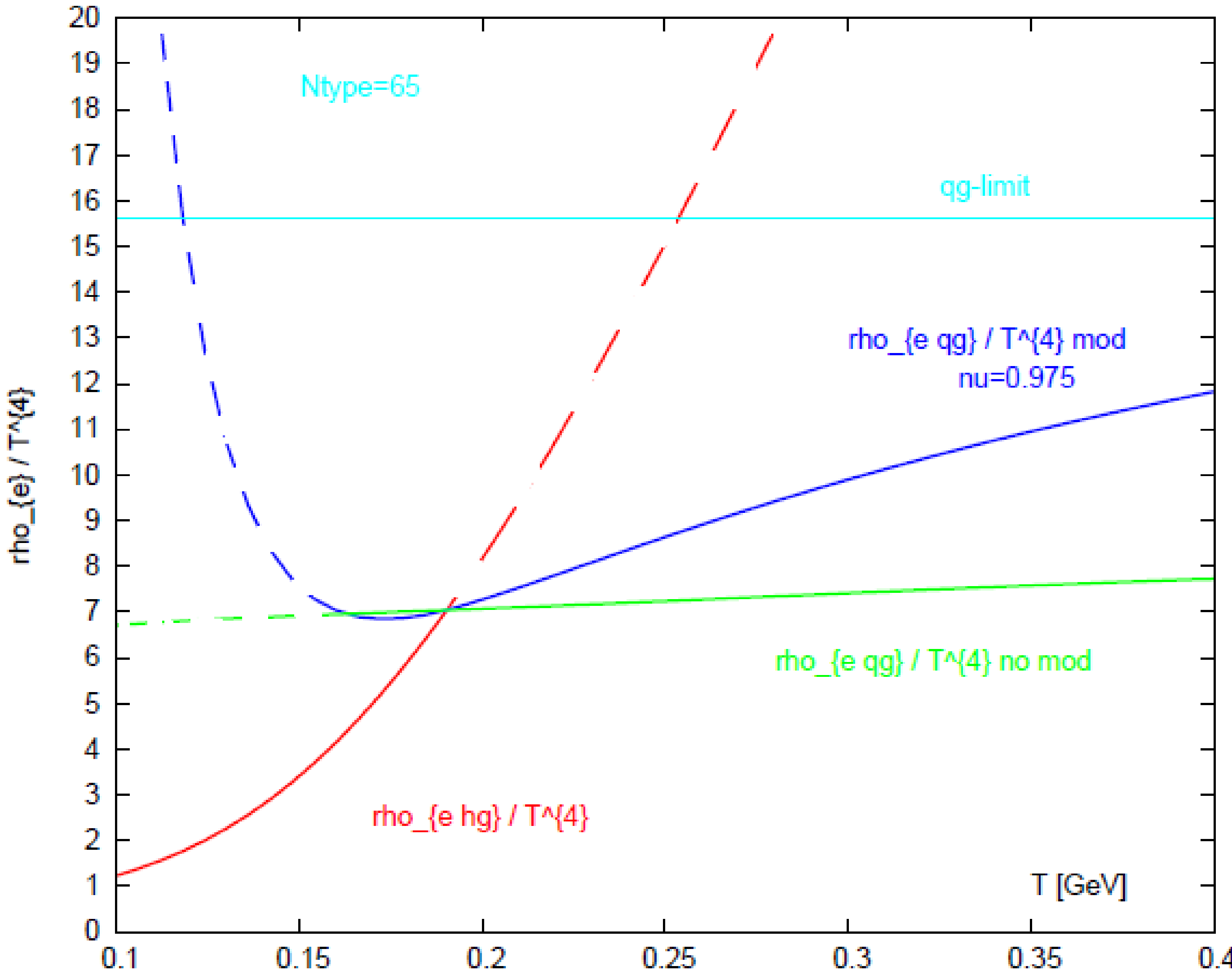
Fig 2 : $\longleftrightarrow$

c-fig2

To Fig 2 : The upper offset case in Fig 1 for Ntype = 65 is displayed separately . Both captions and direct and extended to Fig 1 also refer to Fig 2 . Ntype = 65 is subject of more detailed study in the following .

$\longrightarrow$

fig3



Ntype=65
qg-limit
rho_{e qg} / T^{4} mod
nu=0.975
rho_{e qg} / T^{4} no mod
rho_{e hg} / T^{4}
T [GeV]
rho_{e} / T^{4}
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
0.1
0.15
0.2
0.25
0.3
0.35
0.4

Fig 3 : ⟷

c-fig3

To Fig 3 : The enumeration in subsection 4c with respect to Figure 3 is repeated below .

It shows the second order nature of the transition – unmodified and modified on the quark-gluon-side by subleading critical exponents – of the quantity $\varrho_{\,e}\,(\,T\,)\,/\,T^{\,4}$, piecewise described by $\varrho_{\,e\;hg(-65)}\,/\,T^{\,4}$ for $T\;\leq\;T_{\,c}$ from the HRG side and $\rho_{\,e\;qg\;no\;mod\;(mod)}\,/\,T^{\,4}$ for $T\;\geq\;T_{\,c}$ from the quark-gluon side, as defined in chapter 2 , Appendix 1 and subsection 4b .

The three curves are represented with dashed lines outside their range of validity. The quantity $\rho^{\,\nu}_{\,e\;qg\;mod}$ is defined in eq. 106 , here with $\nu\;=\;0.975$, the same value as used in the subsequent figures 4 - 6.

fig4

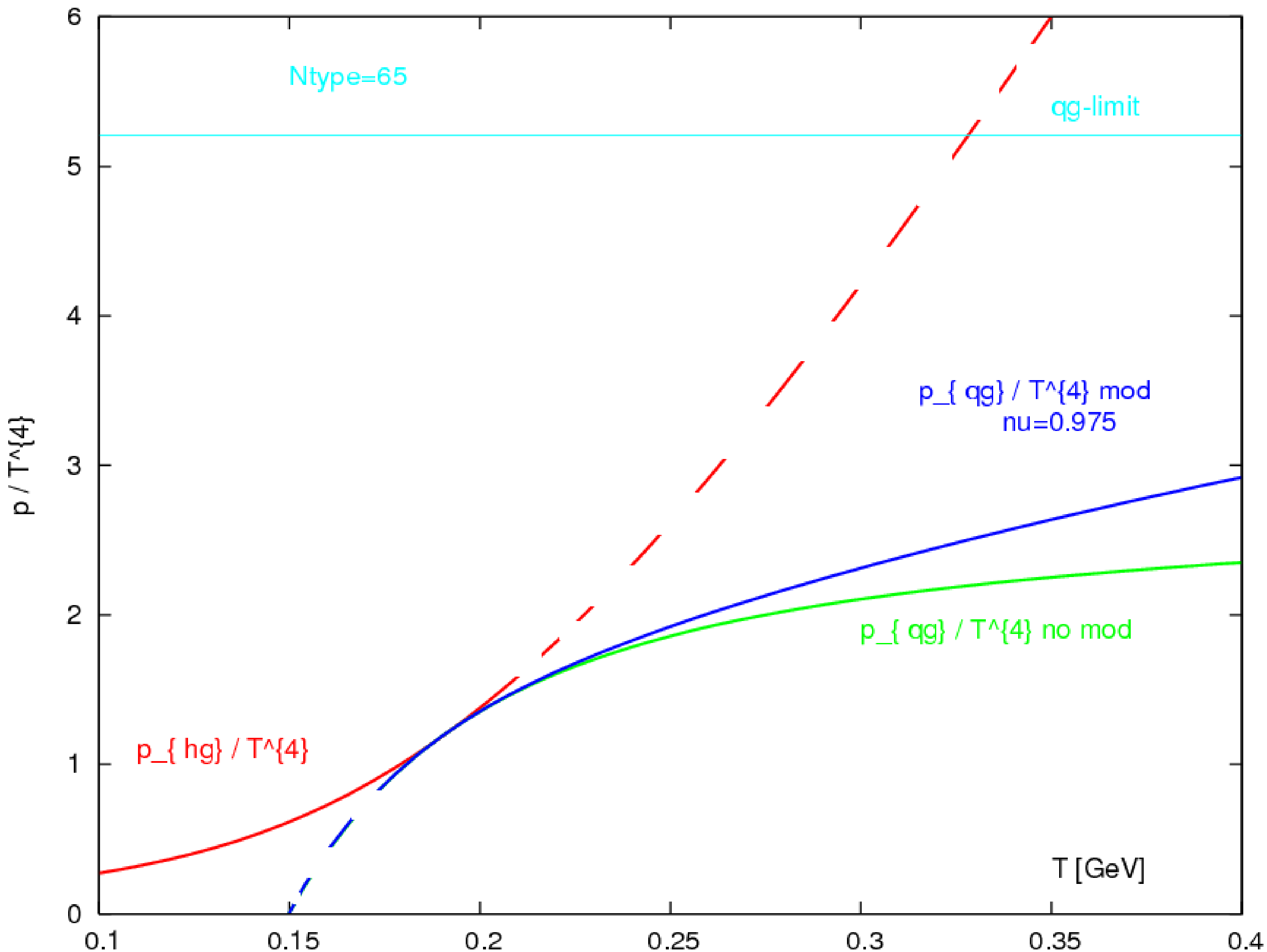

Ntype=65
qg-limit
p_{ qg} / T^{4} mod
nu=0.975
p_{ qg} / T^{4} no mod
p_{ hg} / T^{4}
T [GeV]
p / T^{4}
6
5
4
3
2
1
0
0.1
0.15
0.2
0.25
0.3
0.35
0.4

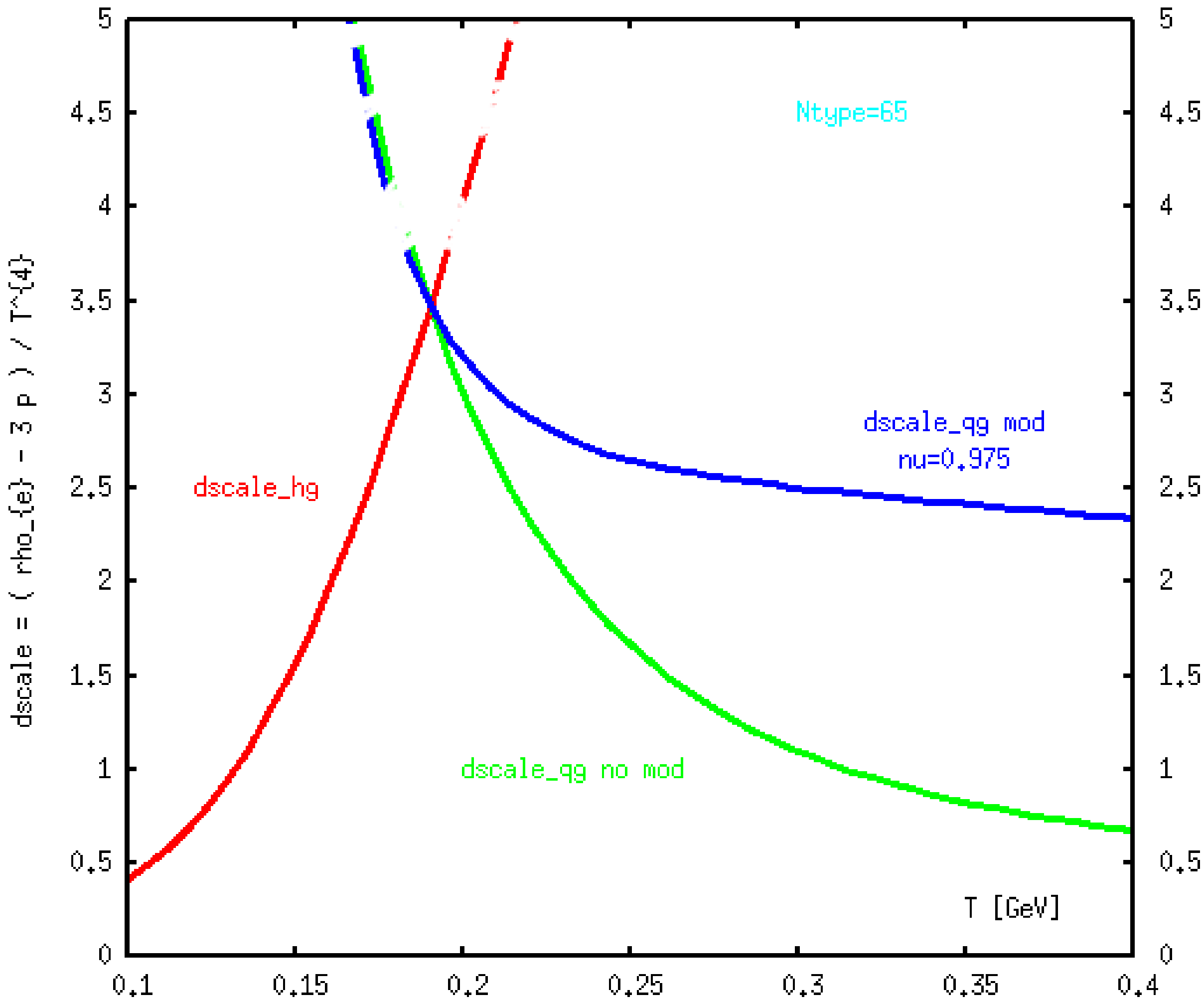
Fig 4 : ⟷

To Fig 4 : The enumeration in subsection 4c with respect to Figure 4 is repeated below .

- **It shows the (dominantly) third order nature of the transition under the same conditions as underlying figure 3 for the quantity $p \, / \, T^{\,4}$, piecewise described by $p_{\,e\;hg(-65)} \, / \, T^{\,4}$ for $T \, \leq \, T_{\,c}$ from the HRG side and $p_{\,qg\;no\;mod\;(mod)} \, / \, T^{\,4}$ for $T \, \geq \, T_{\,c}$ from the quark-gluon side .**

- **The three curves are represented with dashed lines outside their range of validity. The quantity $p^{\,\nu}_{\,qg\;mod}$ is defined in eq. 106 , here with $\nu \, = \, 0.975$, as used in the subsequent figures 5 - 6.**

fig5



dscale = (rho_{e} - 3 p) / T^{4}
T [GeV]
Ntype=65
dscale_hg
dscale_qg mod
nu=0.975
dscale_qg no mod

Fig 5 : ⟷

To Fig 5 : The 'indian tent' .

The enumeration in subsection 4c with respect to Figure 5 is repeated below .

It shows the (dominantly) second order nature of the transition under the same conditions as figures 3 and 4 for the quantity dscale $= (\varrho_e - 3p) / T^4$, displaying the form of an 'indian tent' . The transition is piecewise described by dscale $_{hg}$ for $T \leq T_c$ from the HRG side and dscale $_{qg\ no\ mod\ (mod)}$ for $T \geq T_c$ from the quark-gluon side .
The three curves are represented with dashed lines outside their range of validity. The quantity dscale $^{\nu}_{qg\ mod}$ is obtained from eq. 106 , here with $\nu = 0.975$, as used in the subsequent figure 6.

fig6

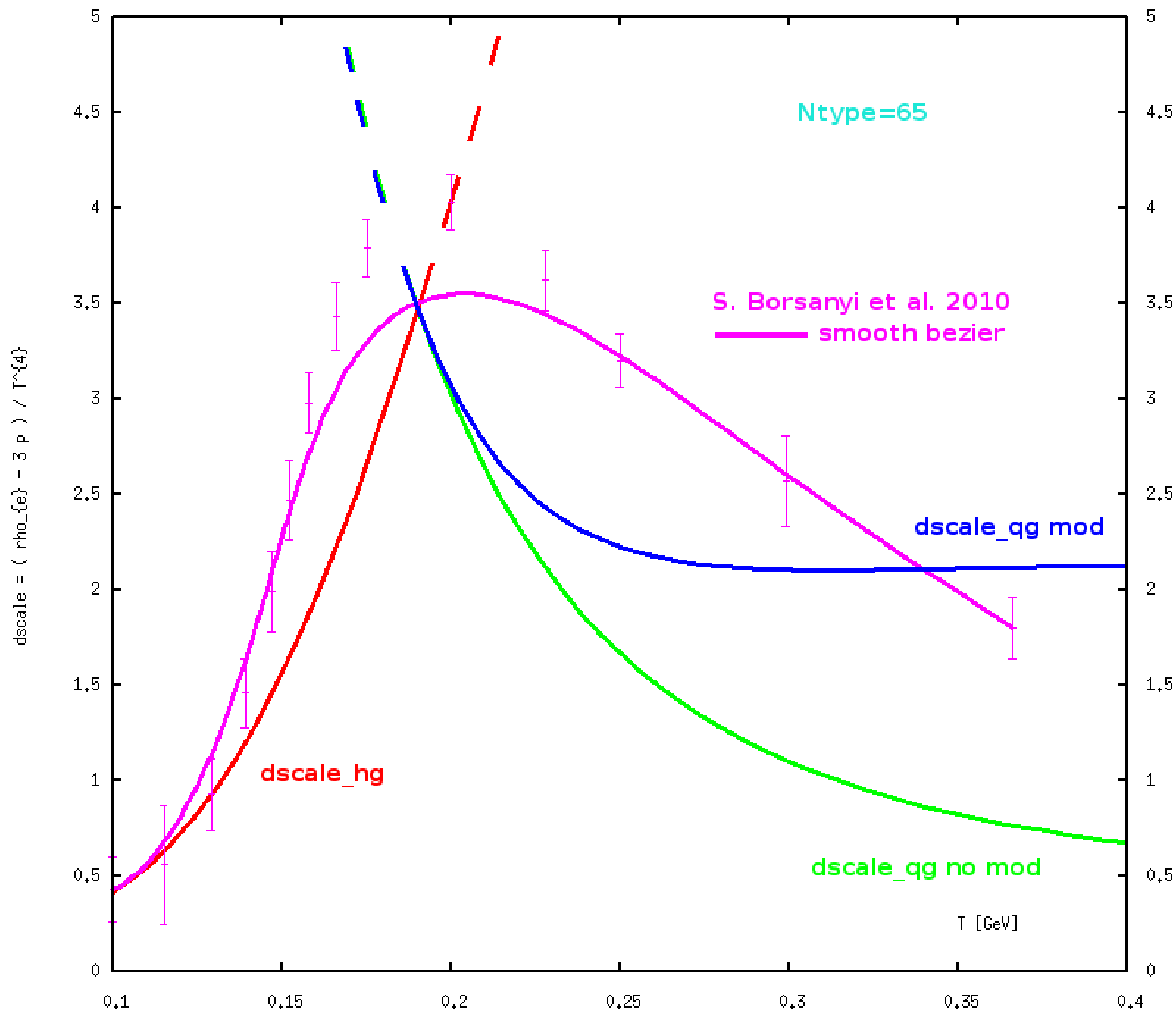

Ntype=65
S. Borsanyi et al. 2010
smooth bezier
dscale_qg mod
dscale_hg
dscale_qg no mod
T [GeV]
dscale = (rho_{e} - 3 p) / T^{4}
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
0.1
0.15
0.2
0.25
0.3
0.35
0.4

Fig 6 : ⟷

c-fig6

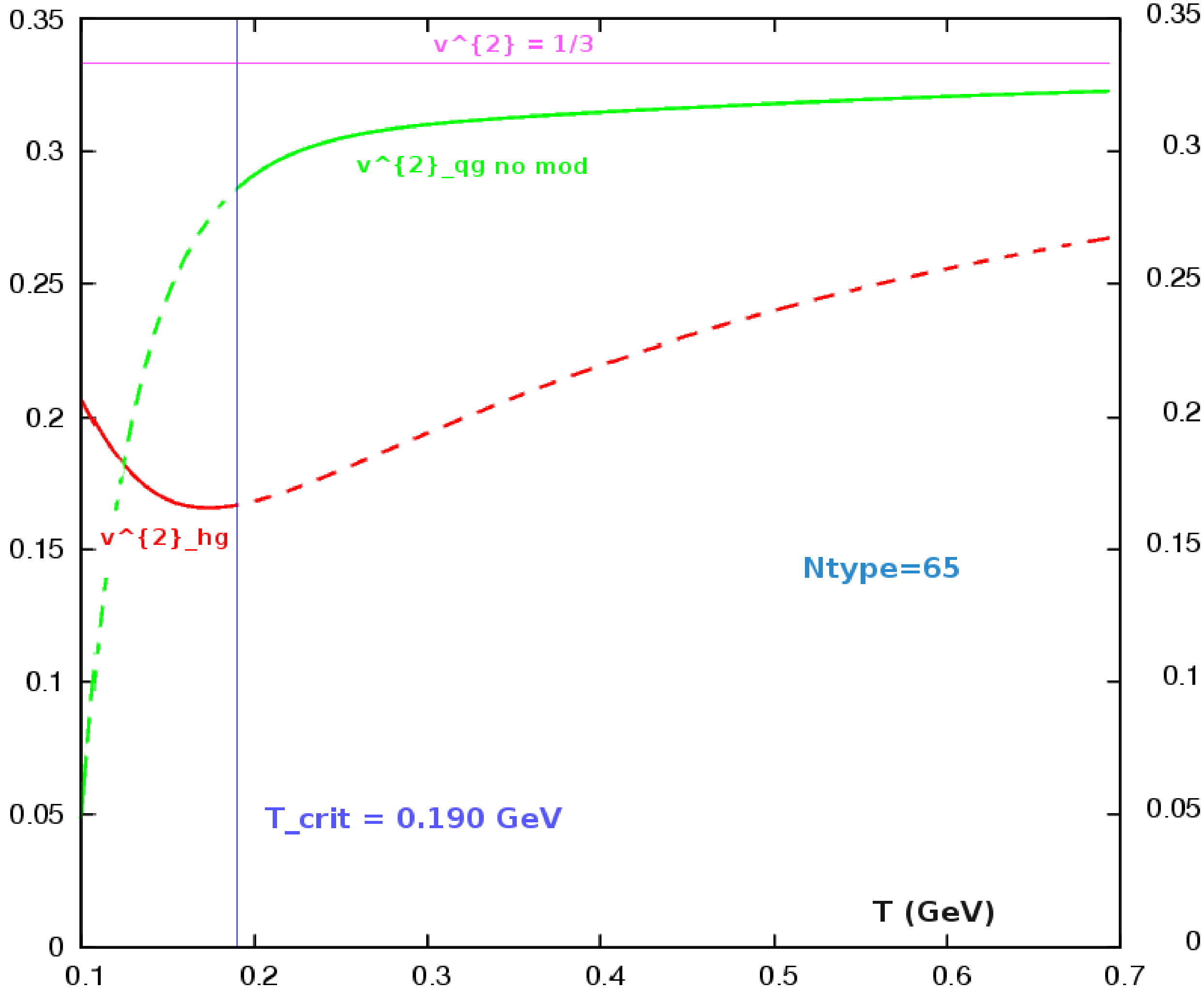
To Fig 6 : The 'indian tent' as compared with the lattice QCD calculation of dscale , also called the trace anomaly , by S. Borsanyi et al. [8-2010] .

The enumeration in subsection 4c with respect to Figure 6 is repeated below .

It shows the quantities (dscale) forming the 'indian tent' identical to figure 5 . The same quantity as obtained in ref. [8-2010] from lattice simulation of QCD under the same thermal conditions is also plotted for comparison .

The three curves forming the 'indian tent' are represented with dashed lines outside their range of validity. The quantity dscale $^{\nu}_{qg\ mod}$ is obtained from eq. 106 , here with $\nu = 0.975$, the same as in figures 3 - 5 .

fig7



0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
v^{2} = 1/3
v^{2}_qg no mod
v^{2}_hg
Ntype=65
T_crit = 0.190 GeV
T (GeV)

Fig 7 : ⟷

To Fig 7 : The square of the velocity of sound .

The enumeration in subsection 4c with respect to Figure 7 is repeated below .

It shows the first order nature of the transition of the square of the velocity of sound , under the same conditions as for figures 3 - 6 , piecewise described by the quantities v_{hg}^2 **for** $T \leq T_c$ **from the HRG side and** $v_{qg\ no\ mod}^2$ **for** $T \geq T_c$ **from the quark-gluon side . Only the unmodified setting is used for** v_{gq}^2 **.**
The two curves for v_{hg}^2 , v_{gq}^2 **are represented with dashed lines outside their range of validity.**

4 d - Discussion and interpretation

Derivations concerning the phase structure of QCD at vanishing chemical potentials , laid out in the present discussion , follow within the *hypothesis* **, that the gauge boson pair condensate is at the origin of stable embeddings of quark flavors – light and heavy – into the (long range) dynamics controled** *primarily* **by gauge boson self-interactions .** *By the same hypothesis* **– this embedding depends in a minimal way on quark masses , including their chiral as well as anti-chiral limits :**
$m_{\chi_1}, \cdots m_{\chi_M} \rightarrow 0 \, ; m_{h_1}, \cdots m_{h_N} \rightarrow \infty$ **.**
Extensive lattice simulations of the thermodynamic state in particular at zero chemical potential have been performed in recent years. With realistic quark masses for the three light flavours u,d,s no phase transition but a cross over region is found. For a review we refer to ref. [13].
For pure gauge theories with gauge group $SU N_c, N_c > 2$ **, a 1st order phase transition and for** $N_c = 2$ **, a second order transition is found [14,15].**
The latter phase transition agrees with the results pertaining to the 3-D Ising model.

The argument for – at least – a second order phase transition with respect to energy density

In this subsection a short outline is given of how spontaneous parameters at $T = 0$ **and** $T > 0$ **are obtained in general renormalizable local field theories.** $\longrightarrow$

We define – first in physical space time – the effective action Γ for the composite classical field $\varphi_{cl} \leftrightarrow \varphi$ associated with the local gauge boson bilinear, introduced in eq. 93

(109)
$$\begin{aligned} \exp\ iW \ &= \ \langle \Omega | \exp \left(i \int d^4 x \left(\mathcal{L} + \mathcal{J} \varphi \right) (x) \right) | \Omega \rangle_c \\ \varphi(x) &= \tfrac{1}{4} (:) F_{\mu\nu}^{A}\ (x)\ F^{\mu\nu\ A}\ (x) (:)\ ;\ \mathcal{J}(x)\ :\ \textbf{source} \end{aligned}$$

in the presence of a classical local source $\mathcal{J}$. $_c$ in eq. 109 denotes the connected part of the generating functionsal W. The effective action $\Gamma(\varphi_c)$ is obtained from the generating functional W through functional Legendre transform

(110)
$$\begin{aligned} &\varphi_{cl}(x) = \delta / \delta \mathcal{J}(x)\, W(\mathcal{J}) \ \rightarrow \\ &\Gamma(\varphi_{cl}) = \int d^4 x \ \left(\ \varphi_{cl}(x) \mathcal{J}(x) \ \right) \ - \ W \ ;\ \varphi_{cl} = \varphi_{cl}(\mathcal{J}) \\ &\delta / \delta \varphi_{cl}(x)\, \Gamma = \mathcal{J}(x)\ ;\ \ \varphi_{cl}(x) |_{\mathcal{J} \rightarrow 0} = \mathcal{B}^2\ ,\ (\textbf{eqs. 94,97}) \\ &\Gamma(\varphi_{cl}) = \int d^4 x\, v(\varphi_{cl}) \end{aligned}$$

The (nonlocal) density functional v defined in eq. 110 determines the spontaneous ground state expected value of φ_{cl} in the limit of vanishing sources as *absolute minimum*. $\longrightarrow$

It is essential to safeguard appropriate boundary and/or integrability conditions for the sources $\mathcal{J}(x)$ which translate into corresponding conditions for the Legendre transforms $\varphi_{cl}(x)$.
In particular as laid out in chapter 3 , the boundary conditions implied by continuity with respect to complete connections , are not (necessarily) just compatible with those used explicitely and implicitely in lattice QCD in general and restricted to thermal ensembles in particular .
The aim of the present outline and discussion is to show the consequences arising from the hypotheses as stated , for the thermodynamic intensive state functions energy density , pressure and 'trace anomaly' or dscale and all equivalent ones , e.g. as divided by $T^{\,4}$.
The proof of stability and nature of the boundary conditions arising from complete connections and the gauge boson pair condensate is a fine task in the future.
An integrability condition arising from this condensate is

(111) $$\lim_{x \to \infty} \mathcal{J}(x) = 0 \;\leftrightarrow\; \int d^4 y \; |\mathcal{J}|^2 \, (y) < \infty$$

as described in ref. [40-1997] . The functionals W , $\mathcal{J}$ in eqs. 109, 110 can be extended to Euclidean space. The absolute thermodynamic limit at zero temperature is reached from the boundary conditions in eq. 111 for spontaneous vacuum parameters allowing sources and classical fields to approach *also* constant values [29-2002], whereas thermal environment at finite temperature T is equivalent to a finite
$\longrightarrow$

Euclidean time extension

(112)
$$\Delta\, t_E = \beta = 1/T$$

In the finite temperature thermal limit new boundary conditions at the bounding times $t_E = 0\ ,\ \beta$ have to be set, periodic for *gauge invariant* bosonic sources and classical fields, as considered above relative to the local operator $\varphi(x)$ defined in eq. 109. Hence we face the two thermodynamic limits, treated here for *just* the associated quantities

(113)
$$\begin{aligned} &\tfrac{1}{4}(:)F_{\mu\nu}^{A}\ (x)\ F^{\mu\nu\ A}\ (x)(:) = \varphi(x)\ \ \rightarrow\ \ (\ \varphi_{cl}(x), \mathcal{J}(x)\ ) \\ &\mathcal{J}(x) = (\ 1/\ g^2\ ) - (\ 1/\ g^2(x)\ ) = -\Delta\ \frac{1}{g^2}\ (x) \end{aligned}$$

The external source $\mathcal{J}(x)$ represents a space time dependent coupling constant, before thermodynamic limits are taken. $\longrightarrow$

Absolute thermodynamic ($T \equiv 0$) – and thermal finite temperature (T finite) limits yield the relations

(114)
$$\begin{array}{ccc} \mathcal{J}(x) & & J \in [-J_*, J_*] \to 0 \\ & \underset{T}{\longrightarrow} & \\ v(\varphi_{cl}(x)) & & v(\varphi_{cl}, T) \end{array} \Bigg| \; T \; : \begin{cases} \equiv 0 \\ \text{finite} \end{cases}$$

$$\partial_{\varphi_{cl}} v(\varphi_{cl}, T) = J \;\to\; 0 \;\to\; \varphi_{cl} = \varphi_{cl}(T)$$

Performing the sequence of limits and letting the (constant) values of the source J vary in a suitable interval $[-J_*, J_*]$ before relaxing it to zero, as shown in eq. 114, defines the hysteresis line of the spontaneous parameter $\varphi_{cl}(J \to 0, T)$. The latter emerges as thermal average for T finite and as vacuum expected value for $T \equiv 0$ (eqs. 109-111).
In the T finite case we have the choice to include chemical potentials , one each for conserved quark flavor neutral currents, which we set to zero here.
The phase transition underlying the present discussion is associated with the vanishinging in a singular way at finite critical temperature, $T = T_{cr} > 0$ of the quantity $\varphi_{cl}(T)$ defined in eq. 114

(115)
$$\varphi_{cl}(T) \to 0 \quad \text{for} \quad T \to T_{cr}$$

Classical configurations leading to this behaviour inherit a 'Watt-less' nature from beeing particular to the ground state of QCD [41-1981, 42-1981] and thus should not generate a step-like behaviour of thermal *energy density*. $\longrightarrow$

This is borne out in the subtraction of the ground state projection of the associated energy momentum density tensor $\vartheta_{\mu\nu}$ as shown in eqs. 95,96.
This however does not imply that the first derivative of energy density changes by a finite amount through the transition, i.e. its corresponding genuinely second order nature. The singularity could well be characterized by critical exponents without relation to a definite step in a given derivative – of thermal energy density.

The mechanism of gauge boson pair condensation discussed here is a new analysis taking its roots in material presented in refs. [41-1981], [39-1988] and [7-2001] in chronological order. It is centered on the thermal average of the field strength bilinear local density operator defined in eq. 93

(116) $$\frac{1}{4}(:)F_{\mu\nu}^{A}\,(x)\;F^{\mu\nu\;A}\,(x)(:)$$

The nonzero vacuum expected value (eq. 94) signals a connection between the localization of color and the structure of Bogoliubov transformations as appropriate for gauge fields at long range.
For QCD, actual calculations of effective potentials at large and intermediary range are not accessible to perturbation theory. Hence analytic control is lacking at present.

5 - Outlook

The theoretical considerations concerning the phase structure of QCD for vanishing chemical potentials lead to the illustration in principle of thermodynamic energy density and pressure in the vicinity of the inferred critical temperature $T_c \sim 200$ Mev, as described between eqs. 101 and 108 .
A comparison with the strong coupling regime observed in central AuAu collisions at 200 GeV at RHIC [4-2005, 43-2005] may well establish through future analyses a direct quantitative derivation of thermal state functions.
The reduction of pressure and energy density relative to noninteracting (anti-) quarks and gauge bosons is interpreted here in the light of a hypothetical phase transition, of (essentially) second order with respect to energy density, as demonstrated in the related thermal quantities shown in figures 3 and 4 , and (essentially) first order with respect to the square of the velocity of sound through its discontinuity as shown in figure 7 – 'pour fixer les idées' .
I hope that the upcoming lower c.m. energy scan at RHIC, new results expected from LHC concerning hadronic physics combined with extended studies in lattice simulations will guide future attempts to establish a link between coherent gauge boson pair condensation and the phase structure of QCD.

Acknowledgements

It is a pleasure to thank the TH-division of CERN for its hospitality, and Anne Perrin as well as Markus Moser, representing the group of computing coordinators at the ITP in Bern, for their logistics support. Topical discussions with Martin Lüscher and Uwe-Jens Wiese are gratefully acknowledged.

Appendix 1 **Tables and figures of thermal densities : ϱ_e / T^4 , p / T^4 and dscale = $(\varrho_e - 3p) / T^4$ for HRG choices Ntype 65 and 26**

A1-2 table densities 65/26

T (MeV)	ϱ_e / T^4 Ntype=65	ϱ_e / T^4 Ntype=26	T (MeV)	ϱ_e / T^4 Ntype=65	ϱ_e / T^4 Ntype=26
100	1.234	1.171	340	29.329	17.382
112	1.561	1.449	352	31.199	18.297
124	1.998	1.803	364	33.039	19.187
136	2.564	2.241	376	34.845	20.050
148	3.279	2.766	388	36.614	20.886
160	4.150	3.378	400	38.344	21.693
172	5.181	4.072	412	40.033	22.472
184	6.366	4.840	424	41.679	23.223
190	7.014	5.250			
196	7.697	5.674	436	43.282	23.946
208	9.159	6.563	448	44.840	24.642
220	10.737	7.496	460	46.354	25.310
232	12.413	8.464	472	47.825	25.953
244	14.169	9.456	484	49.251	26.570
256	15.988	10.464	496	50.635	27.163
268	17.855	11.478	508	51.976	27.732
280	19.754	12.493	520	53.275	28.277
292	21.672	13.501	532	54.534	28.801
304	23.599	14.498	544	55.752	29.303
316	25.522	15.479	556	56.932	29.785
328	27.435	16.441	568	58.074	30.247

Table 9 : ϱ_e / T^4 **in the T range** 100 **MeV** $\leq T \leq 568$ **MeV for HRG choices Ntype 65 and 26 .**

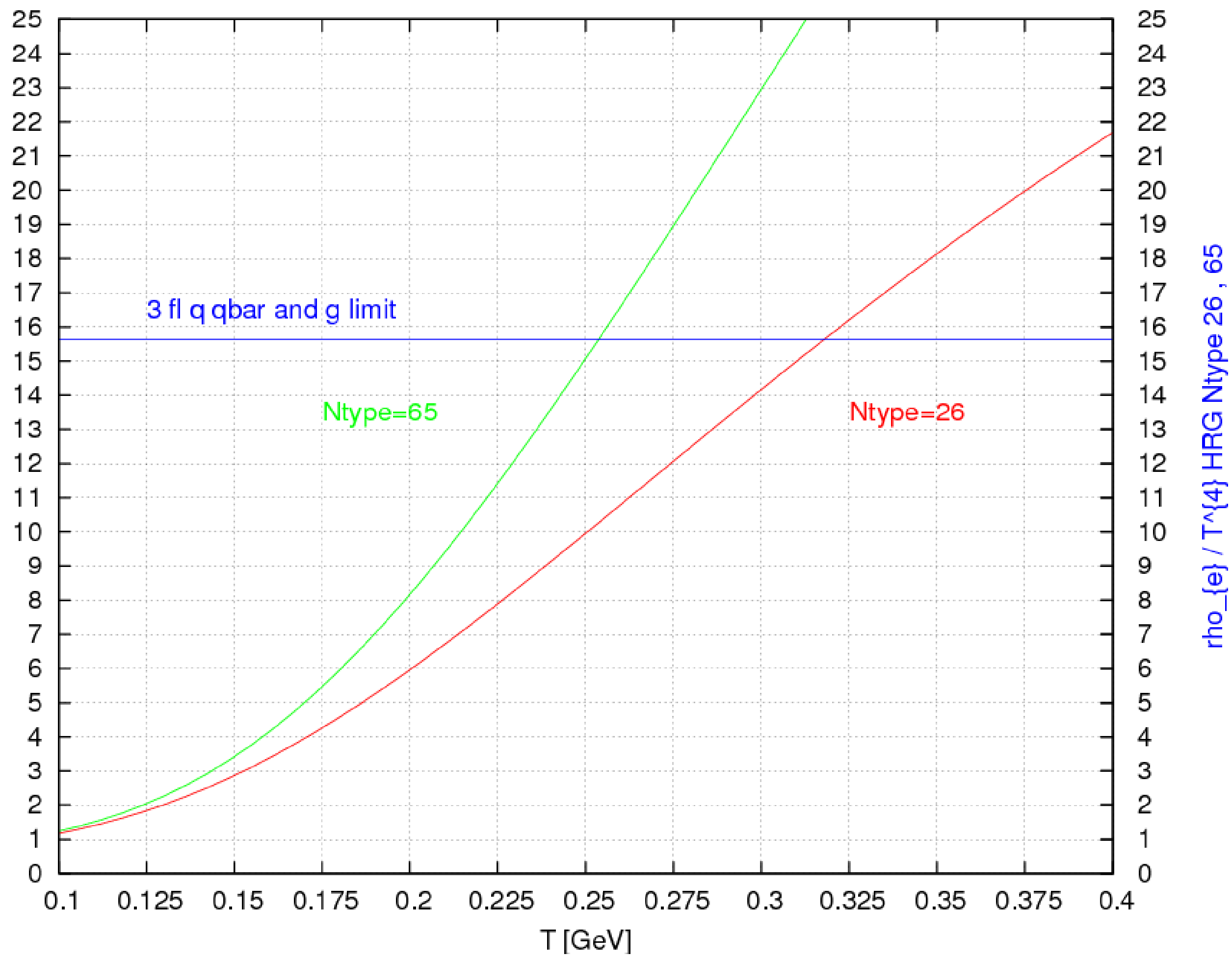


Fig 8 : $\varrho_e \, / \, T^4$ in the T range 0.1 GeV $\leq T \leq 0.4$ GeV for HRG choices Ntype 26 and 65 .

A1-4 table pressures 65/26

T (MeV)	$p\,/\,T^{\,4}$ Ntype=65	$p\,/\,T^{\,4}$ Ntype=26	T (MeV)	$p\,/\,T^{\,4}$ Ntype=65	$p\,/\,T^{\,4}$ Ntype=26
100	0.272	0.264	340	5.637	3.575
112	0.328	0.314	352	6.077	3.810
124	0.397	0.374	364	6.521	4.043
136	0.485	0.446	376	6.965	4.275
148	0.594	0.533	388	7.411	4.504
160	0.728	0.636	400	7.855	4.731
172	0.890	0.754	412	8.297	4.954
184	1.079	0.888	424	8.737	5.174
190	1.185	0.961			
196	1.298	1.038	436	9.173	5.390
208	1.545	1.201	448	9.604	5.602
220	1.819	1.379	460	10.031	5.810
232	2.120	1.568	472	10.453	6.013
244	2.445	1.767	484	10.868	6.212
256	2.792	1.976	496	11.278	6.407
268	3.158	2.192	508	11.681	6.597
280	3.541	2.414	520	12.078	6.782
292	3.940	2.642	532	12.468	6.963
304	4.351	2.872	544	12.851	7.139
316	4.772	3.106	556	13.227	7.311
328	5.201	3.340	568	13.596	7.478

Table 10 : $p\,/\,T^{\,4}$ **in the T range** 100 **MeV** $\leq\ T\ \leq\ 568$ **MeV for HRG choices Ntype 65 and 26 .**

A1-5 fig9

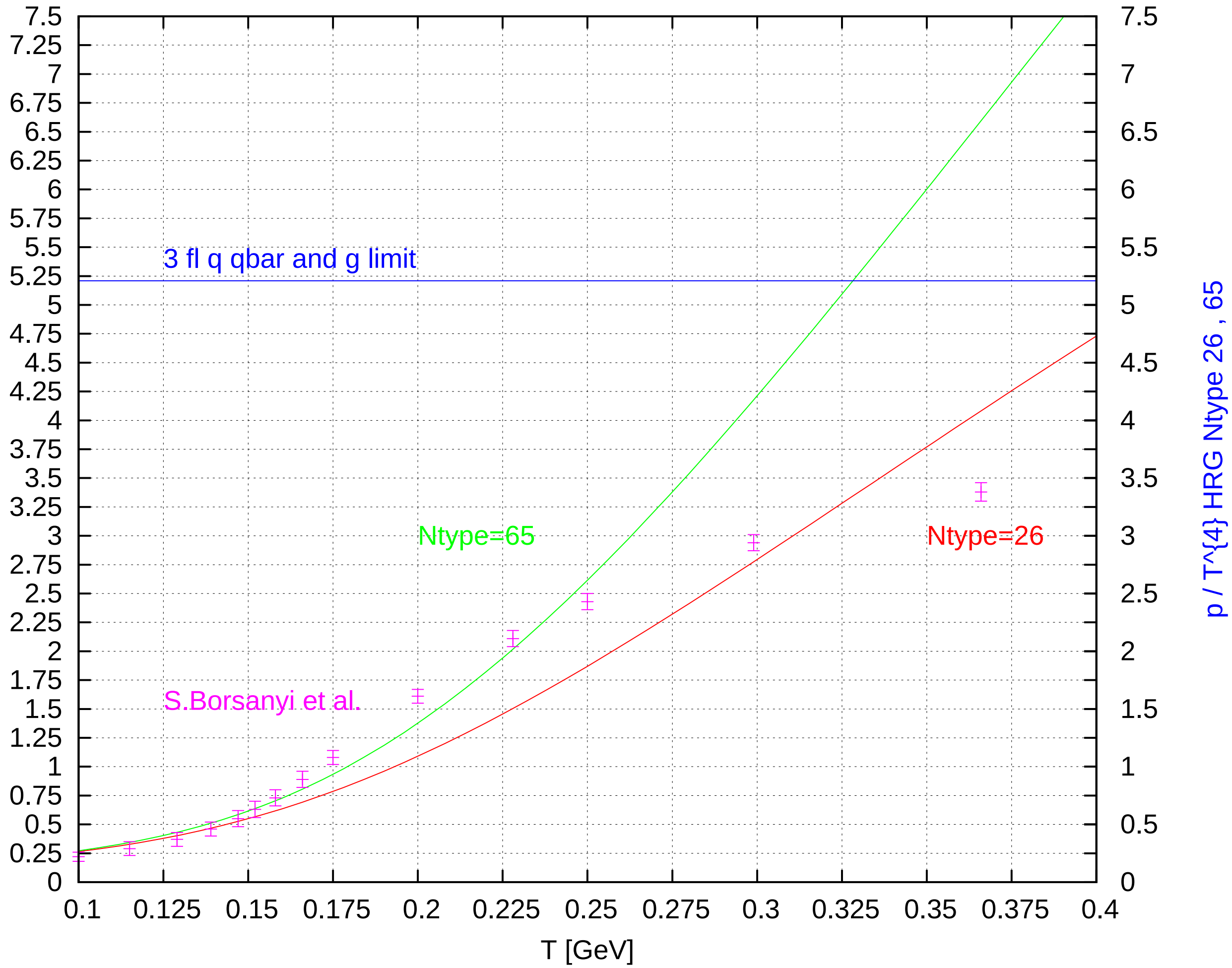


Fig 9 : $p \, / \, T^{\,4}$ **in the T range** 0.1 **GeV** $\leq \; T \; \leq \; 0.4$ **GeV for HRG choices Ntype 26 and 65 ans comparison with S. Borsanyi et al. [8-2010] .**

$\longleftrightarrow$

A1-6 table traces 65/26

T (MeV)	dscale Ntype=65	dscale Ntype=26	T (MeV)	dscale Ntype=65	dscale Ntype=26
100	0.418	0.379	340	12.418	6.656
112	0.578	0.508	352	12.967	6.868
124	0.806	0.682	364	13.477	7.058
136	1.110	0.902	376	13.949	7.226
148	1.496	1.167	388	14.383	7.373
160	1.965	1.471	400	14.780	7.500
172	2.512	1.810	412	15.141	7.609
184	3.129	2.176	424	15.469	7.701
190	3.460	2.366			
196	3.804	2.561	436	15.764	7.775
208	4.525	2.958	448	16.027	7.835
220	5.279	3.361	460	16.261	7.881
232	6.052	3.761	472	16.467	7.913
244	6.834	4.155	484	16.646	7.933
256	7.613	4.536	496	16.801	7.943
268	8.381	4.902	508	16.932	7.942
280	9.130	5.249	520	17.041	7.931
292	9.853	5.576	532	17.130	7.913
304	10.547	5.881	544	17.200	7.886
316	11.207	6.163	556	17.252	7.853
328	11.832	6.421	568	17.287	7.813

Table 11 : dscale = $(\varrho_e - 3p) / T^4$ in the T range 100 MeV $\leq T \leq 568$ MeV for HRG choices Ntype 65 and 26 .

A1-7 fig10

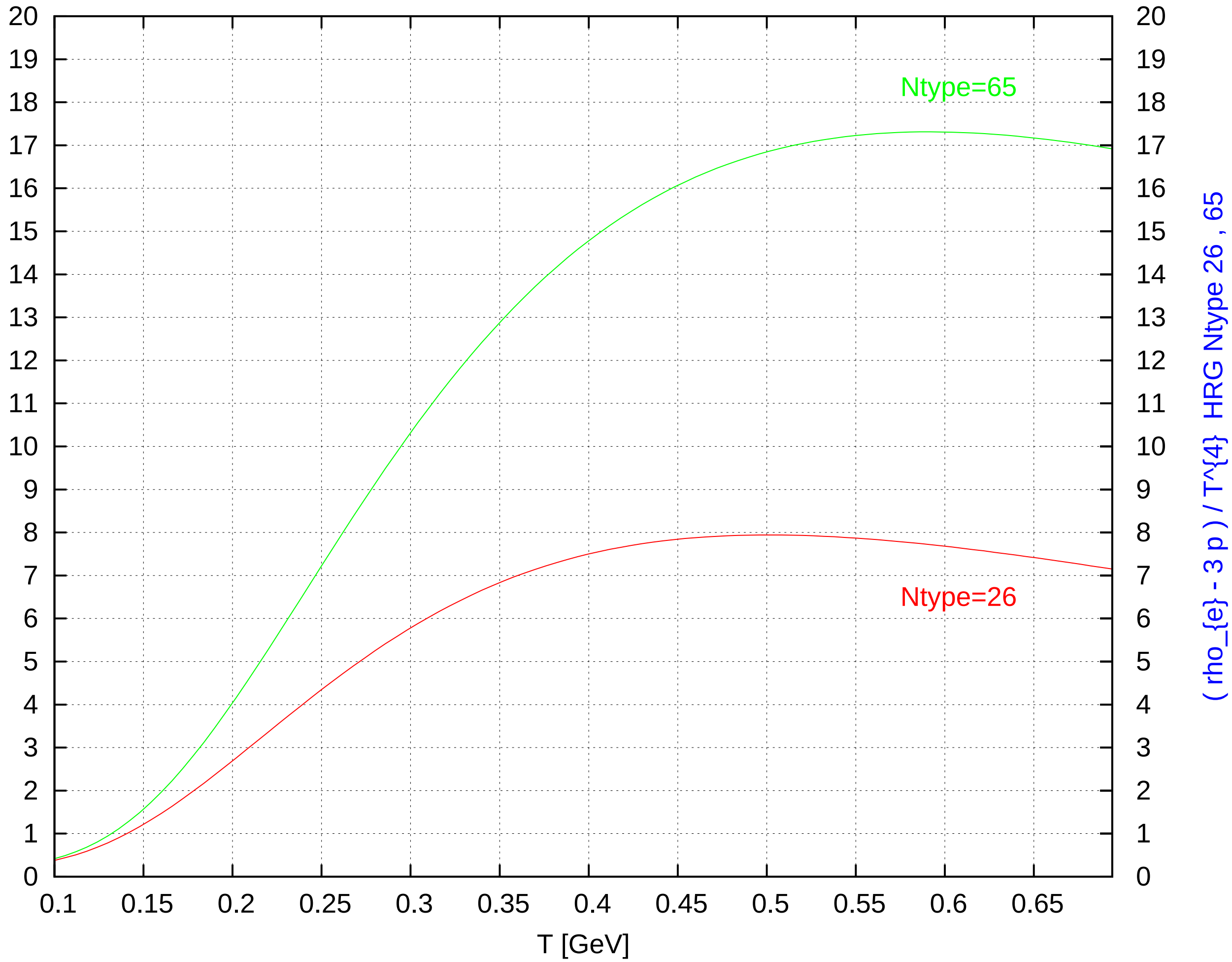


Fig 10 : dscale = $(\varrho_e - 3p) / T^4$ **in the T range** 0.1 **GeV** $\leq T \leq 0.694$ **GeV for HRG choices Ntype 26 and 65 .**

$\longleftrightarrow$

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