Oscillations of three Majorana neutrinos and a possible connection to QCD

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Abstract

The three light neutrino flavors exhibit mass scales ranging from 1 - 100 meV and thus are among elementary local fields closest in mass to the gauge bosons of the unbroken gauge groups of QCD-QED . The discussion of oscillation phenomena is compared and contrasted with coherence properties of QCD gauge bosons (gluons) .

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1 - Neutrino flavors, neutrino mass extension and gravity

1-1 There does not exist a symmetry – within the standard model including gravity and containing only chiral spin $\frac{1}{2}$ 16 families of SO (10) – which could enforce the vanishing of neutrino mass(es) .

Here I follow the exposition of ideas originating around 1973/74 upon my arrival to Caltech, whence we continued our collaboration on gauge theories of strong and electroweak interactions with Harald Fritzsch. A resumé of this extended work is published in ref. [1n-1975] .

The divergence of the current associated to the global charge B - L for three standard model families of 15 base fields – in the left chiral basis removing – to infinite mass – the 16-th components $(\mathcal{N})$ pertaining to one full 16-representation of SO (10) [spin (10)]

(1) $$\left(\begin{array}{cccc|cccc} u^{1} & u^{2} & u^{3} & \nu & \mathcal{N} & \widehat{u}^{3} & \widehat{u}^{2} & \widehat{u}^{1} \\ d^{1} & d^{2} & d^{3} & e^{-} & e^{+} & \widehat{d}^{3} & \widehat{d}^{2} & \widehat{d}^{1} \end{array} \right)^{\dot{\gamma} \rightarrow L} = (f)^{\dot{\gamma}}$$

and admitting a gravitational background field is in this minimal neutrino flavor embedding anomalous , i.e. the global symmetry is broken by winding gravitational fields [2n-2001] , in a form concretized later than 1973 . For a more complete account of left- *and* right-chiral bases see also [3n-2007] . $\longrightarrow$

$$j_\varrho(B-L)|_{3\times 15} =$$

$$\sum_{fam} \left[\begin{array}{c} \frac{1}{3}\Big((u^*)^{\alpha\,\dot c} (\sigma_\mu)_{\alpha\,\dot\gamma} (u)^{\dot\gamma\, c} - (\widehat u^*)^{\alpha\, c} (\sigma_\mu)_{\alpha\,\dot\gamma} (\widehat u)^{\dot\gamma\,\dot c} \Big) \\ + \frac{1}{3}\Big((d^*)^{\alpha\,\dot c} (\sigma_\mu)_{\alpha\,\dot\gamma} (d)^{\dot\gamma\, c} - \left(\widehat d^*\right)^{\alpha\, c} (\sigma_\mu)_{\alpha\,\dot\gamma} \left(\widehat d\right)^{\dot\gamma\,\dot c} \Big) \\ - (e^-)^{*\,\alpha} (\sigma_\mu)_{\alpha\,\dot\gamma} (e^-)^{\dot\gamma} + (e^+)^{*\,\alpha} (\sigma_\mu)_{\alpha\,\dot\gamma} (e^+)^{\dot\gamma} \\ - (\nu)^{*\,\alpha} (\sigma_\mu)_{\alpha\,\dot\gamma} (\nu)^{\dot\gamma} \end{array} \right] e_\varrho^{\ \mu}$$

$g_{\varrho\tau} = e_\varrho^{\ \mu}\, \eta_{\mu\nu}\, e_\tau^{\ \nu}$ **: metric** ; $e_\varrho^{\ \mu}$ **: vierbein** ; * **: hermitian operator conjugation**

$(u^*)^{\alpha\,\dot c} \equiv \left(u^{\dot\alpha\, c}\right)^*$; $\eta_{\mu\nu} = diag(1,-1,-1,-1)$ **: tangent space metric**

$^c\ (^{\dot c})$ **: color and anticolor** ; $c = 1,2,3$

(2)

The contribution of charged fermion (pairs) $q\,,\widehat q\,;\ e^{\mp}$ **can be combined to vector currents – Dirac doubling –** $\overline q\,\gamma_\mu\, q\,;\ \overline e\,\gamma_\mu\, e$ **with** $q \to u,d,c,s,t,b\,;\ e \to e^-,\mu^-,\tau^-$ **.** $\longrightarrow$

The anomalous Ward identy for the B - L current (- density) defined in eq. 2 takes the form

$$d^4 x \sqrt{|g|}\, D^{\varrho}\; j_{\varrho}(B - L)\,|_{3\times 15} = 3\,\widehat{A}_1(X)$$

$$\widehat{A}_1(X) = -\tfrac{1}{24}\, trX^2 \;\; ; \;\; (X)^a{}_b = \tfrac{1}{2\pi}\,\tfrac{1}{2}\, dx^{\varrho} \wedge dx^{\tau}\, (R^a{}_b)_{\varrho\tau}$$

(3) $(R^a{}_b)_{\varrho\tau}$: **Riemann curvature tensor**
mixed components : $^a{}_b \longrightarrow$ **tangent space**
$_{\mu\nu} \longrightarrow$ **covariant space**

$$D^{\varrho}\; j_{\varrho}(B - L)\,|_{3\times(16)} = 0$$ **strictly just non-anomalous** $\longrightarrow$

Before discussing the extension $j_{\varrho}(B - L)\,|_{3\times(15)} \rightarrow j_{\varrho}(B - L)\,|_{3\times(16)}$ **which renders the latter current** *non-anomalous*, **lets define the quantities appearing in eq. 3 :**

$$(R^a{}_b)_{\varrho\tau} = e^a_{\mu}\, e_{b\nu}\, (R^{\mu}{}_{\nu})_{\varrho\tau} \;\; ; \;\; e_{b\nu} = \eta_{bb'}\, e^{b'}_{\nu}$$

(4) $$(R^{\mu}{}_{\nu})_{\varrho\tau} = (\partial_{\varrho}\Gamma_{\tau} - \partial_{\tau}\Gamma_{\varrho} + \Gamma_{\varrho}\Gamma_{\tau} - \Gamma_{\tau}\Gamma_{\varrho})^{\mu}{}_{\nu}$$

$(\Gamma^{\mu}{}_{\nu})_{\tau}$ **: matrix valued** $(GL(4, \mathbb{R}))$ **connection ; minimal here**

$\longrightarrow$

For clarity eq. 3 is repeated below

$$d^4 x \sqrt{|g|} \, D^{\varrho} \, j_{\varrho} (B - L)|_{3 \times 15} = 3 \widehat{A}_1 (X)$$

$$\widehat{A}_1 (X) = - \tfrac{1}{24} \, trX^2 \; ; \; (X)^a{}_b = \tfrac{1}{2\pi} \tfrac{1}{2} \, d x^{\varrho} \wedge d x^{\tau} \left(R^a{}_b \right)_{\varrho \tau}$$

(3) $\left(R^a{}_b \right)_{\varrho \tau}$: $\left\{ \right.$ Riemann curvature tensor
mixed components : $^a{}_b \longrightarrow$ tangent space
$_{\mu \nu} \longrightarrow$ covariant space

$$D^{\varrho} \, j_{\varrho} (B - L) |_{3 \times (16)} = 0$$ strictly just non-anomalous $\longrightarrow$

In eq. 3 $\widehat{A} (X \rightarrow \lambda) = \tfrac{1}{2} \lambda / \sinh (\tfrac{1}{2} \lambda)$ denotes the Atiyah - Hirzebruch character or $\widehat{A} -$ genus [4n-1966] . Its integral over a compact , euclidean signatured closed manifold M_4 , capable of carrying an SO4 - spin structure , becomes the index of the associated *elliptic* Dirac equation

(5) $$\int \widehat{A} (X_E) = n_R - n_L = \text{integer}$$

In eq. 5 $n_{R,L}$ denote the numbers of right - and left - chiral solutions of the Dirac equation on M_4 . The index $_E \rightarrow X_E$ shall indicate the euclidean transposed curvature 2 - form , and is *adapted* here to physical curved and uncurved space time . $\longrightarrow$

For the latter case the first relation in eq. 3 yields the integrated form – in the limit of infinitely heavy $\mathcal{N}_F$ (eq. 67) –

$$\Delta_{R-L} n_\nu = \int d^4 x \sqrt{|g|} D^\mu j_\mu^{B-L(15)} = 3 \Delta n(\widehat{A}) \tag{6}$$

$3 =$ number of families = odd ; $m_{\nu_F} \to 0$

In eq. 6 $\Delta_{R-L} n_\nu$ denotes the difference of right - chiral $(\widehat{\nu})$ [a] and left - chiral (ν) flavors between times $t \to \pm\infty$.

Here a subtlety arises *precisely* because the number of families on the level of G_{SM} is odd , and the light neutrino flavors are not 'Dirac - doubled' , which according to eq. 72 could potentially lead to a change in fermion number being odd , which violates the rotation by 2π symmetry , equivalent to $\widehat{\Theta}^2 = (CPT^2)$, *unless* [b]

$$\Delta n(\widehat{A}) = \text{even} \qquad (\surd \text{ for } dim = 4 \text{ mod } 8) \tag{7}$$

→

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[a] $\widehat{\nu}_\alpha \equiv \varepsilon_{\alpha\beta} (\nu^*)^\gamma$; $\varepsilon = i\sigma_2$; (2nd Pauli matrix) stands for the left-chiral neutrino fields transformed to the right-chiral basis .

[b] The obviously nontrivial relation between the compact Euclidean - and noncompact asymptotic and locality restricted form of the index theorem involves not clearly formulated *boundary conditions* .

We now turn to the SO (10) inspired cancellation of the gravity induced anomaly, giving rise to the completion of neutrino flavors to 3 families of 16-plets , sometimes called 'right-handed' neutrino flavors, denoted $\mathcal{N}$ in the left-chiral basis in eq. 1

(1)
$$\left(\begin{array}{cccccccccc} u^{1} & u^{2} & u^{3} & \nu & | & \mathcal{N} & \widehat{u}^{3} & \widehat{u}^{2} & \widehat{u}^{1} \\ d^{1} & d^{2} & d^{3} & e^{-} & | & e^{+} & \widehat{d}^{3} & \widehat{d}^{2} & \widehat{d}^{1} \end{array} \right)^{\dot{\gamma} \rightarrow L} = (f)^{\dot{\gamma}}$$

(8)
$$j_{\varrho} (B - L) |_{3 \times 15} \rightarrow j_{\varrho} (B - L) |_{3 \times 16}$$

$$d^{4} x \sqrt{| g |} D^{\varrho} j_{\varrho} (B - L) |_{3 \times 15} = 3 \widehat{A}_{1} (X)$$

$$\widehat{A}_{1} (X) = - \frac{1}{24} tr X^{2} \; ; \; (X)^{a}{}_{b} = \frac{1}{2 \pi} \frac{1}{2} d x^{\varrho} \wedge d x^{\tau} \left(R^{a}{}_{b} \right)_{\varrho \tau}$$

(3)
$$\left(R^{a}{}_{b} \right)_{\varrho \tau} :$$

Riemann curvature tensor

mixed components : ${}^{a}{}_{b} \longrightarrow$ **tangent space**

 $\mu \nu \longrightarrow$ **covariant space**

$$D^{\varrho} j_{\varrho} (B - L) |_{3 \times (16)} = 0$$ **strictly just non-anomalous** $\longrightarrow$

$$j_{\varrho}(B - L)|_{3\times 15} \rightarrow j_{\varrho}(B - L)|_{3\times 16} =$$

$$\sum_{fam} \left[\begin{array}{c} \frac{1}{3}\Big((u^{*})^{\alpha\,\dot{c}}(\sigma_{\mu})_{\alpha\,\dot{\gamma}}(u)^{\dot{\gamma}\,c} - (\widehat{u}^{*})^{\alpha\,c}(\sigma_{\mu})_{\alpha\,\dot{\gamma}}(\widehat{u})^{\dot{\gamma}\,\dot{c}}\Big) \\ +\frac{1}{3}\left((d^{*})^{\alpha\,\dot{c}}(\sigma_{\mu})_{\alpha\,\dot{\gamma}}(d)^{\dot{\gamma}\,c} - \left(\widehat{d}^{*}\right)^{\alpha\,c}(\sigma_{\mu})_{\alpha\,\dot{\gamma}}\left(\widehat{d}\right)^{\dot{\gamma}\,\dot{c}}\right) \\ -(e^{-})^{*\,\alpha}(\sigma_{\mu})_{\alpha\,\dot{\gamma}}(e^{-})^{\dot{\gamma}} + (e^{+})^{*\,\alpha}(\sigma_{\mu})_{\alpha\,\dot{\gamma}}(e^{+})^{\dot{\gamma}} \\ -(\nu)^{*\,\alpha}(\sigma_{\mu})_{\alpha\,\dot{\gamma}}(\nu)^{\dot{\gamma}} \quad + \underbrace{(\mathcal{N})^{*\,\alpha}(\sigma_{\mu})_{\alpha\,\dot{\gamma}}(\mathcal{N})^{\dot{\gamma}}} \end{array} \right] e_{\varrho}^{\;\mu}$$

$g_{\varrho\,\tau} = e_{\varrho}^{\;\mu}\,\eta_{\mu\,\nu}\,e_{\tau}^{\;\nu}$ **: metric** ; $e_{\varrho}^{\;\mu}$ **: vierbein** ; * **: hermitian operator conjugation**

$(u^{*})^{\alpha\,\dot{c}} \equiv \left(u^{\dot{\alpha}\,c}\right)^{*}$; $\eta_{\mu\nu} = diag(1, -1, -1, -1)$ **: tangent space metric**

$^{c}\ \left(^{\dot{c}}\right)$ **: color and anticolor ;** $c = 1, 2, 3$

$D^{\varrho}\ j_{\varrho}(B - L)|_{3\times(16)} = 0$ **strictly just non-anomalous** $\rightarrow$

(9)

Let me illustrate the triple-doubling inherent in the elimination of the anomaly in the covariant divergence of $j_{\varrho}(B - L)|_{3 \times 15}$ in eq. 2 as seen through the left-chiral basis , repeating only the ν , $\mathcal{N}$ components of the B - L current in eq. 9

$$j_{\varrho}(B - L)|_{3 \times 16} =$$

$$\sum_{fam} \left[\begin{array}{c} \cdots \\ -(\nu)^{*\alpha}(\sigma_{\mu})_{\alpha\dot{\gamma}}(\nu)^{\dot{\gamma}} \quad + \underbrace{(\mathcal{N})^{*\alpha}(\sigma_{\mu})_{\alpha\dot{\gamma}}(\mathcal{N})^{\dot{\gamma}}} \end{array} \right]$$

(10)

	$\nu_F^{\dot{\gamma}}$	$\mathcal{N}_F^{\dot{\gamma}}$
$B - L$	-1	$+1$

; $F = 1, 2, 3$ **family**

[a]

[a] **In a less definite framework we looked with Wim Brems and Ian Olsen for a symmetry ensuring vanishing neutrino masses at the University of Louvain or Leuven 1968/69 – proving similarly that lepton flavor violating interactions do not allow to generate such a symmetry in general.**

1-2 Consequence from section 1-1

The consequence is clearly that within the structure of known gauge interactions minimally reduced to what is called the Standard Model is not adequate to generate exactly massless – minimally three Majorana type neutrino flavors – with masses well below the 1 eV scale . In addition exact lepton flavor conservation is equally impossible to maintain.

Yet up to here there is no estimate of quantitative levels of the so implied effects – called 'beyond the Standard Model' . To this end present knowledge of neutrino oscillation phenomena *as well as* **specific model dependent schemes can and must be invoked.**

For an alternative but nonrealistic way to guarantee vanishing neutrino masses, already ruled out by the present observations of neutrino oscillations, I refer to ref. [5n-2008] .

Comment on the elliptic Dirac equation in Euclidean space time and the Atiyah - Hirzebruch genus

It may appear illogical , that the characteristic function determining an index pertaining to a system of elliptic differential equations, the book on relativity by Pauli can lead the way [6n-1963]

(11) $$\widehat{A} \left(X \rightarrow \lambda \right) = \frac{1}{2} \lambda / \sinh \left(\frac{1}{2} \lambda \right)$$

in eq. 3 involves a hyperbolic function. This is due to the convention to use antihermition matrices for the generators of a Lie algebra in the mathematical literature . The substitution to *hermitian logic* **yields**

(12) $$X = \frac{1}{i} Y \quad \rightarrow \quad \lambda = \frac{1}{i} \Lambda \quad \rightarrow \quad \frac{1}{2} \lambda / \sinh \left(\frac{1}{2} \lambda \right) = \frac{1}{2} \Lambda / \sin \left(\frac{1}{2} \Lambda \right)$$

2 Quantum interference in vacuo of a neutrino beam from $y \longrightarrow x$

After the preparation of a neutrino beam at y , sufficiently removed from any interaction points – lets say for CNGS at CERN – we deal with a wave of positive frequencies composed of a mixture of mass eigenstates. This wave packet carries the label of an associated charged lepton or antilepton , which we take to be an antilepton to promote the main neutrino component of the beam relative to the antineutrino one. But we have to distinguish states from wave functions here.

So lets consider the mass eigenstates of free neutrino fields α , m_{α} and corresponding momentum eigenstates

$$| Z_{\alpha} , y \rangle =$$

$$\left(\int d^3 p \, (2 \pi)^{-3} \, \Psi_{\alpha} (\vec{p} , h) \, \exp (i \vec{p} \vec{y}) \, \exp(- i E (\vec{p} , m_{\alpha}) y_0) \right) | \vec{p} , h , \alpha \rangle \quad (13)$$

$$E (\vec{p} , m_{\alpha}) = + \sqrt{\vec{p}^{\,2} + m_{\alpha}^2}$$

So $\Psi_{\alpha} (\vec{p} , h)$ is the momentum space wave function, where h shall be chosen to be helicity , i.e. *mainly* **-1 for neutrinos, and which can be rendered covariant by multiplication with momentum base spinors , with suitable normalization conditions.** $\longrightarrow$

Dropping the spinor component A for brevity we have

$$u_A(p,h,\alpha)\;;\;A=1,\cdots,4\;;\;h=\mp$$

(14)
$$(p^{\mu}\gamma_{\mu}-m_{\alpha})\,u(p,h,\alpha)=0\quad;\quad\left(\vec{\Sigma}\,\vec{p}/|\vec{p}|\right)u=h\,u$$

$$\vec{\Sigma}=\begin{pmatrix}\vec{\sigma} & 0\\ 0 & \vec{\sigma}\end{pmatrix}\qquad;\qquad \vec{\sigma}\ :\ \text{Pauli matrices}$$

When considering mixtures of states , it is important to retain the strict orthogonality relations of the momentum states , with respect to the mass flavor α , which can be chosen mass independent , e.g.

(15)
$$\left\langle \vec{p}\,',h\,',\beta\,|\,\vec{p},h,\alpha\right\rangle=\delta_{\beta\,\alpha}\,(2\pi)^3\,\delta^3(\vec{p}\,'-\vec{p})\,\delta_{h'\,h}$$

and *not* superimpose wave functions linearly with different values of α . This is a subtlety with various strange inconsistencies attached. To illustrate this, lets for a moment consider only one given α and the associated *wave function* corresponding to eq. 13

$$(\Psi\,|\,\Psi)=\int d^3y\,\Psi^{\dagger}\,\Psi\quad;\quad\Psi_A(y)=$$

$$\sum_h\int d^3p\,(2\pi)^{-3}\,\Psi_{\alpha}(\vec{p},h)\,\exp(i\,\vec{p}\,\vec{y})\,\exp(-i\,E(\vec{p},m_{\alpha})\,y_0)\,u_A(p,h,\alpha)$$

(16)

→

Then the same wave function propagated to the point x , e.g. within the Opera detector at Gran Sasso, but in time x_0 later than production at y_0 , can be represented by means of the free field propagator dropping the spinor index A

(17) $$\vartheta (x_0 - y_0) \Psi (x) = \int d^3 x' \, S (x - x' , m_\alpha) \gamma_0 \Psi (y_0 , \vec{x}')$$

Properties and m_α dependence of the quantity $S (x - x' , m_\alpha)$ are collected in Appendix A , but here I emphasize , that the large coherence enhancement in neutrino oscillation is negligibly modified by the propagator , as it does further depend on the mass(es) m_α for beam mean momenta much larger than nu- masses and mass differences.

2-1 A shortcut showing the main oscillation pattern

Despite the remarks in the last section (1-3) the oscillation pattern can be rduced up to higher orders with respect to $m_\alpha / | \langle p \rangle |$, where $| \langle p \rangle |$ denotes the absolute value of the mean beam three momentum, to the phase distortion in the scalar product

(18)
$$\int d^3 p \, (2\pi)^{-3} \, \Phi^* (\vec{p}, (')) ^{\ell'} \, u^*_{\ell' \, \beta} \, u_{\ell \, \alpha} \, \Phi (\vec{p})^{\ell} \times$$
$$\times \exp \left(+ i \left(E(p, m_\beta) - E(p, m_\alpha) \right) \Delta t \right)$$

In eq. 18 the wave functions $\Phi^{\ell (', \ell')}$ represent production and detection of neutrino flavors associated with ℓ and ℓ' respectively. Δt can be set to the *mean* light-distance between production and detection for $| \langle p \rangle | \gg m_\alpha$

(19)
$$\Delta t = d_{D-P} / c ; \; (c = 1)$$

$u_{\ell \alpha}, u_{\ell' \beta}$ in eq. 18 refer to mixing parameters deducible from a definite scheme, describing heavy and light neutrino flavors . In the minimal 'seesaw' or 'mass by mixing' scheme, adapted in general form in common work with Harald Fritzsch and also with Murray Gell-Mann [7n-1975] , [8n-1976] there are three, electroweak singlet heavy neutrino flavors, with masses much larger in a specific way than quark and charged lepton ones. $\longrightarrow$

In these papers the central characteristic of the so called 'tilt to the left' was not recognized but instead vecor-like electroweak currents were studied. As far as neutrino oscillation properties are concerned this fact is not relevant. A historical overview is given in ref. [9n-2005] .

Lets now discuss the simple meaning of the energy difference in eq. 18

$$E(p, m_\beta) - E(p, m_\alpha) = \frac{m_\beta^2 - m_\alpha^2}{E(p, m_\beta) + E(p, m_\alpha)}$$

(20)

$$E(p, m_\beta) + E(p, m_\alpha) \sim 2 |\vec{p}| / \langle v \rangle$$

$$\langle v \rangle \sim \left\langle |\vec{p}| / \tfrac{1}{2} \left(E(p, m_\beta) + E(p, m_\alpha) \right) \right\rangle$$

Thus substituting for excellent detector efficiency the mean beam absolute momentum values $\longrightarrow$

the phase factor in eq. 18 becomes

$$\left(E \left(p , m_{\beta} \right) - E \left(p , m_{\alpha} \right) \right) \Delta t \quad \rightarrow \quad \frac{m_{\beta}^{2} - m_{\alpha}^{2}}{2 \left\langle |\vec{p}| \right\rangle} \, d_{D-P} = 2 \pi \left(d_{D-P} / L_{\alpha \beta} \right) \tag{21}$$

$$\left\langle v \right\rangle \Delta t \rightarrow d_{D-P} \quad ; \quad L_{\alpha \beta} = 4 \pi \left\langle |\vec{p}| \right\rangle / \Delta_{\beta \alpha} m^{2}$$

The quantum coherence phenomena behind the phase as derived here in section 2, will not be discussed in detail. Instead let me quote a paper devoted to this topic by Carlo Giunti [10n-2008] and references cited therein.
The tiny long range e.m. dipole-dipole interactions pose a question as to light neutrino flavors developing a vacuum fermionic pair condensate [11n-2008]. For the remaining sections we turn to the apparently disjoint topic of the gauge boson pair condensate in QCD. $\longrightarrow$

3 Bosonic paired oscillator modes

arising from the two central local anomalies in QCD : scale and chiral U1 [1o-2006]

We consider first *one* pair $a\,,\,b\,;\,a\,\neq\,b$ **together with their adjoint operators denoted** $a^{\,*}\,,\,b^{\,*}$ **of associated bosonic oscillators, satisfying the commutation relations**

(22)
$$[\,a\,,\,a^{\,*}\,]\,=\,¶\ \ ,\ \ [\,b\,,\,b^{\,*}\,]\,=\,¶$$
$$[\,a\,,\,a\,]\,=\,[\,b\,,\,b\,]\,=[\,a\,,\,b\,]\,=\,[\,a\,,\,b^{\,*}\,]\,=\,0\ \ \&\ \ a\,,b\,\rightarrow\,a^{\,*}\,,\,b^{\,*}$$

Following John von Neumann [2o-1931] , but extending to a *pair*, the system in eq. 22 is equivalent – modulo unitary transformations – to the following set of operators in the Hilbert space over *one* complex variable $\zeta\,=\,\xi\,+\,i\,\eta$

(23)
$$A\,=\,\sqrt{2}\,a\,=\,\partial_{\,\zeta}\,+\,\overline{\zeta}\ \ ,\ \ A^{\,*}\,=\,\sqrt{2}\,a^{\,*}\,=\,-\,\partial_{\,\overline{\zeta}}\,+\,\zeta$$
$$B\,=\,\sqrt{2}\,b\,=\,\partial_{\,\overline{\zeta}}\,+\,\zeta\ \ ,\ \ B^{\,*}\,=\,\sqrt{2}\,b^{\,*}\,=\,-\,\partial_{\,\zeta}\,+\,\overline{\zeta}$$

The operators $a\,\leftrightarrow\,b$ **represented in eq. 23 are phase related and thus it is useful to specify the phase equivalence, which enlarges to a U2 equivalence of *two* oscillators**

(24)
$$\begin{array}{l} a\,=\,a_{\,1}\ \ \rightarrow\ \ e^{\,i\,\alpha}\,a \\ b\,=\,a_{\,2}\ \ \rightarrow\ \ e^{\,i\,\beta}\,b \end{array}\ \ \longrightarrow\ f_{\,\sigma}\,=\,U_{\,\sigma\tau}\,a_{\,\tau}\,;\,\sigma,\tau\,=\,1,2$$

→

for arbitrary phase factors $e^{i\alpha}$, $e^{i\beta}$ and more generally U2 matrix transformations $U_{\sigma\tau}$, as shown in eq. 24 , which leave the commutation relations in eq. 22 invariant .

The complex and complex conjugate derivatives are related to the variables $\xi , \eta \leftrightarrow \zeta , \overline{\zeta}$

(25)
$$\partial_{\xi} = \partial_{\zeta} + \partial_{\overline{\zeta}} \quad , \quad \partial_{\eta} = i \left(\partial_{\zeta} - \partial_{\overline{\zeta}} \right)$$
$$\partial_{\zeta} = \tfrac{1}{2} \left(\partial_{\xi} - i \partial_{\eta} \right) \quad , \quad \partial_{\overline{\zeta}} = \tfrac{1}{2} \left(\partial_{\xi} + i \partial_{\eta} \right)$$
$$\delta = \partial_{\zeta} \partial_{\overline{\zeta}} = \tfrac{1}{4} \Delta \quad ; \quad \Delta = \partial_{\xi}^{2} + \partial_{\eta}^{2}$$

The paired operators A , B as defined in eq. 23 have the representation

(26)
$$A = \sqrt{2}\, a = e^{-\zeta\overline{\zeta}} \left(\partial_{\zeta} \right) e^{\zeta\overline{\zeta}} \quad , \quad A^{*} = \sqrt{2}\, a^{*} = - e^{\zeta\overline{\zeta}} \left(\partial_{\overline{\zeta}} \right) e^{-\zeta\overline{\zeta}}$$
$$B = \sqrt{2}\, b = e^{-\zeta\overline{\zeta}} \left(\partial_{\overline{\zeta}} \right) e^{\zeta\overline{\zeta}} \quad , \quad B^{*} = \sqrt{2}\, b^{*} = - e^{\zeta\overline{\zeta}} \left(\partial_{\zeta} \right) e^{-\zeta\overline{\zeta}}$$

understood to act from the left on an element of $\psi (\zeta , \overline{\zeta}) \in L^{2} [\xi , \eta]$, as e.g.

(27)
$$A : \psi \rightarrow A \psi = e^{-\zeta\overline{\zeta}} \left(\partial_{\zeta} \right) \left(e^{\zeta\overline{\zeta}} \psi \right)$$

$\longrightarrow$

3-1 'Normal' modes of *one* pair of oscillators

The 'normal' ground state for the representation of the paired oscillator operators defined in eqs. 22 and 23 satisfies the equations

(28)
$$|\,\Omega\,\rangle \;\leftrightarrow\; \psi_{\,\Omega}\,(\,\zeta\,,\,\overline{\zeta}\,)$$
$$A\;|\,\Omega\,\rangle \;=\; 0 \;\;,\;\; B\;|\,\Omega\,\rangle \;=\; 0 \;\;\rightarrow \psi_{\,\Omega} \;=\; \mathcal{N}\,e^{\,-\,\zeta\,\overline{\zeta}}$$

In eq. 28 $\mathcal{N} \;>\; 0$ **denotes a normalization constant chosen here according to the** $L^{\,2}\,(\,\xi\,,\,\eta\,)$ **norm**

(29)
$$\langle\,\Psi\,|\;\;\Psi\,\rangle_{\,\zeta} \;=\; \int d^{\,2}\zeta\;|\Psi\,|^{\,2} \;\;;\;\; d^{\,2}\zeta \;=\; d\,\xi\,d\,\eta$$
$$\int d^{\,2}\zeta\,e^{\,-\,2\,\lambda\,|\zeta\,|^{\,2}} \;=\; \pi\,/\,(\,2\,\lambda\,) \;\;\textbf{for}\;\; \lambda \;>\; 0 \;\;\rightarrow\;\; \mathcal{N} \;=\; (\,2\,/\,\pi\,)^{\,\frac{1}{2}}$$

The associated harmonic oscillator 'Hamiltonian' is formed from the four positive (semi-)definite products

(30)
$$A^{\,*}\,A \;=\; -\,\delta \;+\; |\,\zeta\,|^{\,2} \;+\; \zeta\,\partial_{\,\zeta} \;-\; \partial_{\,\overline{\zeta}}\,\overline{\zeta} \;\;,\;\; A\,A^{\,*} \;=\; -\,\delta \;+\; |\,\zeta\,|^{\,2} \;+\; \partial_{\,\zeta}\,\zeta \;-\; \overline{\zeta}\,\partial_{\,\overline{\zeta}}$$
$$B^{\,*}\,B \;=\; -\,\delta \;+\; |\,\zeta\,|^{\,2} \;-\; \partial_{\,\zeta}\,\zeta \;+\; \overline{\zeta}\,\partial_{\,\overline{\zeta}} \;\;,\;\; B\,B^{\,*} \;=\; -\,\delta \;+\; |\,\zeta\,|^{\,2} \;-\; \zeta\,\partial_{\,\zeta} \;+\; \partial_{\,\overline{\zeta}}\,\overline{\zeta}$$
$$H \;=\; \tfrac{1}{4}\,(\,A^{\,*}\,A \;+\; A\,A^{\,*} \;+\; B^{\,*}\,B \;+\; B\,B^{\,*}\,) \;=\; -\,\delta \;+\; |\,\zeta\,|^{\,2}$$

$\longrightarrow$

Transforming to the normal ordered operators $A^* A$, $B^* B$, H in eq. 30 becomes

(31)
$$A A^* = A^* A + 2 ¶ \quad \rightarrow$$
$$H = a^* a + b^* b + ¶ = -\delta + |\zeta|^2$$

In view of the U2 redundancies shown in eq. 24 the operators a, b are not the same as the pair f, g which decompose harmonic motions along the $\xi\,(f)$ and $\eta\,(g)$ axes respectively .

Even if obvious we choose to make the relation explicit using eqs. 23 and 25

(32)
$$\begin{aligned} A &= +\partial_{\zeta} + \overline{\zeta} = \left(+\tfrac{1}{2}\partial_{\xi} + \xi\right) - i\left(+\tfrac{1}{2}\partial_{\eta} + \eta\right) \\ A^* &= -\partial_{\overline{\zeta}} + \zeta = \left(-\tfrac{1}{2}\partial_{\xi} + \xi\right) + i\left(-\tfrac{1}{2}\partial_{\eta} + \eta\right) \\ B &= +\partial_{\overline{\zeta}} + \zeta = \left(+\tfrac{1}{2}\partial_{\xi} + \xi\right) + i\left(+\tfrac{1}{2}\partial_{\eta} + \eta\right) \\ B^* &= -\partial_{\zeta} + \overline{\zeta} = \left(-\tfrac{1}{2}\partial_{\xi} + \xi\right) - i\left(-\tfrac{1}{2}\partial_{\eta} + \eta\right) \end{aligned} \quad \rightarrow$$

$$f = f_1 = \tfrac{1}{2}\partial_{\xi} + \xi \ , \quad f^* = -\tfrac{1}{2}\partial_{\xi} + \xi$$
$$g = f_2 = \tfrac{1}{2}\partial_{\eta} + \eta \ , \quad g^* = -\tfrac{1}{2}\partial_{\eta} + \eta$$

Substituting f, g in eq. 32 we obtain $\rightarrow$

for the matrix U in eq. 24

(33) $$f_{\sigma} = U_{\sigma\tau}\, a_{\tau} \;\; ; \;\; U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & +i \end{pmatrix} \;\; ; \;\; Det\, U = i$$

For H we have the equivalent forms

(34) $$H = a^{*} a + b^{*} b + \P = f^{*} f + \tfrac{1}{2} \P + g^{*} g + \frac{1}{2\, \P}$$

3-2 Standard single oscillator variables

The standard single oscillator variables are obtained through the substitution

(35)
$$\zeta = \frac{1}{\sqrt{2}} \mathcal{Z} ; \left(\xi , \eta \right) = \frac{1}{\sqrt{2}} \left(\mathcal{X} , \mathcal{Y} \right) \;\; ; \;\; \left(\partial_{\xi} , \partial_{\eta} \right) = \sqrt{2} \left(\partial_{\mathcal{X}} , \partial_{\mathcal{Y}} \right) \;\; \rightarrow$$
$$f = \frac{1}{\sqrt{2}} \left(\partial_{\mathcal{X}} + \mathcal{X} \right) \;\; ; \;\; f^{*} = \frac{1}{\sqrt{2}} \left(-\partial_{\mathcal{X}} + \mathcal{X} \right)$$
$$g = \frac{1}{\sqrt{2}} \left(\partial_{\mathcal{Y}} + \mathcal{Y} \right) \;\; ; \;\; g^{*} = \frac{1}{\sqrt{2}} \left(-\partial_{\mathcal{Y}} + \mathcal{Y} \right)$$
$$f^{*} f + + \tfrac{1}{2} \P = \tfrac{1}{2} \left(-\partial^{2}_{\mathcal{X}} + \mathcal{X}^{2} \right) \;\; ; \;\; g^{*} g + + \tfrac{1}{2} \P = \tfrac{1}{2} \left(-\partial^{2}_{\mathcal{Y}} + \mathcal{Y}^{2} \right)$$

3-3 Coherent pair mode states

We go back to the oscillator variables $\zeta\,,\,\overline{\zeta}$ and the oscillator operators given in eq. 26 and consider the pair mode generators

(36) $$a^{\,*}\,b^{\,*} \;=\; \tfrac{1}{2}\,e^{\,\zeta\,\overline{\zeta}}\;(\,\delta\,)\;e^{\,-\,\zeta\,\overline{\zeta}}\;\;;\;\;\delta\;=\;\partial_{\,\zeta}\,\partial_{\,\overline{\zeta}}\;=\;\tfrac{1}{4}\,\Delta_{\,(\,\xi\,,\,\eta\,)}\;=\;\tfrac{1}{2}\,\Delta_{\,(\,\mathcal{X}\,,\,\mathcal{Y}\,)}$$

The substitution $\zeta\;=\;\frac{1}{\sqrt{2}}\,\mathcal{Z}$ is more suitable in this context , yielding

(37) $$a^{\,*}\,b^{\,*}\;=\;\tfrac{1}{4}\,e^{\,\frac{1}{2}\,|\,\mathcal{Z}\,|^{\,2}}\,\Delta_{\,(\,\mathcal{X}\,,\,\mathcal{Y}\,)}\,e^{\,-\,\frac{1}{2}\,|\,\mathcal{Z}\,|^{\,2}}$$

together with the modified $L^{\,2}\,(\,\mathcal{X}\,,\,\mathcal{Y}\,)$ norm relative to the norm defined in eq. 29

(38) $$\langle\,\Psi\,|\;\;\Psi\,\rangle_{\,\mathcal{Z}}\;=\;\int d^{\,2}\,\mathcal{Z}\;|\Psi\,|^{\,2}\;\;;\;\;d^{\,2}\,\mathcal{Z}\;=\;d\,\mathcal{X}\,d\,\mathcal{Y}\;=\;2\,d^{\,2}\,\zeta$$

This leads to the associated modification of the normalization constant for the ’normal’ oscillator ground state , defined in eq. 28

(39) $$\begin{aligned}&|\,\Omega\,\rangle\;\leftrightarrow\;\Psi_{\,\Omega}\,(\,\mathcal{Z}\,,\,\overline{\mathcal{Z}}\,)\;=\;\frac{1}{\sqrt{2}}\,\psi_{\,\Omega}\,(\,\zeta\,,\,\overline{\zeta}\,)\\&A\,|\,\Omega\,\rangle\;=\;0\;\;,\;\;B\,|\,\Omega\,\rangle\;=\;0\;\;\rightarrow\Psi_{\,\Omega}\;=\;\mathcal{N}^{\,\prime}\,e^{\,-\,\frac{1}{2}\,|\,\mathcal{Z}\,|^{\,2}}\\&\mathcal{N}^{\,\prime}\;=\;\pi^{\,-\,\frac{1}{2}}\end{aligned}$$

→

The coherent (unnormalized) pair mode state shall be defined in analogy with the coherent single mode one as

(40)
$$\psi_{coh-2b}(\tau) = \exp(\tau a^* b^*)\Psi_\Omega \; ; \; \tau \geq 0 , \text{ to start}$$
$$\exp(\tau a^* b^*) = e^{\frac{1}{2}|\underline{\mathcal{Z}}|^2} e^{\frac{\tau}{4}\Delta} e^{-\frac{1}{2}|\underline{\mathcal{Z}}|^2} \; ; \; \Delta = \partial^2_{\mathcal{X}} + \partial^2_{\mathcal{Y}}$$

In eq. 40 $e^{\frac{\tau}{4}\Delta}$ **denotes the heat kernel** $H(\tau; \underline{\mathcal{Z}} - \underline{\mathcal{Z}}')$ **satisying the heat diffusion equation**

(41)
$$\partial_\tau H(\tau; \underline{u}) = \frac{1}{4}\Delta H(\tau; \underline{u}) \; ; \; \underline{u} = (u, v) \; ; \; \Delta = \partial^2_u + \partial^2_v$$
$$\underline{\mathcal{Z}} = (\mathcal{X}, \mathcal{Y}) \; ; \; \underline{u} = \underline{\mathcal{Z}} - \underline{\mathcal{Z}}'$$

with the initial condition $H(0; \underline{u}) = \delta^2(\underline{u})$

We derive the form of H (eq. 65) in eppendix 1 for completeness

(42)
$$\underline{\mathcal{Z}} \to Z \; , \; \underline{\mathcal{Z}}' \to T \; , \; \underline{u} \to u = Z - T$$
$$H(\tau; u) = \frac{1}{\pi\tau} \exp\left[-\frac{u^2}{\tau}\right]$$

→

3-3-3

ψ_{coh-2b} **in eq. 40 thus takes the form**

(43)
$$|Z|^2 \rightarrow Z^2, \cdots \; ; \; \mathcal{N}' = \pi^{-\frac{1}{2}}$$
$$\psi_{coh-2b}(\tau) = \mathcal{N}' e^{\frac{1}{2} Z^2} \int d^2T\, H(\tau; Z - T)\, e^{-T^2}$$

The integral in eq. 43 as worked out in Appendix 2 (eq. 66) becomes

(44)
$$I = \int d^2T\, H(\tau; Z - T)\, e^{-T^2}$$
$$= \frac{1}{\pi\,\tau} \int d^2T \, \exp\left[- \frac{Z^2 - 2ZT + (1+\tau)T^2}{\tau} \right]$$
$$= \frac{1}{1+\tau} \, \exp\left[- \frac{Z^2}{1+\tau} \right]$$

It remains to implement all factors in eqs. 43 and 44 $\longrightarrow$

to obtain $\psi_{coh-2b} = \mathcal{N}' e^{\frac{1}{2} Z^2} I$ and the normalized wave function $\Psi_{coh-2b} = \mathcal{N}'' \psi_{coh-2b} \, ; \, \mathcal{N}'' = \sqrt{1 - \tau^2}$

$$\psi_{coh-2b} = \frac{1}{\sqrt{\pi}\,(1+\tau)} \exp\left[-\frac{1}{2} \frac{1-\tau}{1+\tau} Z^2 \right]$$

(45) $$\Psi_{coh-2b} = \frac{1}{\sqrt{\pi}} \left(\frac{1-\tau}{1+\tau} \right)^{\frac{1}{2}} \exp\left[-\frac{1}{2} \frac{1-\tau}{1+\tau} Z^2 \right]$$

$$\langle \Psi_{coh-2b} \mid \Psi_{coh-2b} \rangle_Z = \int d^2 Z \left| \Psi_{coh-2b} \right|^2 = 1$$

$$\longrightarrow \quad 0 \leq \tau < 1$$

3-4 Extending one pair of oscillators to variable oscillator 'scale' and so extended Bogoliubov transformations [3o-1947] , [4o-2002]

It is straightforward to extend the variable τ introduced in subsection 3-3 (osc-1c) (to be checked) to complex values in the strip

(46) $$\tau = \chi + i\vartheta \quad : \quad 0 \leq \Re\tau = \chi < 1$$

Next we consider the operators $\exp(\tau a^* b^*)$ defined in eq. 40 for $\Re\tau = 0$

$$\Re\tau = 0 \rightarrow \partial_\tau = \frac{1}{i}\partial_\vartheta$$

(47) $$X(\vartheta) = \exp(i\vartheta a^* b^*) \quad \rightarrow$$

$$\{a\}(\vartheta) = X(-\vartheta)\, a\, X(\vartheta) = \exp(-i\vartheta a^* b^*)\, a\, \exp(i\vartheta a^* b^*)$$

The ϑ family of operators $\{a\}(\vartheta)$ satisfies the differential equation and initial condition

(48) $$\partial_\vartheta \{a\}(\vartheta) = i\; X(-\vartheta)\, [\, a, \, a^* b^* \,]\, X(\vartheta) = i\, b^* \quad ; \quad \{a\}(0) = a$$

Eq. 48 can be integrated and gives

(49) $$\{a\}(\vartheta) = a + i\vartheta b^* \leftrightarrow \{a\}^*(\vartheta) = a^* - i\vartheta b$$

$$\leftrightarrow \quad \{a\}^*(\vartheta) = \{a^*\}(\vartheta) = (\{a\}(\vartheta))^*$$

→

Similarly to $\{a\}(\vartheta)$ defined in eq. 47 we form the family $\{b\}(\vartheta)$

(50)
$$\{b\}(\vartheta) = X(-\vartheta)\, b\, X(\vartheta) = \exp(-i\vartheta a^* b^*)\, b \exp(i\vartheta a^* b^*)$$
$$\partial_\vartheta \{b\}(\vartheta) = i a^* \longrightarrow \{b\}(\vartheta) = b + i\vartheta a^* \;\;\&\;\; \{b\} \to \{b^*\}$$

Eqs. 48 and 50 yield for the four associated families $\{a\} \leftrightarrow \{a^*\}$, $\{b\} \leftrightarrow \{b^*\}$

(51)
$$\{a\}_\vartheta = a + \tau b^* \leftrightarrow \{a^*\}_\vartheta = a^* + \tau^* b$$
$$\{b\}_\vartheta = b + \tau a^* \leftrightarrow \{b^*\}_\vartheta = b^* + \tau^* a$$
$$\tau = i\vartheta$$

I call the associations in eq. 51 $\longrightarrow$

'scale'-extended Bogolioubov transformations , rewritten in eq. 52 below . The quotes in 'scale' are to indicate , that the variables $\mathcal{Z}$ describing an oscillator paired system do not have any immediate relation to the length scale of configuration space variables .

$$\begin{array}{c} a \rightarrow \{a\}_{\vartheta} = a + \tau\, b^{*} \quad \leftrightarrow \quad a^{*} \rightarrow \{a^{*}\}_{\vartheta} = a^{*} + \tau^{*}\, b \\ b \rightarrow \{b\}_{\vartheta} = b + \tau\, a^{*} \quad \leftrightarrow \quad b^{*} \rightarrow \{b^{*}\}_{\vartheta} = b^{*} + \tau^{*}\, a \\ \tau = i\,\vartheta = -\tau^{*} \end{array} \tag{52}$$

We verify first the (nontrivial) homogeneous commutation relations

$$\left[\, \{a\}_{\vartheta} \,,\, \{b\}_{\vartheta} \,\right] = \left[\, \{a\}_{\vartheta} \,,\, \{b^{*}\}_{\vartheta} \,\right] = 0 \;\;\&\;\; \{a\}_{\vartheta} \leftrightarrow \{b\}_{\vartheta} \;\forall\, \vartheta \tag{53}$$

The inhomogeneous ones reveal a subtlety beyond re'scaling'

$$\left[\, \{a\}_{\vartheta} \,,\, \{a^{*}\}_{\vartheta} \,\right] = \left[\, \{b\}_{\vartheta} \,,\, \{b^{*}\}_{\vartheta} \,\right] = \left(1 - |\,\tau\,|^{2} \right) \P \tag{54}$$

The regions $|\,\tau\,| < 1$ and $|\,\tau\,| > 1$ reverse the role of creation and destruction operators assigned to $\{a\,,\,b\}_{\vartheta}\;;\;|\,\tau\,| < 1$ and $\{a^{*}\,,\,b^{*}\}_{\vartheta}\;;\;|\,\tau\,| > 1$ respectively , while for $|\,\tau\,| = 1$ all commutation relations become homogeneous. $\longrightarrow$

3-4-4

Hence we devise the following identifications , not modifying the definitions of the four families in eqs. 51 , 52

(55)

$$\{a\}_{\vartheta} = \left\{ \begin{array}{c} c_{\vartheta} \textbf{ for } | \tau \ \le \ 1 \\ c^{*}_{\vartheta} \textbf{ for } | \tau | \ > \ 1 \end{array} \right\} \ , \ \{a^{*}\}_{\vartheta} = \left\{ \begin{array}{c} c^{*}_{\vartheta} \textbf{ for } | \tau \ \le \ 1 \\ c_{\vartheta} \textbf{ for } | \tau | \ > \ 1 \end{array} \right\}$$

$$\{b\}_{\vartheta} = \left\{ \begin{array}{c} d_{\vartheta} \textbf{ for } | \tau \ \le \ 1 \\ d^{*}_{\vartheta} \textbf{ for } | \tau | \ > \ 1 \end{array} \right\} \ , \ \{b^{*}\}_{\vartheta} = \left\{ \begin{array}{c} d^{*}_{\vartheta} \textbf{ for } | \tau \ \le \ 1 \\ d_{\vartheta} \textbf{ for } | \tau | \ > \ 1 \end{array} \right\}$$

Using the 4 families as assigned in eq. 55 $c_{\vartheta} \leftrightarrow c^{*}_{\vartheta} \, d_{\vartheta} \leftrightarrow d^{*}_{\vartheta}$ **commutation rules in eqs. 55 and 54 take the form**

(56)

$$\left[c_{\vartheta} , \ d_{\vartheta} \right] = \left[c_{\vartheta} , d^{*}_{\vartheta} \right] = 0 \ \ \& \ \ c_{\vartheta} \leftrightarrow d_{\vartheta}$$

$$\left[c_{\vartheta} , c^{*}_{\vartheta} \right] = \left[d_{\vartheta} , d^{*}_{\vartheta} \right] = \left| 1 - | \tau |^{2} \right| \ ¶ \qquad \forall \vartheta$$

We first eliminate the phase induced in eq. 56 by applying the transformations $X(\pm \vartheta) , \ X^{*}(\pm \vartheta)$ **defined in eq. 50** $\longrightarrow$

through the substitution

(57)
$$\begin{array}{ccccccc} a & \longrightarrow & a + \tau\, b^* & \leftrightarrow & a^* & \longrightarrow & a^* + \tau^*\, b \\ & & \| & & & & \| \\ & & a + \vartheta\, \widetilde{b}^* & & & & a^* + \vartheta\, \widetilde{b} \\ -i\, b & \longrightarrow & -i\, b + \vartheta\, a^* & \leftrightarrow & i\, b^* & \longrightarrow & i\, b^* + \vartheta\, a \\ & & \| & & & & \| \\ & & \widetilde{b} + \vartheta\, a^* & & & & \widetilde{b}^* + \vartheta\, a \end{array}$$

$$\tau = i\,\vartheta = -\,\tau^* \;\; ; \;\; \widetilde{b} = -\,i\, b \;\; , \;\; \widetilde{b}^* = i\, b^*$$

As a consequence the substitutions inferred from eq. 57

(58)
$$\{a\}_\vartheta \longrightarrow \{a\}_\vartheta \;\leftrightarrow\; \{a^*\}_\vartheta \longrightarrow \{a^*\}_\vartheta$$

$$\{b\}_\vartheta \longrightarrow \left\{\widetilde{b}\right\}_\vartheta \;\leftrightarrow\; \{b^*\}_\vartheta \longrightarrow \left\{\widetilde{b}^*\right\}_\vartheta$$

with $\left\{\widetilde{b}\right\}_\vartheta = -\,i\,\{b\}_\vartheta \;\leftrightarrow\; \left\{\widetilde{b}^*\right\}_\vartheta = i\,\{b^*\}_\vartheta \;\; ; \;\; \tau = i\,\vartheta$

$\longrightarrow$

render real the so phase-transformed relations in eq. 52 [a]

$$a \longrightarrow \{a\}_{\vartheta} = a + \tau\, b^{*} \quad \leftrightarrow \quad a^{*} \longrightarrow \{a^{*}\}_{\vartheta} = a^{*} + \tau^{*}\, b$$

$$b \longrightarrow \{b\}_{\vartheta} = b + \tau\, a^{*} \quad \leftrightarrow \quad b^{*} \longrightarrow \{b^{*}\}_{\vartheta} = b^{*} + \tau^{*}\, a$$

$$\downarrow$$

(59)

$$a \longrightarrow \{a\}_{\vartheta} = a + \vartheta\, \widetilde{b}^{*} \quad \leftrightarrow \quad a^{*} \longrightarrow \{a^{*}\}_{\vartheta} = a^{*} + \vartheta\, \widetilde{b}$$

$$\widetilde{b} \longrightarrow \left\{\widetilde{b}\right\}_{\vartheta} = \widetilde{b} + \vartheta\, a^{*} \quad \leftrightarrow \quad \widetilde{b}^{*} \longrightarrow \left\{\widetilde{b}^{*}\right\}_{\vartheta} = \widetilde{b}^{*} + \vartheta\, a$$

$$\left\{\widetilde{b}\right\}_{\vartheta} = -i\, \{b\}_{\vartheta} \quad \leftrightarrow \quad \left\{\widetilde{b}^{*}\right\}_{\vartheta} = i\, \{b^{*}\}_{\vartheta} \quad ; \quad \tau = i\, \vartheta = -\tau^{*}$$

$\longrightarrow$

---

[a] This amounts to a phase transformation as defined in eq. 24 .

The associations in eq. 55 are adapted accordingly

$$\{a\}_{\vartheta} = \left\{ \begin{array}{c} c_{\vartheta} \text{ for } |\tau \leq 1 \\ c^{*}_{\vartheta} \text{ for } |\tau| > 1 \end{array} \right\} , \quad \{a^{*}\}_{\vartheta} = \left\{ \begin{array}{c} c^{*}_{\vartheta} \text{ for } |\tau \leq 1 \\ c_{\vartheta} \text{ for } |\tau| > 1 \end{array} \right\}$$

(60)
$$\{b\}_{\vartheta} = \left\{ \begin{array}{c} d_{\vartheta} \text{ for } |\tau \leq 1 \\ d^{*}_{\vartheta} \text{ for } |\tau| > 1 \end{array} \right\} , \quad \{b^{*}\}_{\vartheta} = \left\{ \begin{array}{c} d^{*}_{\vartheta} \text{ for } |\tau \leq 1 \\ d_{\vartheta} \text{ for } |\tau| > 1 \end{array} \right\}$$

$$\downarrow$$

$$\left\{\widetilde{b}\right\}_{\vartheta} = \left\{ \begin{array}{c} \widetilde{d}_{\vartheta} \text{ for } |\tau \leq 1 \\ \widetilde{d}^{*}_{\vartheta} \text{ for } |\tau| > 1 \end{array} \right\} , \quad \left\{\widetilde{b}^{*}\right\}_{\vartheta} = \left\{ \begin{array}{c} \widetilde{d}^{*}_{\vartheta} \text{ for } |\tau \leq 1 \\ \widetilde{d}_{\vartheta} \text{ for } |\tau| > 1 \end{array} \right\}$$

$\longrightarrow$

Eq. 56 then becomes

$$\left[c_{\vartheta} \, , \; \widetilde{d}_{\vartheta} \right] = \left[c_{\vartheta} \, , \widetilde{d}^{\,*}_{\vartheta} \right] = 0 \;\; \& \;\; c_{\vartheta} \leftrightarrow \widetilde{d}_{\vartheta}$$

$$\left[c_{\vartheta} \, , \, c^{\,*}_{\vartheta} \right] = \left[\widetilde{d}_{\vartheta} \, , \, \widetilde{d}^{\,*}_{\vartheta} \right] = \left| 1 - \vartheta^{2} \right| \, \P \qquad \forall \, \vartheta$$

with

(61)
$$\left. \begin{array}{l} c_{\vartheta} = \begin{cases} a + \vartheta \, \widetilde{b}^{\,*} & \textbf{for } 0 \le |\vartheta| \le 1 \\ a^{*} + \vartheta \, \widetilde{b} & \textbf{for } |\vartheta| > 1 \end{cases} \\ \widetilde{d}_{\vartheta} = \begin{cases} \widetilde{b} + \vartheta \, a^{*} & \textbf{for } 0 \le |\vartheta| \le 1 \\ \widetilde{b}^{\,*} + \vartheta \, a & \textbf{for } |\vartheta| > 1 \end{cases} \end{array} \right\} \quad ; \quad \& \, c_{\vartheta} \to c^{\,*}_{\vartheta} \;\; , \;\; \widetilde{d}_{\vartheta} \to \widetilde{d}^{\,*}_{\vartheta}$$

Finally – in this section – we can rescale the inhomogeneous commutation rules in eq. 61 →

to conventional 'scale' (= 1) but only for $|\vartheta| \neq 1$

$$|\vartheta| \neq 1 \quad : \quad \begin{pmatrix} C_\vartheta \\ \widetilde{D}_\vartheta \end{pmatrix} = \frac{1}{|1-\vartheta^2|^{\frac{1}{2}}} \begin{pmatrix} c_\vartheta \\ \widetilde{d}_\vartheta \end{pmatrix} \quad \forall \vartheta \ \text{except } |\vartheta| = 1$$

with

$$\left. \begin{aligned} C_\vartheta &= \begin{cases} \dfrac{1}{|1-\vartheta^2|^{\frac{1}{2}}}\, a + \dfrac{\vartheta}{|1-\vartheta^2|^{\frac{1}{2}}}\, \widetilde{b}^* & \text{for } 0 \leq |\vartheta| < 1 \\ \dfrac{1}{|1-\vartheta^2|^{\frac{1}{2}}}\, a^* + \dfrac{\vartheta}{|1-\vartheta^2|^{\frac{1}{2}}}\, \widetilde{b} & \text{for } |\vartheta| > 1 \end{cases} \\ \widetilde{D}_\vartheta &= \begin{cases} \dfrac{1}{|1-\vartheta^2|^{\frac{1}{2}}}\, \widetilde{b} + \dfrac{\vartheta}{|1-\vartheta^2|^{\frac{1}{2}}}\, a^* & \text{for } 0 \leq |\vartheta| < 1 \\ \dfrac{1}{|1-\vartheta^2|^{\frac{1}{2}}}\, \widetilde{b}^* + \dfrac{\vartheta}{|1-\vartheta^2|^{\frac{1}{2}}}\, a & \text{for } |\vartheta| > 1 \end{cases} \end{aligned} \right\} \quad (62)$$

→

A parametric representations of the coefficients in eq. 62 is , separately for the two ranges of $| \vartheta |$

$$\frac{1}{| 1 - \vartheta^2 |^{\frac{1}{2}}} = \begin{cases} \cosh \Xi & \text{for } | \vartheta | < 1 \\ \sinh \Xi & \text{for } | \vartheta | > 1 \end{cases}$$

(63)

$$\frac{\vartheta}{| 1 - \vartheta^2 |^{\frac{1}{2}}} = \begin{cases} \sinh \Xi & \text{for } | \vartheta | < 1 \\ sign (\vartheta) \cosh \Xi & \text{for } | \vartheta | > 1 \end{cases}$$

The conventional form for the bosonic Bogoliubov transformation is the one corresponding to $| \vartheta | < 1$.

Let me remark , that neither the real form nor the conventinal normalization (= 1) are representing the full structure associated with *one* bosonic pair of oscillators .

Epilogue

The embedding of chiral symmetry depends in a nontrivial way on the strength of the *gauge field strength pair-* Bose condensate as does the excitation of binary and higher gauge boson compounds ('glueballs') *and* the phase structure of QCD .

There is some way to go. I hope to come back to this theme soon.

— **Thank you** —

osc-1c Appendix 1 : the heat kernel , Schrödinger kernel for imaginary τ

Eq. 41 is solved from the Fourier-Laplace transform

$$\underline{u} \;\longrightarrow\; u \,, \;\cdots$$

$$H(\tau\,;\,u) \;=\; \int d\,E\;\frac{1}{4\,\pi^{\,2}}\;d^{\,2}\,p\,F\,(\,E\,;\,p\,)\,e^{\,i\,p\,u\,-\,E\,t}$$

$$\left(\,E \;-\; \tfrac{1}{4}\,u^{\,2}\,\right)\;F \;=\; 0 \quad\rightarrow\; F \;=\; \delta\;\left(\,E \;-\; \tfrac{1}{4}\,u^{\,2}\,\right)$$

(64)

$$H(\,\tau\,;\,u\,) \quad=\quad \frac{1}{4\,\pi^{\,2}}\;\int d^{\,2}\,p\;\exp\;\left[\;-\;\tfrac{\tau}{4}\,p^{\,2} + \,i\,p\,u\;\right]$$

$$=\quad \frac{1}{4\,\pi^{\,2}\,\tau}\;\int d^{\,2}\,P\;\exp\;\left[\;-\;\tfrac{1}{4}\,P^{\,2} + \,i\,P\,U\;\right]$$

$$P \;=\; \sqrt{\tau}\,p\,;\,U \;=\; \frac{1}{\sqrt{\tau}}\,u \quad;\quad -\;\tfrac{1}{4}\,P^{\,2} \;+\; i\,P\,U \;=\; -\;\left(\,\tfrac{1}{2}\,P \;-\,i\,U\,\right)\;-\;U^{\,2}$$

$\longrightarrow$

Substituting $\frac{1}{2}\Pi = \frac{1}{2} P - i U$ **we obtain**

$$H(\tau\,;\,u) = \frac{K}{4\pi^2\tau}\exp\left[-\frac{u^2}{\tau}\right]$$

(65) $$K = \int d^2\Pi\, e^{-\frac{1}{4}\Pi^2} = 4\pi \quad \rightarrow$$

$$H(\tau\,;\,u) = \frac{1}{\pi\tau}\exp\left[-\frac{u^2}{\tau}\right] \qquad (\surd)$$

osc-1c Appendix 2 : the heat kernel convolution (eq. 44)

The heat kernel convolution (Schrödinger kernel for imaginary τ) in eq. 44 becomes using the substitutions

$$\sigma = \tau^{-1} \; ; \; \sqrt{1+\sigma}\, T = S$$

$$I = \frac{1}{\pi(1+\tau)} \int d^2 S \exp\left[-\sigma Z^2 + 2\frac{\sigma}{\sqrt{1+\sigma}} Z S - S^2 \right]$$

$$= \frac{1}{\pi(1+\tau)} \int d^2 S \exp\left[-\left(\sigma - \frac{\sigma^2}{1+\sigma}\right) Z^2 - \left(S - \frac{\sigma}{\sqrt{1+\sigma}} Z \right)^2 \right]$$

$$= \frac{1}{1+\tau} \exp\left[-\frac{Z^2}{1+\tau} \right]$$

(66)

References

[1n-1975] H. Fritzsch and P. Minkowski, ’Unified interactions of leptons and hadrons’ , Annals Phys.93 (1975) 193 .

[2n-2001] P. Minkowski, ’On amplified and tiny signatures of neutrino oscillations’, NO-VE International Workshop on Neutrino Oscillations, Venice, Italy 24.-26. July 2001, published in ”Venice 2001, Neutrino Oscillations”, p. 291 .

[3n-2007] P. Minkowski, ’Neutrino flavors light and heavy – how heavy ?’, in nus-albufeira2007.pdf , URL : http://www.mink.itp.unibe.ch/ , and references cited therein .

[4n-1966] F. Hirzebruch, ’Topological methods in algebraic geometry’ in ”Die Grundlehren der mathematischen Wissenschaften” Band 131 , Springer Verlag, Berlin 1966 .

[5n-2008] P. Minkowski, ’The origin of neutrino mass, stations along the path of cognition’, Contribution to – Discrete’08 – Symposium on the Prospects in the Physics of Discrete Symmetries, 11.-16. December 2008, IFIC, Valencia, Spain , published in the proceedings ; also in http://www.mink.itp.unibe.ch/ , nuorigin08.pdf

[6n-1963] W. Pauli, ’Relativitätstheorie’, Paolo Boringhieri ed. , Torino , 1963 .

References

[7n-1975] H. Fritzsch, M. Gell-Mann and P. Minkowski , ’Vector - like weak currents and new elementary fermions’ , Phys.Lett.B59 (1975) 256 .

[8n-1976] H. Fritzsch and P. Minkowski, ’Vector - like weak currents, massive neutrinos, and neutrino beam oscillations’ , Phys.Lett.B62 (1976) 72 .

[9n-2005] P. Minkowski, ’Neutrino oscillations: a historical overview and its projection’, Contributed to 11th International Workshop on Neutrino Telescopes, Venice, Italy, 22-25 Feb 2005, *Venice 2005, Neutrino telescopes* 7-27, hep-ph/0505049,
extended version in venice30.pdf (unpublished) .

[10n-2008] C. Giunti, ’Neutrino Flavor States and the Quantum Theory of Neutrino Oscillations’, DFTT-2-2008, Jan 2008. 16pp., talk given at 11th Mexican Workshop on Particles and Fields 2007, Tuxtla Gutierrez, Mexico, 7-12 Nov 2007, AIP Conf.Proc.1026 (2008) 3-19, arXiv:0801.0653 [hep-ph] .

[11n-2008] R. Horvat, P. Minkowski and J. Trampetic, ’Dark consequences from light neutrino condensations’, Sep 2008. 5pp., Phys.Lett.B671 (2009) 51-54, arXiv:0809.0582 [hep-ph] .

References

[1o-2006] P. Minkowski, lecture notes 2005/2006, Ausgewählte Kapitel aus der Teilchenphysik

1 WS 2005/06 : axialanomaly.pdf
2 SS 2006 : neutrino2006.pdf
} in http://www.mink.itp.unibe.ch

and references cited therein .

[2o-1931] J. von Neumann, 'Die Eindeutigkeit der Schrödngerschen Operatoren', Vol. II, pp. 221-229 in Collected works, Vols. I-VI, A. H. Taub ed., Pergamon Press, Oxford 1961 , original with the same title in Math. Ann. 104 (1931) 570-578 .

[3o-1947] N. N. Bogoliubov, J. Phys. (USSR) 11 (1947) 23 .

[4o-2002] J.M. Vogels,K. Xu,C. Raman,J.R. Abo-Shaeer and W. Ketterle, 'Experimental observation of the Bogolyubov transformation for a Bose-Einstein condensed gas', Phys.Rev.Lett.88 (2002) 060402, cond-mat/0109205 .

Additional material

1-a Abelian constant field-strengths configurations

In view of the phase structure analysis of QCD , in the neigbourhood of vanishing chemical potentials, we begin with generic thermal quantities appropriate for this limit

(67)

extensive potentials $S = s V \; ; \; \Phi = p \beta V \; ; \; \mathcal{E} = \varrho_{e} V$

1 intensive independent variable $T = \beta^{-1}$

S : **entropy** ; Φ : **Gibbs potential** ; $\mathcal{E}$: **energy** ; V : **spatial volume**

p : **pressure**

The equations of state coalesce for zero chemical potentials to the form

(68)

$$T s = \varrho_{e} + p \; ; \; s = \partial_{T} p$$

$$(s , p , \varrho_{e}) (T)$$

With the substitution $T = T^{*} e^{\tau}$ **eq. 68 becomes**

(69)

$$d / d\tau = \cdot \; , \; \tau = \log (T / T^{*}) \; \rightarrow$$

$$\dot{p} = \varrho_{e} + p$$

$\rightarrow$

a2

In eq. 69 – T^* denotes an a priory arbitrary characteristic energy scale of QCD .
The functional simplicity of the single thermal quantity – the pressure as a function of one variable, T (or τ), is *in contrast* to all well known systems forming the phenomenological base of thermodynamics.

It is obviously quite demanding to adapt the nontrivial thermal behaviour characterizing QCD in the envisaged limit of vanishing chemical potentials to this kinematical simplicity.
As a consequence the derivation of the eventually multiple phase structure is, whence guided by thermodynamic considerations alone, simply impossible.
To illustrate the above, I quote a recent paper [1-2008] not related to the vanishing chemical potential region directly : ”The extrapolation of the instanton-induced crossover deduced here to lower baryon density is a nontrivial question, which we cannot address within our framework. ...”
Yet things become – at least appear to become – perfectly clear from a review paper by Urs Heller [2-2006], where the complete phase structure at zero chemical potentials is detailed, as shown in figure 1 below $\longrightarrow$

a3

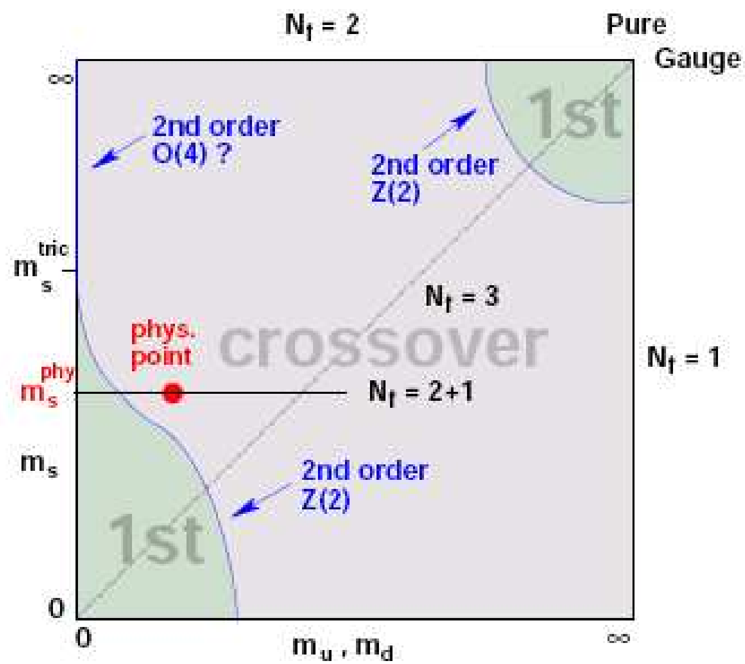

Figure 1: Sketch of the QCD phase diagram in the $m_{ud} - m_s$ plane.

Fig 1 : from ref. [2-2006] $\longleftrightarrow$

For quark masses with ratios near the characteristic values for the three light flavor of quarks

(70)
$$m_u : m_d : m_s \sim 3 : 5 : 100$$
$$\widehat{m} = \frac{1}{2}(m_u + m_d) : m_s = 1 : 25$$

Lattice simulations of thermal QCD indicate the existence of a unique phase [2-2006] for vanishing chemical potentials.

This is contrary to the conjectured analogy of ’color superfluidity’ arising from the condensates pertaining to the (T = 0) ground state, as discussed in ref. [3-1981] . The approximate calculation of the pressure for a noninteracting ensemble of hadron resonances and noninteracting quarks and gluons confirmed this conjecture, yet could not reproduce the superfluidity associated *second* order transition with respect to energy density, as discussed in ref. [4-2001] .

The expected absence of latent heat in the phase transition shall be further discussed. Its corresponding (maximally-) second order nature is illustrated in an extension to the energy density in units of T^4 as calculated in ref. [5-2007] by the MILC collaboration in Fig. 2 below $\longrightarrow$

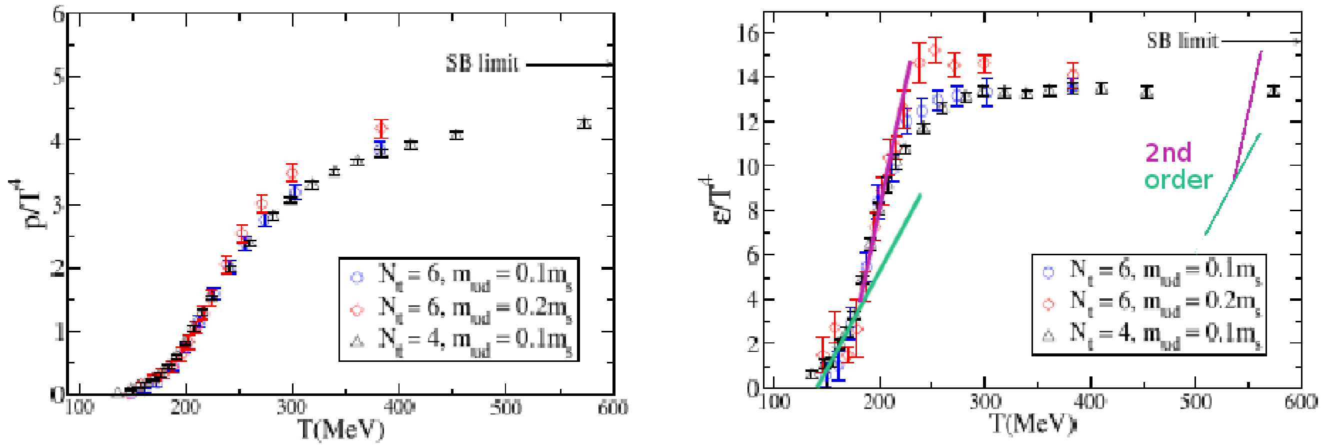

Fig 2 : $p\,/\,T^{\,4}$; $\varepsilon\,/\,T^{\,4}$ **from ref. [5-2007] and to guide the eye**

a second order phase transition with respect to energy density superimposed. $\longleftrightarrow$

Several comments are required as to what is indicated by '2nd order' in Fig. 2 :

1) In no way I mean to imply any phenomenological consequence or evidence from the calculated curve $\varepsilon\,/\,T^{\,4}$ **for a second order phase transition.**

2) The existence of specifically this second order with respect to ε **transition is not derived from first principles, but by analogy with the phenomena of superconductivity and/or superfluidity and the condensation of pairs of quanta , fermionic- as well as bosonic pairs in the ground state.**

$\longrightarrow$

2) (continued) The condensate composite, local operators are

(71) $$\frac{1}{4} F^{A}_{\mu\nu} F^{\mu\nu\ A}(x) \ ; \ \sum_c \overline{q}^{\,a}_{c} q^{\,a}_{c}(x) \ \text{for} \ a = u, d, s$$
$$c = 1, 2, 3 \ : \ \text{color triplet} \ ; \ A = 1, \cdots, 8 \ : \ \text{color octet}$$

The operators in eq. 71 are understood to be (re-)normalized in a renormalization group invariant way.

3) While we give an argument for zero latent heat of the phase transition below, the analytic form of the singularity at $T = T_c$ may occur for higher than first derivative of the energy density, or be in a more subtle form except for vanishing latent heat.

In fact the bridge between local field dynamics and global thermal parameters, as messengers of phase transitions and microscopic dynamics has evolved away from the initially prevailing hope of a resolution of basic problems [6-1973] .

As a consequence of points 1-3 discussed above I am induced to *redefine* the phase transition pertaining to superconductivity/superfluidity like 'pairing' as well as 'tripling' , 'quadrupling' $\cdots$ of color octet bosonic as well as color triplet quark and antitriplet antiquark modes [7-1988] as 'second' order with respect to energy density, which just shall mean any order including infinite one *except* first order with respect to the same quantity. $\longrightarrow$

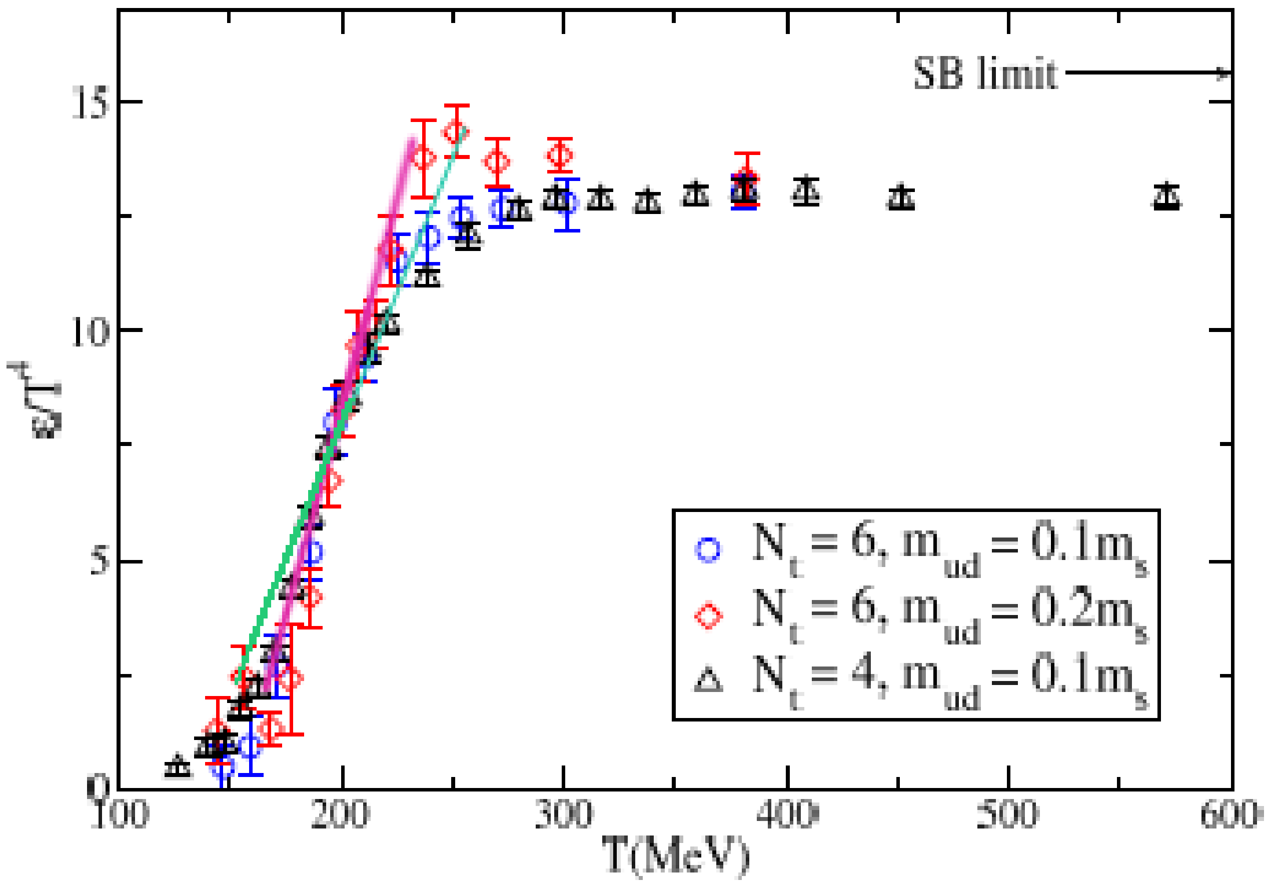

Fig 2a : $\varepsilon\,/\,T^{\,4}$ **from ref. [5-2007] and with slope break opposite to Fig. 2 a second order phase transition with respect to energy density superimposed.** $\longleftrightarrow$

Here a distant yet related phase structure of He4 atoms serves as illustration [8-1993] . The phase related to 'color superfluidity' is superfluid He4 , with phase boundaries as shown in ref. [8-1993] and in Fig. 3 below

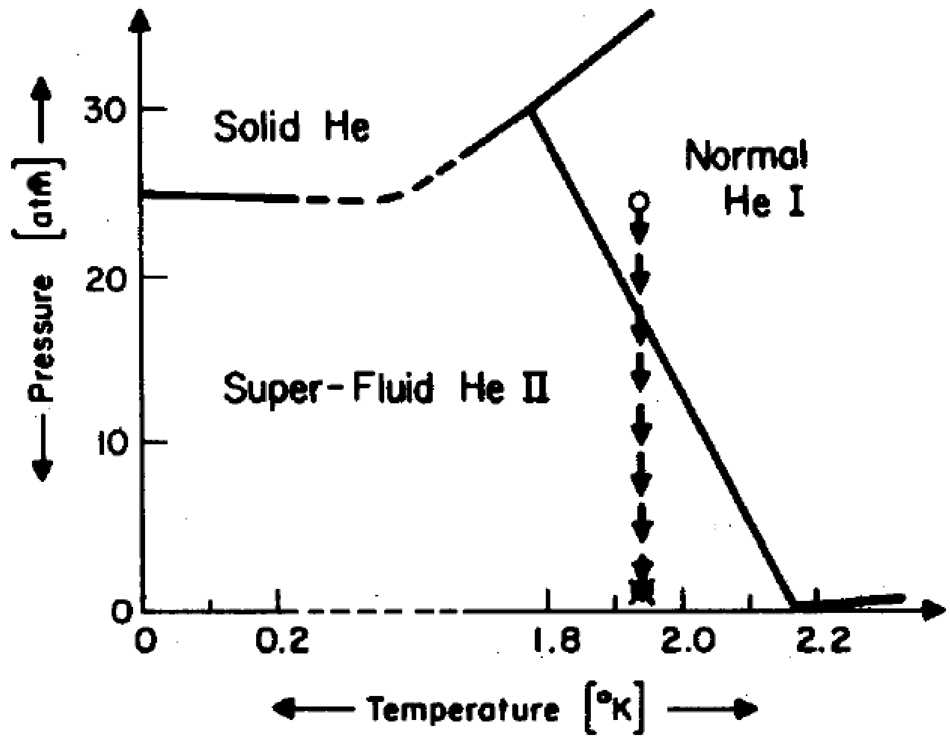

Fig 3 : The phase diagram of He4 and the trajectory of the quench in the proposed experiment , from ref. [8-1993] . ⟷

The QCD systems with $\mu_{\,u,d,s\;(\cdots)} = 0$ *and* $0 \leq m_{\,u,d,s\;(\cdots)} \leq \infty$ with both limits for the quark masses : $0\;,\;\infty$ *included* and with condensate operators given in eq. 71 and thermal equation of state in eqs. 67 , 68 are very different also from any mixture of superfluid He4 and He3 .

Yet despite the essential role of chemical potentials – associated with conservation of baryonic number – played for the low temperature phases of He3&4 [9-1937] , certain analogies exist. In particular condensation of single , double , multiple products of atom-related fields – even multiples for fermions – , leading to perfect superfluid behaviour have something in common with the fully relativistic counterparts of QCD-phases (also and in particular for vanishing chemical potentials) .
It was L. D. Landau [10-1941] , [11-1998] , [12-1961] , [13-195(6)7] who pioneered the theoretical derivation of phases of both Bose- *and* Fermi liquids .
Bardeen, Cooper and Schrieffer [14-195(6)7] introduced dynamical pairing of conduction electron-related fields (Cooper pairs) in their original theory of superconductivity, while the general transformations of creation and annihilation operators pertinent to either fermions or bosons are due to N. N. Bogoliubov [15-1947] and carry his name . It is interesting to remark here that bosonic pairing condensation is today readily observable [16-2002] . $\longleftrightarrow$

1-a1 Classical gauge boson modes (in QCD) in path integral representations obstructing the gauge invariant nature of associated boson 'pair' condensation

Within QCD , assuming completely unbroken local gauge invariance , the most direct way to assess bosonic multiple mode condensation follows the short distance expansion of pairs of gauge invariant operators , for which the most convenient set is formed by chiral $q\,\overline{q}$ currents . The simplest case arises for currents with perturbatively vanishing anomalous dimensions, and in order to simplify flavor dependence to restrict ourselves to the vectorial quark number current with a projection on N_{fl} (e.g. 3) according to their masses

$$\begin{matrix} T \\ \\ (\Pi) \end{matrix} \quad \left\{ j_{\mu\,\mathcal{A}} (x + \tfrac{1}{2} z) \, j_{\nu\,\mathcal{B}} (x - \tfrac{1}{2} z) \right\} \underset{z \,\to\, 0}{\sim} \sum_{\mathcal{O}} C^{\,T\,(\Pi)}_{\mu\,\nu\,\mathcal{A}\,\mathcal{B}\,\mathcal{O}} (z) \, \mathcal{O} (x)$$

(72)

$$\left\{ \begin{matrix} j_{\mu\,\mathcal{A}} \\ \\ j_{\nu\,\mathcal{B}} \end{matrix} \right\} \rightarrow J_{\mu} = \sum_{f}^{N_{fl}} \sum_{c} \; : \overline{q}^{\,f}_{\,c} \, \gamma_{\mu} \, q^{\,f}_{\,c} \; :$$

In eq. 72 $\mathcal{A}$, $\mathcal{B}$ label color : c , flavor : f , and chiral projections $\frac{1}{2}$ (¶ $\pm\ \gamma_{5}$) respectively , whereas T (Π) denote the time ordered and simple product of the two currents respectively . The $: :$ signs indicate that some normal ordering is necessary to remove local singularities of simple products of operators . $\longrightarrow$

This brings eq. 72 to the form

(73) $$\mathop{T}_{(\Pi)} \{ J_\mu (x + \tfrac{1}{2} z) J_\nu (x - \tfrac{1}{2} z) \} \underset{z \to 0}{\sim} \sum_{\mathcal{O}} C^{T\,(\Pi)}_{\mu\,\nu\,\mathcal{O}} (z) \mathcal{O} (x)$$

The local gauge invariant operators $\mathcal{O}$ in eqs. 72 , 73 can be ordered according their *twist* (tw) , and treating separately the special case of the unit operator $\mathcal{O} = ¶$

(74) $$tw = \text{mass dimension} - \text{spin} \quad : \quad \begin{cases} 0 & \text{for } \mathcal{O} = ¶ \\ \geq 2 & \text{for genuinely local operators in QCD} \end{cases}$$

Since we will need some subtleties inherent to the short distance expansion , a standard within the perturbative treatment of ultraviolet stable (asymptotically free) QCD [17-197(3)4] some results are collected in Appendix 2 .

b1 - The genus of the Einstein tensor $E_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$

Here I follow the conventions used mainly by mathematicians, in the definition of the Riemann- , Ricci- and Einstein tensors as obtained from a minimal, metric preserving connection, as restricted to $D_{\mathcal{P}} = 1 + 3$ dimensions

$$(R^{\mu}{}_{\nu})_{\varrho \tau} = (\partial_{\varrho} \Gamma_{\tau} - \partial_{\tau} \Gamma_{\varrho} + \Gamma_{\varrho} \Gamma_{\tau} - \Gamma_{\tau} \Gamma_{\varrho})^{\mu}{}_{\nu}$$

(75)

$(\Gamma^{\mu}{}_{\nu})_{\tau}$: matrix valued $(GL (4 , \mathbb{R}))$ connection ; minimal here $\longrightarrow$

$$(\Gamma^{\mu}{}_{\nu})_{\tau} = g^{\mu\sigma} \Gamma_{\sigma ; \nu\tau} \quad ; \quad \Gamma_{\sigma ; \nu\tau} = \frac{1}{2} (\partial_{\nu} g_{\sigma\tau} + \partial_{\tau} g_{\sigma\nu} - \partial_{\sigma} g_{\nu\tau})$$

and conversely $\partial_{\nu} g_{\sigma\tau} = \Gamma_{\sigma ; \nu\tau} + \Gamma_{\tau ; \sigma\nu}$

In eq. 75 $g_{\sigma \tau}$ denotes the metric, for which a signature ambiguity is subject to another convention, for which I use, departing from the most common one in the mathematical literature

(76)

$$sign\, g = (+ , - , - , -) \rightarrow \{ + \} \quad ; \quad g = g \{ + \}$$

with $g \{ - \} \equiv - g \{ + \}$

Next we construct the Ricci tensor , which does not depend on the signature(s) $\{ \pm \}$ $\rightarrow$

$$R_{\nu\tau} = \left(R^{\varrho}{}_{\nu} \right)_{\varrho\,\tau} = \left(\partial_{\varrho} \Gamma_{\tau} - \partial_{\tau} \Gamma_{\varrho} + \Gamma_{\varrho} \Gamma_{\tau} - \Gamma_{\tau} \Gamma_{\varrho} \right)^{\varrho}{}_{\nu}$$

(77)

$$= g^{\varrho\mu} \left(\partial_{\varrho} \Gamma_{\mu\,;\,\nu\tau} - \partial_{\tau} \Gamma_{\varrho\,;\,\mu\nu} \right) + O\left(\Gamma^{2} \right)$$

For the signature discussion it is sufficient to use – for classical field configurations – Riemann normal coordinates adapted to the argument of the Ricci tensor , which amounts to neglect the $O\left(\Gamma^{2} \right)$ **terms and substitute the flat space metric** $g^{\varrho\mu} \rightarrow \eta^{\varrho\mu} = diag\left(1, -1, -1, -1 \right)$ **in the expression for the Ricci tensor in eq. 77**

$$R_{\nu\tau} = \partial^{\mu} \Gamma_{\mu\,;\,\nu\tau} - \partial_{\tau} \eta^{\varrho\mu} \Gamma_{\varrho\,;\,\mu\nu} \quad ; \quad \eta^{\varrho\mu} \partial_{\varrho\mu} = \partial^{\mu}$$

(78)

$$= \frac{1}{2} \left[\begin{array}{l} \left(\partial_{\nu} G_{\tau} + \partial_{\tau} G_{\nu} - \Box g_{\nu\tau} \right) + \\ \left(\partial_{\tau} G_{\nu} - \partial_{\tau} G_{\nu} - \partial_{\nu} \partial_{\tau} G \right) \end{array} \right]$$

$$G_{\nu} = \partial^{\mu} g_{\mu\nu} \;,\; G = \eta^{\mu\varrho} g_{\varrho\mu} \;,\; \Box = \partial_{t}^{2} - \Delta \;,\; \Delta = \sum_{i=1}^{3} \partial_{i}^{2}$$

Collecting terms we obtain along the curvature scalar $R = \eta^{\nu\tau} R_{\nu\tau} \,;\, R\{+\} = -R\{-\}$

$$R_{\nu\tau} = \frac{1}{2} \left(- \Box g_{\nu\tau} - \partial_{\nu} \partial_{\tau} G + \partial_{\nu} G_{\tau} + \partial_{\tau} G_{\nu} \right)$$

(79)

$$R = - \Box G + \partial^{\varrho} G_{\varrho}$$

→

The Einstein tensor becomes

$$E_{\nu\tau} = R_{\nu\tau} - \tfrac{1}{2}\,\eta_{\nu\tau}\,R$$

(80) $$= \tfrac{1}{2}\left(-\Box g_{\nu\tau} + (\eta_{\nu\tau}\Box G - \partial_\nu \partial_\tau G) + \partial_\nu G_\tau + \partial_\tau G_\nu - \eta_{\nu\tau}\,\partial^\varrho G_\varrho \right)$$

$$G_\nu = \partial^\mu g_{\mu\nu} \;,\; G = \eta^{\mu\varrho} g_{\varrho\mu}$$

In eq. 80 we can substitute in all second derivatives the deviations of the metric tensor from its flat form, the expression becoming approximate if used for all x

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \longrightarrow$$

(81) $$2\,E_{\nu\tau} \sim \left[\begin{array}{l} -\Box h_{\nu\tau} + (\eta_{\nu\tau}\Box h - \partial_\nu \partial_\tau h) + \\ +\partial_\nu \partial^\varrho h_{\varrho\tau} + \partial_\tau \partial^\varrho h_{\varrho\nu} - \eta_{\nu\tau}\,\partial^\varrho \partial^\sigma h_{\varrho\sigma} \end{array} \right]$$

$$h = \eta^{\varrho\sigma} h_{\varrho\sigma}$$

The further substitution reduces the 6 terms on the right hand side of eq. 81

(82) $$h_{\varrho\sigma} = f_{\varrho\sigma} - \tfrac{1}{2}\,\eta_{\varrho\sigma}\,f \;\; ; \;\; f_{\varrho\sigma} = h_{\varrho\sigma} - \tfrac{1}{2}\,\eta_{\varrho\sigma}\,h \;\; ; \;\; h = -f$$

→

(83) $$2\,E_{\nu\tau} \sim \left[\, - \Box f_{\nu\tau} + \partial_{\nu}\,\partial^{\varrho} f_{\varrho\tau} + \partial_{\tau}\,\partial^{\varrho} f_{\varrho\nu} - \eta_{\nu\tau}\,\partial^{\varrho}\,\partial^{\sigma} f_{\varrho\sigma} \, \right]$$

The genus of the Einstein tensor in the title of this section denotes the sign $\gamma = \pm$ in the equation

(84) $$E_{\nu\tau} = \gamma\,K\,\vartheta_{\mu\nu} \;\; ; \;\; \gamma = \pm \, , \; K > 0$$

which in the Newtonian limit goes over in the Poisson equation for the gravitational potential in our $\{ + \}$ convention

Eq. 79 becomes in the static case and keeping the velocity of light general (c)

$$R_{00} \sim \frac{1}{2}\,\Delta\,h_{00} \sim \gamma\,K\,\frac{1}{2}\,\vartheta_{00} \;\; ; \;\; \vartheta^{\;\mu}_{\;\mu} \sim \vartheta_{00} \;\; ; \;\; h_{00} \rightarrow 2\,\varphi\,/\,c^{2}$$

(85) $$\Delta\,\varphi \sim \frac{1}{2}\,K\,c^{4}\,\rho_{m} = 4\,\pi\,G_{N}\,\rho_{m} \rightarrow \begin{cases} \gamma = +1 \\ K = 8\,\pi\,G_{N}\,/\,c^{4} \end{cases}$$

In eq. 85 φ denotes the gravitational potential , ϱ_{m} the mass density and G_{N} Newtons constant.

z1 - The fibred torus as a rectangular lattice in $D_{\mathcal{P}} \rightarrow D_{E}$ dimensions and gauge potentials

	Gauge potentials are related to a local enlargement of a fixed gauge group	G
(86)	G **is extended into a** *collection* **of local gauge group actions**	$\{ G \} = \{ G \}_{M}$
	with respect to the D dimensional base-manifold	M

In conjunction with the notion of *flavor* **and the exclusion from our discussion of gravitational interactions the three quantities** G , $\{ G \}_{M}$, M **in eq. 86 are** *initially* **characterized as follows**

M

The dimension D decomposes into

(87) $$D = D_{\mathcal{P}} + d \ ; \ D_{\mathcal{P}} = (1 + 3) \leftrightarrow D_{E} = (0 + 4)$$

where d denote a form of compactified dimensions , which hitherto have not been experimentally *resolved* **, while to start with** $D_{\mathcal{P}} = (1 + 3)$ **denotes fourdimensional** *physical* **time-space , supporting in appropriate coordinates** $x = (t , \vec{x})$ **an unbroken rigid Poincaré invariance under the linear substitutions**

$\longrightarrow$

$M \; = \; M^{\,D \,=\, D_{\mathcal{P}} \,+\, d}$ **(continued)**

$$X \;=\; (\, x^{\mu} \,;\, \xi^{\,r} \,) \;\in\; M^{\,D} \quad ; \quad \begin{cases} x \;=\; \left(x^{0} \,,\, x^{1} \,,\, x^{2} \,,\, x^{3} \right) \;\in\; M^{\,D_{\mathcal{P}}} \\ \xi \;=\; \left(\xi^{1} \,,\, \cdots \,,\, \xi^{d} \right) \;\in\; M^{\,d}\big|_{compactified} \\ \vec{x} \;=\; \left(x^{1} \,,\, x^{2} \,,\, x^{3} \right) \;\in\; R^{3} \end{cases}$$

action of Poincaré group : $(\, \Lambda \,,\, a \,) \;\in\; \mathcal{P} \quad ; \quad \mathcal{P} \;=\; L \circledless Tr$

$$x \;\rightarrow\; \Lambda\, x \;+\; a \quad ; \quad \Lambda \;\in\; L \,,\, a \;\in\; Tr \;=\; Tr^{\,D_{\mathcal{P}}}$$

(88)

In eq. 88 two additional symbols appear, denoting (rigid) groups

(89) $$\left. \begin{array}{ll} L & : \textbf{ Lorentz group} \\ Tr & : \textbf{ Translation group} \end{array} \right\} \text{ in } D_{\mathcal{P}} \;=\; 1 \;+\; 3 \text{ dimensions}$$

whereas $\circledless$ **stands for semidirect product.**

M_E **the Euclidean version** M

In many field theory applications it is assumed that as a consequence of locality it is legitimate to transform $M \;\rightarrow\; M_E$ **by a complex substitution involving only the time variable** x^{0}

$\longrightarrow$

$M = M^{D = D_{\mathcal{P}} + d}$ **(continued)**

(90) $$x^0 = i\tau \quad ; \quad \tau : \text{real}$$

The Poincaré group then becomes the Euclidean motion group and the Lorentz group is replaced by the orthogonal rotation group in $D_{\mathcal{P}} \longrightarrow D_E$ dimensions. We shall not distinguish here physical and Euclidean space-time explicitely, unless this becomes essential.

A1 Appendix 1 - Nonabelian local gauge invariance

Let me proceed along the path outlined in ref. [A11-2008] .

So we consider the adjoint *real, unitary* **i.e.** *orthogonal* **representation** $Ad \, (\, g \,)$ **of a local, compact, (semi-) simple gauge group, for the case of QCD the gauge group is** $SU3 \,_c$**, and the associated** *antihermitian, real* **representation of its Lie-algebra** $ad \, (\, L \,)$

$$\alpha, \beta, \cdots \, = \, 1 \, , \, \cdots \, , \, dim \, (\, G \,) \;\; ; \;\; G \, = \, SU3 \,_c \, \rightarrow \, dim \, (\, G \,) \, = \, 8$$

(91) $$ad \,_\kappa \, \equiv \, (\, ad \,_\kappa \,) \,^\sigma \,_\varrho \, = \, f \,_{\sigma \kappa \varrho} \;\; ; \;\; \kappa, \sigma, \varrho \, = \, 1, \cdots, dim \, (\, G \,) \, = \, 8$$

$$f \,_{\sigma \kappa \varrho} \, = \, f \,_{[\sigma \kappa \varrho]} \quad : \quad \textbf{totally antisymmetric, real structure coefficients of L}$$

Hence the 8 *real antisymmetric, i.e. antihermitian* **matrices** $ad \,_\kappa \, = \, ad \, (\, L \,_\kappa \,)$ **as defined in eq. 91 form a basis of the Lie-algebra of G, generated by the associated elements** $L \,_\kappa$**, forming together with real coefficients** $\omega \,^\kappa$ **the full linear space of** $Lie \, (\, G \,)$

(92) $$ad \,_\kappa \, \leftrightarrow \, L \,_\kappa \;\; ; \;\; L \, = \, L \, (\, \omega \,) \, \equiv \, \omega \,^\kappa \, L \,_\kappa$$

$$\left[\, L \,_\alpha \, , \, L \,_\beta \, \right] \, = \, L \,_\alpha \, L \,_\beta \, - \, L \,_\beta \, L \,_\alpha \, = \, f \,_{\alpha \, \beta \, \gamma} \, L \,_\gamma$$

Eqs. 91 and 92 contain two very nontrivial consequences, which may – incorrectly – appear obvious $\longrightarrow$

1) structure constants can be chosen constant : $f_{\alpha\beta\,\gamma}$ independent of group coordinates

2) structure constants can be chosen totally antisymmetric .

On continuous transformation groups

As an entry point I quote the lecture notes edited in the CERN Yellow report series from lectures given by Giulio Racah in Princeton 1951 [A12-1951-1961] . From this reference for the sake of historical correctness the following literature shall be cited [A13-1893] - [A17-1933].

The base space $\mathcal{B}$ shall be a manifold with (local) coordinates $x\ ,\ y\ ,\ \cdots$ chosen eventually with enumerating index sets from the end of the alphabet, whereas the group space $\mathcal{G}$ be equally a manifold with coordinates $a\ ,\ b\ ,\ \cdots$ chosen likewise from its beginning.

The action of the group transformations on $\mathcal{B}$ becomes

(93)
$$\begin{array}{l} T_a\, x \ = \ y \ = \ y\,(\,x\,;\,a\,) \ \ ; \ \ y^{\,j} \ = \ y^{\,j}\,(\,x^{\,1}\,,\ \cdots\ x^{\,B}\,;\,a^{\,1}\,,\ \cdots\ ,\ a^{\,G}\,) \\ B \ = \ dim\,(\,\mathcal{B}\,) \ \ ; \ \ G \ = \ dim\,\mathcal{G} \ \ ; \ \ j \ = \ 1\,,\ \cdots\ ,\ B \end{array}$$

with the transformation-(group property)

(94)
$$\begin{array}{l} T_a\, T_b\, x \ = \ T_{a\,.\,b}\, x \ \ \rightarrow \ \ y\,(\,y\,(\,x\,;\,b\,)\,;\,a\,) \ = \ y\,(\,x\,;\,a\,.\,b\,) \\ a\,.\,b \ \rightarrow \ c \ = \ c\,(\,a\,,\,b\,) \ \ ; \ \ c^{\,\nu} \ = \ c^{\,\nu}\,(\,a^{\,1}\,,\ \cdots\ a^{\,G}\,;\,b^{\,1}\,,\ \cdots\ ,\ b^{\,G}\,) \end{array}$$

$\longrightarrow$

The set of base space transformations $\{ T_a \mid \forall a \}$ shall be truly (or essentially) dependent on the set of group coordinates $\{ a \}$, which translates into the condition

(95) $$C1 : \quad \text{if} \;\; y(x;a) = y(x;b) \,\forall x \rightarrow a = b$$

A second condition to be valid for transformation group as well as group concerns the unique inverse to any given group element a as well as to its representation T_a

$$\begin{array}{lll} & a \leftrightarrow a^{-1} & \textit{and} \quad T_a \leftrightarrow T_{a^{-1}} = (T_a)^{-1} \longrightarrow \\ C2 : & \left\{ \begin{array}{r} c(a; a^{-1}) \\ = c(a^{-1}; a) \end{array} \right\} = e & \textit{and} \quad \left\{ \begin{array}{r} T_a T_{a^{-1}} x \\ = T_{a^{-1}} T_a x \end{array} \right\} = T_e x = x \, (\forall x) \end{array}$$

(96)

The associated unit elements $e \rightarrow T_e \; (\doteq ¶)$ defined in eq. 96 have the property from eq. 94

(97) $$e.a = a.e = a \;\leftrightarrow\; T_a T_e = T_e T_a = T_a \; (\forall a)$$

It is no loss of generality to assign the neutral element e the coordinates in $\mathcal{G}$

(98) $$e = \left(e^1, \cdots, e^G \right) \; ; \; e^\nu = 0 \; , \; \nu = 1, \cdots, G$$

→

Infinitesimal increments and their transport – from the 'right' and from the 'left'

Lets first consider the group element around a given a (in G)

(99) $$b = a + da = a\,.\,(de + e) \;\rightarrow\; de = a^{-1}\,.\,b = e + a^{-1}\,.\,da$$

Eq. 93 implies

(100)
$$y(x;\, a + da) = y(y(x;\, de);\, a)$$
$$y(x;\, de) = y(x;\, e) + v_{(\alpha)}\, de^{\alpha} = x + v_{(\alpha)}\, de^{\alpha}$$
$$v^{j}_{(\alpha)}(x) = \partial_{b^{\alpha}}\left(y^{j}(x;\, b)\right)\big|_{b=0}$$

In eq. 100 $v_{(\alpha)}(x);\ \alpha = 1, \cdots, G$ represent G vectorfields on $\mathcal{B}$ – with a priory unspecified rank, except for condition C1 – and (local) tangent-space coordinates

(101) $$\left[\, v_{(\alpha)} = \left(v^{1}_{(\alpha)}, \cdots, v^{G}_{(\alpha)} \right) \right](x) \;\rightarrow\; v^{j}_{(\alpha)}(x) = \partial_{b^{\alpha}}\left(y^{j}(x;\, b)\right)\big|_{b=0}$$

Differentiating the first relation in eq. 100 we obtain

(102)
$$\partial_{c^{\nu}}\left(y^{k}(x;\, c)\right)\big|_{c=a}\, da^{\nu} = \left(\partial_{x^{j}}\left(y^{k}(x;\, a)\right)\right) v^{j}_{(\alpha)}(x)\, de^{\alpha}$$

for $a + da = c(a;\, de)$ **here de from the 'right'** $\rightarrow$

Next we transpose de in eq. 99 to de' from the 'left'

$$b = a + da = a\,.\,(de + e) \quad \rightarrow \quad de = a^{-1}\,.\,b = e + a^{-1}\,.\,da \quad \longrightarrow$$

(103) $$b = a + da = \left(de' + e\right)\,.\,a \quad \rightarrow \quad de' = b\,.\,a^{-1} = e + da\,.\,a^{-1}$$

$$de' = a\,.\,de\,.\,a^{-1}$$

Eq. 100 implies

(104) $$y\,(x\,;\,a + da) = y\left(y\,(x\,;\,a)\,;\,de'\right) = y\left(x\,;\,c\,(de'\,;\,a)\right)$$

Lets – going step by step – substitute

(105) $$c\,(de'\,;\,a) = a + f_{(\alpha)}\,(a)\,de'^{\,\alpha}$$
$$\left[f_{(\alpha)} = \left(f^{\,1}_{(\alpha)}\,,\,\cdots\,f^{\,G}_{(\alpha)}\right)\right](a) \;\rightarrow f^{\,\nu}_{(\alpha)}\,(a) = \partial_{b^\alpha}\;c^{\nu}\,(b\,;\,a)\,|_{b=0}$$

It shall here be just mentioned as an exercise to clarify , why the left transformations on $\mathcal{B}$: $\{T_a \mid \cup\; a\}$ transform the same way as the left *or* **right transformations on $\mathcal{G}$: $\{L_a \mid \cup\; a\}$** *or* $\{D_a \mid \cup\; a\}$

(106) $$L_a\, b = a\,.\,b \;\;;\;\; D_a\, b = b\,.\,a^{-1}$$

$\rightarrow$

For simplicity lets choose besides the transformation group $\{ T_a \mid \cup\ a \}$ on $\mathcal{B}$ the *left* transformation group on $\mathcal{G}$: $\{ L_a \mid \cup\ a \}$, as defined in eq. 106 .
Then the tangent vector fields associated with $\{ T_a \mid \cup\ a \}$ of motions on $\mathcal{B}$: $v_{(\alpha)}(x)$ defined in eqs. 100 and 101 are associated with $\{ L_a \mid \cup\ a \}$ inducing motions on $\mathcal{G}$ which generate the tangent vector fields $f_{(\alpha)}(b)$ as defined in eq. 105 proper care beeing taken to rename the differentiating variable in eq. 105 $b \rightarrow b'$ to distinguish it from the general coordinate – b above on $\mathcal{G}$. Eq. 103 becomes

(107)
$$\begin{aligned} v^{k}_{(\alpha)} \{ y(x; a) \} \, de'^{\alpha} &= v^{k}_{(\beta)}(x) f^{\beta}_{(\alpha)}(a) \, de^{\alpha} \\ &= \partial_{c^{\nu}} \left(y^{k}(x; c) \right) \Big|_{c=a} da^{\nu} \end{aligned}$$

for $\quad a + da = c(de'; a) \equiv c(a; de) \ ; \ de' = a \,.\, de \,.\, a^{-1}$

The vector fields in eq. 108 below are called Killing fields (on $\mathcal{B}$ and 'left' on $\mathcal{G}$ respectively) [A18-1890] .

(108)
$$\left[v_{(\alpha} = \left(v^{1}_{(\alpha)}, \cdots, v^{B}_{(\alpha)} \right) \right](x) \ ; \ \textbf{on } \mathcal{B}$$
$$\left[f_{(\alpha)} = \left(f^{1}_{(\alpha)}, \cdots, f^{G}_{(\alpha)} \right) \right](a) \ ; \ \textbf{on } \mathcal{G}$$

$$v^{j}_{(\alpha)}(x) = \partial_{b^{\alpha}} \left(y^{j}(x; b) \right) \Big|_{b=0} \ ; \ f^{\nu}_{(\alpha)}(a) = \partial_{b^{\alpha}} c^{\nu}(b; a) \Big|_{b=0}$$

→

A1-7

On the notion of motion

Fig A11 : Aristarchos from Samos, $\sim$ 310-230 BC $\longleftrightarrow$

The exponential mapping

We first look at a one parameter family of group transformations, depending on a 'time' parameter τ, which traces out a path on $\mathcal{B}$ as well as on $\mathcal{G}$ with the properties corresponding to the Killing (left) vector fields outlined in eq. 108

$$0 \leq \tau \leq t \; : \; b \to b(\tau)$$

(109)
$$b(\tau + d\tau) = b(\tau) + de'^{\,\alpha} f^{\,\nu}_{(\alpha)}(b) \quad \textbf{with} \quad de'^{\,\alpha} = d\tau\, \omega^{\alpha}$$

$$\omega^{\alpha} = de'^{\,\alpha} / d\tau \quad \textbf{independent of} \quad \tau$$

Thus to the tangent vector in the neighbourhood of $e \in \mathcal{G}$

(110)
$$\omega = (\omega^{1}, \cdots, \omega^{G})$$

corresponds the infinitesimal 'left' transformation constructed from $\{ L_a \mid \cup \; a \}$

(111)
$$\omega \to \widehat{\omega} = \frac{L_{e + de'} - L_e}{d\tau} \; ; \; de' = d\tau\, \omega \; ; \; L_e \doteq \P_{\mathcal{G}}$$

$$\widehat{\omega}\, b = \omega^{\alpha} f_{(\alpha)}(b) \; ; \; \left[f_{(\alpha)} = \left(f^{\,\nu}_{(\alpha)} \right) \right](b)$$

→

So from the second relation in eq. 109 we obtain the system of G linear differential equations (first on $\mathcal{G}$)

(112) $$\dot{b}^{\,\nu} = \omega^{\,\alpha} f^{\,\nu}_{\,(\alpha)} (b (\tau)) \qquad ; \quad \dot{} = \frac{d}{d\,\tau}$$

and choosing the initial condition(s)

(113) $$b_{\,0} = b (\tau = 0) = e \;\rightarrow\; f^{\,\nu}_{\,(\alpha)} (e) = \delta^{\,\nu}_{\,(\alpha)}$$

ω **is – using eq. 109 – correctly identified with the initial direction in the tangent space at e , or – associating τ with a time – with the initial velocity .**
With the initial conditions as specified in eq. 113 we thus find a one-parametric, abelian subgroup of $\mathcal{G}$, as a solution to the so completed set of *first order differential equations* **(eq. 112)**

(114) $$b (\tau_{\,2}) \,.\, b (\tau_{\,1}) = b (\tau_{\,1} + \tau_{\,2}) \;;\; b = b (\tau \,; \omega) \;;\; \dot{b} (0 \,; \omega) = \omega$$

for : $0 \le \tau_{\,1} , \tau_{\,2} , \tau_{\,1} + \tau_{\,2} \le t_{\,max}$

The range of regularity and uniqueness of the above system of differential equations depends on the range of differentiability of the Killing fields $f_{\,(\alpha)} (g)$ [on $\mathcal{G}$] . $\longrightarrow$

Pro- or in-jecting the exponential mapping from $\mathcal{G}$ to $\mathcal{B}$

We now consider the trajectory in $\mathcal{B}$ associated with $b(\tau)$ as constructed in eq. 112 - 114

(115) $$b(\tau) \longrightarrow \overline{y}(\tau) = y(x_0; b(\tau)) \;\; ; \;\; \overline{y}(\tau = 0) = x_0$$

Next we recast eq. 100 to the 'left' transformation on $\mathcal{G}$

$$y(x_0; b + db) = y\left\{ y(x_0; b); e + de' \right\} = y(x_0; b) + v_{(\alpha)}(\overline{y})\, de'^{\,\alpha}$$
$$= y(x_0; b) + v_{(\alpha)}(x_0)\, de^{\alpha}$$

$$\overline{y} = y(x_0; b) \;\; ; \;\; de = b^{-1} . de' \, b$$

(116)

Using eq. 107 , we infer the differential equation on $\mathcal{B}$

(117) $$\dot{\overline{y}} = v_{(\alpha)}(\overline{y}(\tau))\, \omega^{\alpha}$$
$$v_{(\alpha)}\{ y(x_0; b(\tau)) \} = v_{(\beta)}(x_0)\, \widetilde{f}^{\,\beta}_{(\alpha)}(b(\tau)) \;\; ; \;\; \widetilde{f} \neq f$$

but because of the interplay between 'left' and 'right' transformations on $\mathcal{G}$ it remains to establish, there, a relation between de and de'

(118) $$de = b^{-1} . de' . b$$

→

The adjoint *linear* representation of $\mathcal{G}$ from $de = b^{-1} . de' . b$

Eq. 118 paves the way from the operator version of the Lie generators of – to be definite – 'left' transformations on $\mathcal{G}$ according to eq. 111 extended below

(119)

$$\omega \rightarrow \widehat{\omega} = \frac{L_{e + de'} - L_e}{d\tau} \; ; \; de' = d\tau \, \omega \; ; \; L_e \doteq \P_{\mathcal{G}}$$

$$\widehat{\omega} \, b = \omega^{\alpha} f_{(\alpha)}(b) \; ; \; \left[f_{(\alpha)} = \left(f^{\nu}_{(\alpha)} \right) \right](b) \longrightarrow$$

$$\underline{\omega} = \omega^{\alpha} \underline{I}_{(\alpha)}$$

In eq. 119 the underlined quantities $\underline{\omega}$; $\underline{I}_{(\alpha)}$ denote linear (derivative) operators on function space(s) over $\mathcal{G}$

$$\{ \Phi \mid \Phi = \Phi[b], b \in \mathcal{G} \} \rightarrow$$

$$g(\widehat{\omega}) \sim e + de' \; ; \; g^{-1}(\widehat{\omega}) \sim e - de' \; : \; de' = d\tau \, \omega$$

$$d\tau \, \underline{\Delta}(\omega) \, \Phi[b] \sim \Phi \left[\left(e - de' \right) . b \right] - \Phi[b] \sim - d\tau \, \omega^{\alpha} f^{\nu}_{(\alpha)}(b) \, \partial_{b^{\nu}} \Phi[b]$$

$$\longrightarrow \; \underline{\Delta}(\omega) \equiv \underline{\omega} = \omega^{\alpha} \underline{I}_{(\alpha)} = - \omega^{\alpha} f^{\nu}_{(\alpha)}(b) \, \partial_{b^{\nu}}$$

(120)

$\longrightarrow$

In eq. 120 operators acting on the function space $\{\Phi \mid \Phi = \Phi[b], b \in \mathcal{G}\}$ are underlined except for the partial derivative operators $\partial_{b^{\nu}}$ to maintain clarity. We shall use (covariant) vector notation for partial derivatives

(121)
$$\nabla_b = (\partial_{b^1}, \cdots, \partial_{b^G}) \rightarrow$$
$$\underline{\omega} = -\omega^{\alpha} \left(f_{(\alpha)}(b) \nabla_b \right) \Big|_{\mathcal{G}}$$

From $\underline{\omega}\,|_{\mathcal{G}}$ to $\underline{\omega}\,|_{\mathcal{B}}$ inducing $\left[\underline{I}_{(\alpha)}\Big|_{\mathcal{G}} \leftrightarrow \underline{I}_{(\alpha)}\Big|_{\mathcal{B}} \right]$

Here we recall eq. 107 and define in clear association of $\mathcal{G} \rightarrow \mathcal{B}$ the function space(s) , Killing fields and partial derivatives on $\mathcal{B}$

(122)
$$\{\Phi \mid \Phi = \Phi[b], b \in \mathcal{G}\} \rightarrow \{\Psi \mid \Psi = \Psi[y], y \in \mathcal{B}\}$$
$$\left(f^{1}_{(\alpha)}, \cdots, f^{G}_{(\alpha)} \right)[b] \rightarrow \left(v^{1}_{(\alpha)}, \cdots, v^{B}_{(\alpha)} \right)[y]$$
$$\nabla_b = (\partial_{b^1}, \cdots, \partial_{b^G}) \rightarrow \nabla_y = \left(\partial_{y^1}, \cdots, \partial_{y^B} \right) \rightarrow$$
$$\underline{\omega} = -\omega^{\alpha} \left(f_{(\alpha)}(b) \nabla_b \right) \Big|_{\mathcal{G}} \rightarrow \underline{\omega} = -\omega^{\alpha} \left(v_{(\alpha)}(y) \nabla_y \right) \Big|_{\mathcal{B}}$$
$$\underline{I}_{(\alpha)} = -f_{(\alpha)}(b) \nabla_b \Big|_{\mathcal{G}} \rightarrow \underline{I}_{(\alpha)} = -v_{(\alpha)}(y) \nabla_y \Big|_{\mathcal{B}}$$

$\rightarrow$

It is interesting to note here, that the original idea of Sophus Lie [A13-1893] was in the reverse association $($ from $\mathcal{B}$ to $\mathcal{G}$ $)$ than displayed in eq. 122 , more in line with Élie Cartan [A14-1894] .

Now we return to the abelian one parameter subrgoups constructed from the differential equations eqs. 109 - 112 on $\mathcal{G}$ and eqs. 115 - 117 on $\mathcal{B}$, collected in abbreviation below

$$\dot{b}^{\,\nu} = \omega^{\,\alpha} f^{\,\nu}_{\,(\alpha)} (b (\tau)) \;\; ; \;\; \dot{b}^{\,\nu} (\tau = 0) = \omega^{\,\nu} \qquad \text{on } \mathcal{G}$$

(123) $$\dot{\overline{y}}^{\,\varrho} = \omega^{\,\alpha} v^{\,\varrho}_{\,(\alpha)} (\overline{y} (\tau)) \;\; ; \;\; \dot{\overline{y}}^{\,\varrho} (\tau = 0) = \omega^{\,\alpha} v^{\,\varrho}_{\,(\alpha)} (x_{\,0}) \quad \text{on } \mathcal{B}$$

$$\overline{y}^{\,\varrho} (\tau = 0) = x^{\,\varrho}_{\,0}$$

Example : $\mathcal{B} = R_{\,3} \, , \, \mathcal{G} = SU2 \equiv S_{\,3}$

Let the coordinates on $\mathcal{B} = R_{\,3}$ be denoted by $\vec{y} \, , \, \vec{x}_{\,0} \, , \, \cdots$, following the notation used in the general case in eqs. 122 - 123 . *Not knowing yet* **which group $\mathcal{G}$ will emerge, we take the rotation group acting on $R_{\,3}$ infinitesimally**

(124) $$\underline{\omega} \, |_{\,\mathcal{B}} = - \, (\vec{\omega} \wedge \vec{y} \,) \, \vec{\nabla}_{\,y} = - \, \omega^{\,\alpha} \, \varepsilon_{\,\alpha\beta\,\gamma} \, y^{\,\beta} \, \partial_{\,y^{\,\gamma}} = - \, \vec{\omega} \left(\vec{y} \wedge \vec{\nabla}_{\,y} \right)$$

In eq. 124 the vector-product of two three vectors $\vec{x} \, , \, \vec{y}$ is denoted $\vec{x} \wedge \vec{y}$. Further ε is the totally antisymmetric three tensor on $R_{\,3}$ implying $\alpha, \beta, \gamma = 1, 2, 3$. →

Hence we learn from well known kinematical formulae that there must be a continuous group $\mathcal{G}$ of dimension three, and consequently that there are three infinitesimal generators

(125)
$$\underline{\vec{I}} = \left(\underline{I}_{(1)} , \underline{I}_{(2)} , \underline{I}_{(3)} \right) \Big|_{\mathcal{B}} \quad \textbf{with}$$
$$\underline{\vec{I}} = - \vec{y} \wedge \vec{\nabla}_{y} \Big|_{\mathcal{B}}$$

The formulae derived from eqs. 124 - 125 are in their classical phase space variant usually embedded topically in a treatise on mechanics of rigid bodies [A19-1897] and thus not immediately identified with the

search for a continuous group.

Here let us give the three components of $\underline{\vec{I}} \Big|_{\mathcal{B}}$, dropping for clarity the label $\big|_{\mathcal{B}}$

(126)
$$\underline{I}_{(1)} = - \left(y^{2} \partial_{y^{3}} - y^{3} \partial_{y^{2}} \right) \quad 1\,23$$
$$\underline{I}_{(2)} = - \left(y^{3} \partial_{y^{1}} - y^{1} \partial_{y^{3}} \right) \quad 2\,31$$
$$\underline{I}_{(3)} = - \left(y^{1} \partial_{y^{2}} - y^{2} \partial_{y^{1}} \right) \quad 3\,12$$

It follows – and shall be given as an exercise – that the Lie-algebra commutators induced on $\mathcal{B}$ are of the form

(127)
$$\left[I_{(\alpha)} , I_{(\beta)} \right] = \varepsilon_{\alpha\beta\gamma} I_{(\gamma)} \quad ; \quad \varepsilon_{123} = 1 \, ; \, \alpha, \beta, \gamma = 1, 2, 3$$

Back to the adjoint representation and its relation to second order derivatives along the abelian one parameter subgroups on $\mathcal{B}$ and $\mathcal{G}$

We take up the consequences of eqs. 115 - 118 and distinguish two one-parameter abelian subgroubs using the following notation both on $\mathcal{B}$ and $\mathcal{G}$

(128)
$$\begin{array}{lll} \textbf{on } \mathcal{G} : \; a(\tau) \, ; \, b(\vartheta) & g : & \textbf{general coordinate} \\ \textbf{on } \mathcal{B} : \; \underline{T}_{\,a(\tau)} \, ; \, \underline{T}_{\,b(\vartheta)} & x : & \textbf{general coordinate} \end{array}$$

Next we represent the tranformation group acting on $\mathcal{G}$ by left-multiplication in accordance with eqs. 119 - 121 with respect to $\mathcal{G}$ and eqs. 122 - 123 with respect to (also) $\mathcal{B}$, defining for *one* one-parameter subgroup the (operator-) better transformation-valued quantities $\underline{\omega} \,|_{\mathcal{G}}$ to $\underline{\omega} \,|_{\mathcal{B}}$ inducing $\left[\underline{I}_{(\alpha)} \Big|_{\mathcal{G}} \leftrightarrow \underline{I}_{(\alpha)} \Big|_{\mathcal{B}} \right]$

(129)
$$\begin{aligned} o(\tau\, ;\, \omega) &\rightarrow \exp\left(\tau\, \omega^{\,\nu}\, \underline{I}_{(\nu)} \right) \Big|_{\mathcal{G}\,(\mathcal{B})} \longrightarrow \\ a(\tau\, ;\, \alpha) &\rightarrow \exp\left(\tau\, \alpha^{\,\nu}\, \underline{I}_{(\nu)} \right) \Big|_{\mathcal{G}\,(\mathcal{B})} \\ b(\vartheta\, ;\, \beta) &\rightarrow \exp\left(\vartheta\, \beta^{\,\nu}\, \underline{I}_{(\nu)} \right) \Big|_{\mathcal{G}\,(\mathcal{B})} \end{aligned}$$

$\longrightarrow$

Next we come back to the active form of left-multiplication as defined in eq. 119

(130) $$g \longrightarrow L_a(g) = a \, . \, g \quad ; \quad \left\{ \begin{array}{l} a \longrightarrow a(\tau\,;\alpha) \\ b \longrightarrow b(\vartheta\,;\alpha) \end{array} \right\}$$

The group representation takes the form

(131) $$g \longrightarrow g_1 = L_{a_1}(g) \longrightarrow g_2 = L_{a_2}(g_1) \longrightarrow$$
$$L_{a_2}(L_{a_1}(g)) = a_2 \, . \, (a_1 \, . \, g) = (a_2 \, . \, a_1) \, . \, g$$
$$L_{a_2} L_{a_1} = L_{a_2 \, . \, a_1}$$

The one parameter abelian subgoups $\{ a(\tau\,;\alpha) \}$ **and** $\{ b(\vartheta\,;\beta) \}$ **on** $\mathcal{G}$

We consider the unique one parameter abelian subgroup associated with the pair $\{ b \} \; ; \; \{ a \}$

(132) $$d(\vartheta\,;\gamma) = a(\tau\,;\alpha) \, . \, b(\vartheta\,;\beta) \, . \, a^{-1}(\tau\,;\alpha) \longrightarrow$$
$$d(\vartheta_1\,;\gamma) \, d(\vartheta_2\,;\gamma) = d(\vartheta_1 + \vartheta_2\,;\gamma)$$

It is easier to represent the conjugation $($ **of** b **by** a $)$ **defined in eq. 132 through the associated operators than through the active transforming relations .** $\longrightarrow$

This is achieved by the substitutions

(133) $$a \, , \, b \, , \, c \; \longrightarrow \; \underline{a} \, , \, \underline{b} \, , \, \underline{c}$$

For infinitesimal $\vartheta \;\rightarrow\; d\vartheta$ eq. 133 takes the *associated* form , remembering that $\underline{e} \;=\; \P \, \big|_{\,\mathcal{G}}$

(134) $$\begin{aligned} &\underline{a} \, (\, \tau \, ; \, \alpha \,) \, \Big(\, \P \, + \, d\vartheta \, \underline{\beta} \, \Big) \, \underline{a}^{\,-1} \, (\, \tau \, ; \, \alpha \,) \; \simeq \; \P \, + \, d\vartheta \, \underline{\gamma} \\ &\underline{a} \;=\; \exp \, \Big(\, \tau \, \alpha^{\,\nu} \, \underline{I}_{\,(\nu)} \, \Big) \, \Big|_{\,\mathcal{G}} \quad ; \quad \cdots \\ &\underline{\gamma} \;=\; \gamma^{\,\nu} \, \underline{I}_{\,(\nu)} \, \Big|_{\,\mathcal{G}} \quad \text{and} \quad \gamma \;\rightarrow\; \beta \, , \, \alpha \end{aligned}$$

In eq. 134 the quantities $\alpha \, , \, \beta \, , \, \gamma$ stand for *tangent vectors* on/of $\mathcal{G}$

(135) $$\alpha \;=\; \left(\, \alpha^{\,1} \, , \, \cdots \, \alpha^{\,G} \, \right) \quad \text{and} \quad \alpha \;\rightarrow\; \beta \, , \, \gamma$$

It follows from the structure of eqs. 130 - 135 *through several steps*

(136) $$\begin{aligned} &\gamma^{\,\nu} \;=\; F^{\,\nu} \, (\, \beta \, ; \, a \,) \;\rightarrow\; F^{\,\nu} \, (\, \beta \, ; \, a \,) \;=\; [\, Ad \, (\, a \,) \,]^{\,\nu}_{\;\;\mu} \, \beta^{\,\mu} \\ &a \;=\; a \, (\, \tau \;=\; 1 \, ; \, \alpha \,) \quad ; \quad {}^{\nu} \, , \, {}_{\mu} \;=\; 1 \, , \, \cdots \, G \end{aligned}$$

Furthermore choosing a as an endpoint of a one parameter abelian subgroup of $\mathcal{G}$ is not essential in the matrix association , rewriting eq. 136 in matrix form $\longrightarrow$

(137)
$$\gamma = F(\beta ; a) = Ad(a)\,\beta \longrightarrow$$
and for $a = a_2 . a_1 \quad : \quad Ad(a_2 . a_1) = Ad(a_2)\,Ad(a_1)$

The set of $G \times G$ **matrices**

(138)
$$\{ Ad(a) \,|\, a \in \mathcal{G}) \}$$

with the group relation defined in eq. 137 is called the adjoint representation *of/over* $\mathcal{G}$ **.**

A1-ad From adjoint representation to its infinitesimal form ; $Ad(a)\,|_{\mathcal{G}} \rightarrow ad(\alpha)\,|_{\mathcal{T}}$

In A1-ad we have defined (a new symbol) $\mathcal{T}$ **denoting the tangent space at the unit element of** $\mathcal{G}$

(139)
$$\underline{a} = \exp\left(\tau\,\alpha^{\nu}\,\underline{I}_{(\nu)}\right)\Big|_{\mathcal{G}} \longrightarrow$$
$$a \in \mathcal{G} \rightarrow \alpha \in \mathcal{T} = \mathcal{T}\,|_{e}$$

Now we expand eq. 134 with respect to $\tau \rightarrow d\tau$ *and* $d\vartheta$

(140)
$$\underline{a}(\tau ; \alpha)\left(¶ + d\vartheta\,\underline{\beta}\right)\underline{a}^{-1}(\tau ; \alpha) \simeq ¶ + d\vartheta\,\underline{\gamma} \longrightarrow$$
$$(¶ + d\tau\,\underline{\alpha})\left(¶ + d\vartheta\,\underline{\beta}\right)(¶ - d\tau\,\underline{\alpha}) \simeq ¶ + d\vartheta\,d\tau\,\underline{\delta}$$
$$\underline{\delta} = \underline{\delta}\left(\underline{\alpha}, \underline{\beta}\right) = \left[\,\underline{\alpha}, \underline{\beta}\,\right] \equiv \underline{\alpha}\,\underline{\beta} - \underline{\beta}\,\underline{\alpha}$$

→

Eq. 140 can be cast to the form using eq. 136

(141)
$$\left(Ad \left(e + d\tau \, \alpha \right) - ¶ \right)^{\sigma}{}_{\varrho} \, \beta^{\varrho} = d\tau \, \alpha^{\kappa} \left(ad_{(\kappa)} \right)^{\sigma}{}_{\varrho} \, \beta^{\varrho} \longrightarrow$$
$$\alpha^{\kappa} \left(ad_{(\kappa)} \right)^{\sigma}{}_{\varrho} \, \beta^{\varrho} \, \underline{I}_{(\sigma)} = \left[\alpha^{\nu} \, \underline{I}_{(\nu)} \, , \, \beta^{\mu} \, \underline{I}_{(\mu)} \right] \quad \forall \, \alpha \, , \, \beta \longrightarrow$$
$$\left(ad_{(\kappa)} \right)^{\sigma}{}_{\varrho} \, \underline{I}_{(\sigma)} = \left[\underline{I}_{(\kappa)} \, , \, \underline{I}_{(\varrho)} \right]$$

We thus cannot use the apperantly 'evident' definition in eq. 91 , but rather set

(142)
$$\left[\underline{I}_{(\kappa)} \, , \, \underline{I}_{(\varrho)} \right] = f^{\sigma}{}_{\kappa \varrho} \, \underline{I}_{(\sigma)} \quad ; \quad f^{\sigma}{}_{\kappa \varrho} = - f^{\sigma}{}_{\varrho \kappa} \longrightarrow$$
$$\left(ad_{(\kappa)} \right)^{\sigma}{}_{\varrho} \equiv f^{\sigma}{}_{\kappa \varrho}$$

While the antisymmetric nature of the structure constants $f^{\sigma}{}_{\kappa \varrho}$ **with respect to the two lower case compenents** $\kappa \, \varrho$ **is straightforward, the overall threefold antisymmetric form only obtains for compact groups and only after suitable coordinates on** $Lie_{\mathcal{G}}$ **are introduced.**

We first note the quadratic covariant tangent space metric with respect to the adjoint $\left\{ Ad \left(a \right) \, | \, a \in \mathcal{G} \right\}$ **representation**

(143)
$$\eta_{\gamma \kappa} = - tr \, ad_{(\gamma)} \, ad_{(\kappa)} = - f^{\varrho}{}_{\gamma \sigma} \, f^{\sigma}{}_{\kappa \varrho} = \eta_{\kappa \gamma}$$

$\longrightarrow$

We note here the counting of continous derivatives in the transformation group functional forms on $\mathcal{B}$ and $\mathcal{G}$: in order to define the action on tangent space of $\mathcal{T}_{\mathcal{G}}\,(\,e\,)$ of the adjoint representation of $Lie_{\,\mathcal{G}}$ as defined in eqs. 141 - 142 we need two continous derivatives and for any further power with respect to $ad_{\,(\kappa)} \in Lie_{\,\mathcal{G}}$, one more continous derivative beyond 2 is needed , e.g. a total of 3 in eq. 143 .

Next, continuing at third order , as defined above, we verify the representation associated Lie algebra commutation rules

(144)
$$\underline{I}_{\,(\varrho)} \;\rightarrow\; ad_{\,(\varrho)} \quad \stackrel{?}{\longrightarrow}$$
$$\left[\, ad_{\,(\varrho)}\, , ad_{\,(\sigma)} \,\right] \;=\; f^{\;\alpha}_{\;\;\varrho\,\sigma}\;\; ad_{\,(\alpha)}$$

To this end we use the Jacobi identity involving double commutators

$$\left[\,\underline{I}_{\,(\varrho)}\,,\,\left[\,\underline{I}_{\,(\sigma)}\,,\,\underline{I}_{\,(\tau)}\,\right]\,\right] + \left[\,\underline{I}_{\,(\tau)}\,,\,\left[\,\underline{I}_{\,(\varrho)}\,,\,\underline{I}_{\,(\sigma)}\,\right]\,\right] + \left[\,\underline{I}_{\,(\sigma)}\,,\,\left[\,\underline{I}_{\,(\tau)}\,,\,\underline{I}_{\,(\varrho)}\,\right]\,\right] \;=\; 0$$
$$\left[\,\underline{I}_{\,(\varrho)}\,,\,\underline{I}_{\,(\alpha)}\,\right] f^{\;\alpha}_{\;\;\sigma\,\tau} \;+\; \left[\,\underline{I}_{\,(\tau)}\,,\,\underline{I}_{\,(\alpha)}\,\right] f^{\;\alpha}_{\;\;\varrho\,\sigma} \;+\; \left[\,\underline{I}_{\,(\sigma)}\,,\,\underline{I}_{\,(\alpha)}\,\right] f^{\;\alpha}_{\;\;\tau\,\varrho} \;=\; 0$$
$$\left(\, f^{\;\nu}_{\;\;\varrho\,\alpha} f^{\;\alpha}_{\;\;\sigma\,\tau} \;+\; f^{\;\nu}_{\;\;\tau\,\alpha} f^{\;\alpha}_{\;\;\varrho\,\sigma} \;+\; f^{\;\nu}_{\;\;\sigma\,\alpha} f^{\;\alpha}_{\;\;\tau\,\varrho}\,\right) \;=\; 0 \;\longrightarrow$$
$$\left(\, ad_{\,(\varrho)}\,\right)^{\,\nu}_{\;\;\alpha}\left(\, ad_{\,(\sigma)}\,\right)^{\,\alpha}_{\;\;\tau} \;-\; f^{\;\alpha}_{\;\;\varrho\,\sigma}\left(\, ad_{\,(\alpha)}\,\right)^{\,\nu}_{\;\;\tau} \;-\; \left(\, ad_{\,(\sigma)}\,\right)^{\,\nu}_{\;\;\alpha}\left(\, ad_{\,(\varrho)}\,\right)^{\,\alpha}_{\;\;\tau} \;=\; 0$$
(145)

The relation in the last line of eq. 145 verifies eq. 144 . $\longrightarrow$

A1-Ad Finite adjoint representations $\{ Ad(a) \mid a \in \mathcal{G} \}$ leave the metric $\eta_{\gamma\kappa}$ in eq. 143 invariant

Eq. 133 can be transcribed to the basis formed by the association

$$\underline{I}_{(\nu)} \rightarrow (ad_{(\nu)})^{\lambda}{}_{\sigma}$$

(146)

$$\underline{a} = \exp\left(\tau\,\alpha^{\nu}\,\underline{I}_{(\nu)}\right)\Big|_{\mathcal{G}} \rightarrow D(a(\tau)) = \exp\left(\tau\,\alpha^{\nu}\,ad_{(\nu)}\right) \longrightarrow$$

$$D(a(\tau))\,\beta^{\varrho}\,ad_{(\varrho)}\,D^{-1}(a(\tau)) = (D(a(\tau))^{\sigma}{}_{\varrho}\,\beta^{\varrho}\,ad_{(\sigma)} \rightarrow$$

$$D(a(\tau))\,ad_{(\varrho)}\,D^{-1}(a(\tau)) = (D(a(\tau))^{\sigma}{}_{\varrho}\,ad_{(\sigma)}$$

From eq. 146 we infer

(147)

$$ad_{(\lambda)} = D^{-1}(a(\tau))\,ad_{(\varrho)}\,D(a(\tau))\,(D(a(\tau))^{\varrho}{}_{\lambda}$$

$$ad_{(\kappa)} = D^{-1}(a(\tau))\,ad_{(\sigma)}\,D(a(\tau))\,(D(a(\tau))^{\sigma}{}_{\kappa}$$

$$D(a(\tau)) \equiv Ad(a(\tau)) \rightarrow D \longrightarrow$$

$$ad_{(\lambda)}\,ad_{(\kappa)} = D^{-1}\,ad_{(\varrho)}\,D\,D^{-1}\,ad_{(\sigma)}\,D\,\left(D^{\varrho}{}_{\lambda}\,D^{\sigma}{}_{\kappa}\right) \rightarrow$$

$$tr\,ad_{(\lambda)}\,ad_{(\kappa)} = tr\,ad_{(\varrho)}\,ad_{(\sigma)}\,\left(D^{\varrho}{}_{\lambda}\,D^{\sigma}{}_{\kappa}\right) \longrightarrow$$

$$\eta_{\lambda\kappa} = D^{\varrho}{}_{\lambda}(a(\tau))\,D^{\sigma}{}_{\kappa}(a(\tau))\,\eta_{\varrho\sigma} \quad qed$$

$\longrightarrow$

In order to exponentiate $Ad \,(\, a \,(\, \tau \,)\,) \;=\; \exp\; \left(\, \tau\, \alpha^{\,\nu}\, ad_{\,(\nu)} \,\right)$ **as displayed in eq. 146 convergent transformation group parametrization is needed to infinite order** $($ **on** $\mathcal{B}\; and\; \mathcal{G}\;)$ **. This is tantamount to demand (real) analytic properties of these transformations .**

Here we reexpand the last relation in eq. 147 , tantamount to four continuous derivatives ***only***

(148)
$$\begin{array}{l} Ad \,(\, a \,(\, \tau \,)\,) \;\sim\; \P \;+\; d\tau\, \alpha^{\,\nu}\, ad_{\,(\nu)} \\ \eta \;=\; Ad^{\,T}\, \eta\, Ad \;\;\rightarrow\;\; 0 \;=\; ad^{\,T}_{\,(\nu)}\, \eta \;+\; \eta\, ad_{\,(\nu)} \end{array}$$

In eq. 148 the superfix ^{T} **denotes the transposed of a given matrix.**

The last relation in eq. 148 in components becomes

(149)
$$\begin{array}{l} \left(\, ad_{\,(\nu)} \,\right)^{\,\varrho}_{\;\;\lambda}\, \eta_{\,\varrho\,\kappa} \;+\; \eta_{\,\lambda\,\varrho}\, \left(\, ad_{\,(\nu)} \,\right)^{\,\varrho}_{\;\;\kappa} \;=\; 0 \;\;;\;\; \left(\, ad_{\,(\nu)} \,\right)^{\,\varrho}_{\;\;\kappa} \;=\; f^{\,\varrho}_{\;\;\nu\kappa} \;\;\longrightarrow \\ \eta_{\,\kappa\,\varrho}\, f^{\,\varrho}_{\;\;\nu\lambda} \;+\; \eta_{\,\lambda\,\varrho}\, f^{\,\varrho}_{\;\;\nu\kappa} \;=\; 0 \quad ; \quad \eta_{\,\lambda\,\kappa} \;=\; -\, f^{\,\varrho}_{\;\;\lambda\sigma}\, f^{\,\sigma}_{\;\;\kappa\varrho} \end{array}$$

A1-ENA Condition of resolvability of $\mathcal{G}$ characterizing essentially nonabelian continuous groups

The condition of essential nonabelian nature of the group $\mathcal{G}$ contains the *insufficient* condition that every infinitesimal generator shall be representable also as the Lie product of two such

(150) $$\forall \text{ given } \alpha = \alpha^{\nu}\, ad_{(\nu)} \;\; \exists \;\; \beta, \gamma \text{ with } \alpha = [\beta, \gamma]$$

The main condition however , also denoted semi-simple restriction of $\mathcal{G}$ demands that $\mathcal{G}$ does not contain an abelian normal subgroup $\mathcal{A}$, i.e. an abelian subgroup invariant under conjugation by the entire group $\mathcal{G}$. On $Lie_{\mathcal{G}}$ an abelian normal subgroup displays the following features

$$Lie_{\mathcal{A}} \;\; : \;\; ad_{(a)} \;\; ; \;\; (a) = 1, \cdots, A$$

(151) $$Lie_{\mathcal{G}} \;\; : \;\; ad_{(a)}, ad_{(\nu)} \;\; ; \;\; (a) = 1, \cdots, A; \, (\nu) = 1, \cdots, G - A \;\; \longrightarrow$$

$$f^{c}{}_{ab} = f^{\nu}{}_{ab} = 0 \;\; ; \;\; f^{\nu}{}_{a\mu} = f^{\nu}{}_{\mu b} = 0$$

The metric quantities $\eta_{\gamma\kappa}$ defined in eq. 143 thus have the following structure , upon a change of basis adapted to the separation into $Lie_{\mathcal{G}} = Lie_{\mathcal{A}} \oplus R$ as implied by eq. 151

$$\eta_{\gamma\kappa} = -f^{\varrho}{}_{\gamma\sigma} f^{\sigma}{}_{\kappa\varrho}$$

(152) $$\eta_{ab} = -f^{c}{}_{a\mu} f^{\mu}{}_{bc} = 0 \;\; ; \;\; \eta_{a\varrho} = -f^{c}{}_{a\mu} f^{\mu}{}_{\varrho c} = 0 \;\; \longleftrightarrow$$

$$det\, \eta = 0$$

It follows that for a semi-simple *continuous* group $\mathcal{G}$ $det\, \eta \neq 0$.

A1-Cpct The semi-simple case and eventual reduction to compact nature of $\mathcal{G}$

The invertible metric η allows to treat the scalar product within $Lie_{\mathcal{G}}$ as nondegenerate .

We return to the notation for the elements of $Lie_{\mathcal{G}}$ in line with those of eq. 146

(153)
$$\underline{a} = \exp\left(\tau\,\alpha^{\nu}\,\underline{I}_{(\nu)}\right)\Big|_{\mathcal{G}} \overset{\tau\,\rightarrow\,1}{\longrightarrow} \underline{\alpha} = \alpha^{\nu}\,\underline{I}_{(\nu)} \quad \leftrightarrow \quad \alpha = \alpha^{\nu}\,ad_{(\nu)}$$
$$\left(ad_{(\nu)}\right)^{\varrho}{}_{\sigma} \equiv f^{\varrho}{}_{\nu\sigma}$$

The *real, symmetric* **scalar product shall be denoted**

(154)
$$\left(\ \beta\ ,\ \alpha\ \right)_{\eta} = \beta^{\lambda}\,\eta_{\lambda\,\kappa}\,\alpha^{\kappa}$$

In the semi-simple restriction there are no vanishing eigenvalues of η , which are all ***real*** **. Hence η can be diagonalized**

(155)
$$\eta_{\lambda\,\kappa} = O_{\lambda\nu}\,\eta_{(\nu)}\,O_{\kappa\,\nu} \ ; \ O\,O^{T} = \P \ ; \ O \ : \ \textbf{real orthogonal}$$

The eigenvalues $\eta_{(\nu)}$ in eq. 154 must by no means be positive , leading to an indefinite metric.

It is here , where the additional restriction to compact groups comes in, which as will be shown below, implies , in addition to the semi-simple restriction

(156)
$$\eta_{(\nu)} = \left(e_{(\nu)}\right)^{2} > 0 \ ; \ e_{(\nu)} \ : \ \textbf{real} \ ; \ \nu = 1, \cdots, G$$

→

As a first step we make the orthogonal substitution rendering diagonal the scalar product in eq. 154

(157)
$$\left(\ \beta \ , \ \alpha \ \right)_{\eta} = \left(\ \widetilde{\beta} \ , \ \widetilde{\alpha} \ \right)_{\widetilde{\eta}}$$

$$\eta_{\lambda\kappa} = O_{\lambda\nu}\, \eta_{(\nu)}\, O_{\kappa\nu} \longrightarrow \ \widetilde{\eta}_{\widetilde{\lambda}\,\widetilde{\kappa}} = \delta_{\widetilde{\lambda}\,\widetilde{\kappa}} \left(e_{(\widetilde{\lambda})} \right)^{2}$$

$$\begin{pmatrix} \widetilde{\alpha} \\ \widetilde{\beta} \end{pmatrix}^{\widetilde{\kappa}} = O_{\sigma\,\widetilde{\kappa}} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}^{\sigma}$$

The orthogonal substition $\alpha^{\sigma} \ \rightarrow \ \widetilde{\alpha}^{\widetilde{\kappa}}$ **in eq. 157 implies a tensor-like substitution of the structure constants** $f^{\varrho}{}_{\sigma\tau}$ **, derived next .**

(158)
$$\alpha = \alpha^{\nu}\, ad_{(\nu)} = \widetilde{\alpha}^{\widetilde{\mu}}\, \widetilde{ad}_{(\widetilde{\mu})} = O_{\nu\,\widetilde{\mu}}\, \alpha^{\nu}\, \widetilde{ad}_{(\widetilde{\mu})} \longrightarrow$$

$$ad_{(\nu)} = O_{\nu\,\widetilde{\sigma}}\, \widetilde{ad}_{(\widetilde{\sigma})} \ \rightarrow \ \widetilde{ad}_{(\widetilde{\mu})} = O_{\nu\,\widetilde{\mu}}\, ad_{(\nu)}$$

From eq. 158 it follows – ***not*** **changing component basis**

(159)
$$\left[\widetilde{ad}_{(\widetilde{\sigma})} \, , \, \widetilde{ad}_{(\widetilde{\tau})} \right] = O_{\nu\,\widetilde{\sigma}}\, O_{\mu\,\widetilde{\tau}} \left[ad_{(\nu)} \, , \, ad_{(\mu)} \right] = O_{\nu\,\widetilde{\sigma}}\, O_{\mu\,\widetilde{\tau}}\, f^{\varrho}{}_{\nu\mu}\, ad_{(\varrho)} =$$

$$= O_{\nu\,\widetilde{\sigma}}\, O_{\mu\,\widetilde{\tau}}\, O^{\varrho\,\widetilde{\varphi}}\, f^{\varrho}{}_{\nu\mu}\, \widetilde{ad}_{(\widetilde{\varphi})} \ ; \ O^{\varrho\,\widetilde{\varphi}} \equiv O_{\varrho\,\widetilde{\varphi}}$$

$\longrightarrow$

It becomes obvious how to redefine the structure constants

(160) $$\widetilde{f}^{\,\widetilde{\varphi}}{}_{\widetilde{\sigma}\widetilde{\tau}} = O_{\nu\,\widetilde{\sigma}}\, O_{\mu\,\widetilde{\tau}}\, O^{\varrho\,\widetilde{\varphi}}\, f^{\,\varrho}{}_{\nu\mu} \quad \longrightarrow \quad \widetilde{f}^{\,\widetilde{\varphi}}{}_{\widetilde{\sigma}\widetilde{\tau}} = -\,\widetilde{f}^{\,\widetilde{\varphi}}{}_{\widetilde{\tau}\widetilde{\sigma}}$$

but **we should then also transform the** *components* **of the matrices starting from the original basis of** $\left(ad_{\,(\nu} \right)^{\chi}{}_{\psi}$ **.**
In order to make this explicit we rewrite eq. 159 using the valid transformation of structure constants given in eq. 160

(161) $$\left(\left[\widetilde{ad}_{\,(\widetilde{\sigma})} \,,\, \widetilde{ad}_{\,(\widetilde{\tau})} \right] \right)^{\chi}{}_{\psi} = \widetilde{f}^{\,\widetilde{\varphi}}{}_{\widetilde{\sigma}\widetilde{\tau}} \left(\widetilde{ad}_{\,(\widetilde{\varphi})} \right)^{\chi}{}_{\psi}$$

The choice of appropriate change of *judiciuosly chosen* **coordinates on** $\mathcal{G}$ **relative to** *general* **such, reconstructed from a complete set of base spaces** $\mathcal{B}$ **, each with similar associated** *choices* **of coordinates, gives rise to the distinctive discussion of the next subsection .**

A1-IOA Inner and outer- automorphisms

In order to see the notions of automorphisms arise let me repeat the *generic* **matrix relation in eq. 161 independent of the matrix element basis**

(162) $$\left[\widetilde{ad}_{\,(\widetilde{\sigma})} \,,\, \widetilde{ad}_{\,(\widetilde{\tau})} \right] = \widetilde{f}^{\,\widetilde{\varphi}}{}_{\widetilde{\sigma}\widetilde{\tau}}\, \widetilde{ad}_{\,(\widetilde{\varphi})}$$

The general matrix solution to eq. 162 , given $\widetilde{f}^{\,\widetilde{\varphi}}{}_{\widetilde{\sigma}\widetilde{\tau}}$ **, is** *not unique* **, rather of the form** →

obtained from any *singled out* solution, denoted by $\widetilde{\textbf{ad}}_{\,(\widetilde{\nu})}$ by a similarity *matrix-* transformation, denoted by S

(163) $$\widetilde{ad}_{\,(\widetilde{\lambda})} \;=\; S \, \widetilde{\textbf{ad}}_{\,(\widetilde{\lambda})} \, S^{\,-1} \quad \leftrightarrow \quad \widetilde{\textbf{ad}}_{\,(\widetilde{\lambda})} \;=\; S^{\,-1} \, \widetilde{ad}_{\,(\widetilde{\lambda})} \, S$$

We write eq. 163 in components

$$\left(\widetilde{\textbf{ad}}_{\,(\widetilde{\lambda})} \right)^{\widetilde{\mu}}_{\;\;\widetilde{\nu}} \;=\; \left(S^{\,-1} \right)^{\widetilde{\mu}\chi} \, S_{\,\psi\widetilde{\nu}} \left(\widetilde{ad}_{\,(\widetilde{\lambda})} \right)^{\chi}_{\;\;\psi}$$

(164) $$\left(\left[\, \widetilde{\textbf{ad}}_{\,(\widetilde{\sigma})} \, , \, \widetilde{\textbf{ad}}_{\,(\widetilde{\tau})} \, \right] \right)^{\widetilde{\mu}}_{\;\;\widetilde{\nu}} \;=\; \widetilde{f}^{\,\widetilde{\varrho}}_{\;\;\widetilde{\sigma}\widetilde{\tau}} \left(S^{\,-1} \right)^{\widetilde{\mu}\chi} \, S_{\,\psi\,\widetilde{\nu}} \left(\widetilde{ad}_{\,\widetilde{\varrho}} \right)^{\chi}_{\;\;\psi}$$

$$=\; \widetilde{f}^{\,\widetilde{\varrho}}_{\;\;\widetilde{\sigma}\widetilde{\tau}} \left(\widetilde{\textbf{ad}}_{\,\widetilde{\varrho}} \right)^{\widetilde{\mu}}_{\;\;\widetilde{\nu}}$$

In order to be covariantly in line with the transformation of the structure constants $f \;\rightarrow\; \widetilde{f}$ in eq. 160 we choose

(165) $$S \;=\; O \quad , \quad S^{\,-1} \;=\; O^{\,T}$$

in components : $S_{\,\psi\widetilde{\nu}} \;=\; O_{\,\psi\widetilde{\nu}} \quad , \quad \left(S^{\,-1} \right)^{\widetilde{\mu}\chi} \;=\; O^{\,\chi\widetilde{\mu}} \;\equiv\; O_{\,\chi\widetilde{\mu}}$

To clarify and establish the similarity transformation $S \;=\; O$ in eqs. 163 - 165 as outer automorphism relative to the inner ones, generated by the finite adjoint representation $\{ \; Ad \, (\, a \,) \; | \; a \;\in\; \mathcal{G} \; \}$, we go back to subsection A1-Ad . →

We recapitulate the steps from $ad_{(\nu)} \rightarrow \widetilde{ad}_{(\widetilde{\mu})} \rightarrow \widetilde{\mathbf{ad}}_{(\widetilde{\lambda})}$ according to eqs. 146 , 153 $\rightarrow$ 158 $\rightarrow$ 163 - 165.

Motto : General coordinates are a blessing and an obstacle reciprocally conseqential

I) the starting point ; general coordinates (eq. 146)

$$\underline{I}_{(\nu)} \rightarrow \left(ad_{(\nu)} \right)^{\lambda}{}_{\sigma}$$

(166)
$$\underline{a} = \exp\left(\tau\, \alpha^{\nu}\, \underline{I}_{(\nu)} \right)\Big|_{\mathcal{G}} \rightarrow \; Ad\,(\,a\,(\,\tau\,)\,) = \exp\left(\tau\,\alpha^{\nu}\, ad_{(\nu)}\right) \longrightarrow$$
$$Ad\,(\,a\,(\,\tau\,)\,)\,\beta^{\varrho}\, ad_{(\varrho)}\, Ad^{-1}(\,a\,(\,\tau\,)\,) = (\,Ad\,(\,a\,(\,\tau\,)\,)^{\sigma}{}_{\varrho}\,\beta^{\varrho}\, ad_{(\sigma)} \rightarrow$$
$$Ad\,(\,a\,(\,\tau\,)\,)\, ad_{(\varrho)}\, Ad^{-1}(\,a\,(\,\tau\,)\,) = (\,Ad\,(\,a\,(\,\tau\,)\,)^{\sigma}{}_{\varrho}\, ad_{(\sigma)}$$

Further we retain from eqs. 142 and 144

(167)
$$\left[\,\underline{I}_{(\kappa)}\,,\,\underline{I}_{(\varrho)}\,\right] = f^{\sigma}{}_{\kappa\,\varrho}\,\underline{I}_{(\sigma)} \;\;;\;\; f^{\sigma}{}_{\kappa\,\varrho} = -\,f^{\sigma}{}_{\varrho\,\kappa} \longrightarrow$$
$$\left[\,ad_{(\kappa)}\,,\,ad_{(\varrho)}\,\right] = f^{\sigma}{}_{\kappa\,\varrho}\, ad_{(\sigma)} \;\;;\;\; \left(ad_{(\kappa)}\right)^{\sigma}{}_{\varrho} \equiv f^{\sigma}{}_{\kappa\,\varrho}$$

and the invariant scalar product from eqs. 143 , 147 and 155 - 156

(168)
$$\eta_{\gamma\,\kappa} = -\,tr\, ad_{(\gamma)}\, ad_{(\kappa)} = -\,f^{\varrho}{}_{\gamma\,\sigma}\, f^{\sigma}{}_{\kappa\,\varrho} = \eta_{\kappa\,\gamma}$$
$$\eta = Ad^{T}(\,a\,(\,\tau\,)\,)\,\eta\, Ad\,(\,a\,(\,\tau\,)\,) \;\;;\;\; \forall\, a\,(\,\tau\,)$$
$$\eta = O\,\eta_{diag}\,O^{T} \;\;;\;\; \left(\eta_{diag}\right)_{\nu\,\mu} = \delta_{\nu\,\mu}\,\left(e_{(\nu)}\right)^{2}$$

$\rightarrow$

II) from $\eta \;\longrightarrow\; \eta_{\,diag} \;\equiv\; \widetilde{\eta}$ $\big($ **eqs. 157 and 168** $\big)$

The transformation to orthogonal , *not yet orthonormal* **, axes of** $Lie_{\,\mathcal{G}}$ **, begins with eqs. 157 - 161 in subsection A1-Cpct , leaving the matrix element basis** $(\;)^{\,\sigma}{}_{\,\tau}$ **fixed in the transformation** $\big($ **eq. 158** $\big)$

$$\underline{\alpha} \;=\; \alpha^{\,\nu}\,\underline{I}_{\,(\nu)} \;=\; \widetilde{\alpha}^{\,\nu}\,\widetilde{\underline{I}}_{\,(\widetilde{\nu})}$$

$$\Big(\alpha \;=\; \alpha^{\,\nu}\,ad_{\,(\nu)} \;=\; \widetilde{\alpha}^{\,\widetilde{\mu}}\,\widetilde{ad}_{\,(\widetilde{\mu})} \;=\; O_{\,\nu\,\widetilde{\mu}}\,\alpha^{\,\nu}\,\widetilde{ad}_{\,(\widetilde{\mu})} \Big)^{\,\sigma}{}_{\,\tau} \;\longrightarrow$$

$$\begin{pmatrix} \underline{I}_{\,(\nu)} \\ ad_{\,(\nu)} \end{pmatrix} = O_{\,\nu\,\widetilde{\sigma}} \begin{pmatrix} \widetilde{\underline{I}}_{\,(\widetilde{\sigma})} \\ \widetilde{ad}_{\,(\widetilde{\sigma})} \end{pmatrix} \;\leftrightarrow\; \begin{pmatrix} \widetilde{\underline{I}}_{\,(\widetilde{\mu})} \\ \widetilde{ad}_{\,(\widetilde{\mu})} \end{pmatrix} = O_{\,\nu\,\widetilde{\mu}} \begin{pmatrix} \underline{I}_{\,(\nu)} \\ ad_{\,(\nu)} \end{pmatrix}$$

(169)

The transformations displayed in eq. 169 are incomplete but aim at bringing about a *basis change* **in the induced invariant tangent space metric** $\big($ **eq. 157** $\big)$

$\longrightarrow$

II) continued $\cdots$

$$\left(\beta \;,\; \alpha \right)_{\eta} = \left(\widetilde{\beta} \;,\; \widetilde{\alpha} \right)_{\widetilde{\eta}} = \sum_{(\widetilde{\kappa})} e^{2}_{(\widetilde{\kappa})} \, \widetilde{\beta}^{\widetilde{\kappa}} \, \widetilde{\alpha}^{\widetilde{\kappa}}$$

$$\eta_{\lambda\kappa} = O_{\lambda\nu} \, \eta_{(\nu)} \, O_{\kappa\nu} \longrightarrow \widetilde{\eta}_{\widetilde{\lambda}\widetilde{\kappa}} = \delta_{\widetilde{\lambda}\widetilde{\kappa}} \left(e_{(\widetilde{\lambda})} \right)^{2}$$

(170)

$$\begin{pmatrix} \widetilde{\alpha} \\ \widetilde{\beta} \end{pmatrix}^{\widetilde{\kappa}} = O_{\sigma\widetilde{\kappa}} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}^{\sigma}$$

We define here the inner automorphisms beeing generated under step I) from the original structure constants and original $\{ Ad(a) \mid a \in \mathcal{G} \}$ **generated from the Lie algebra** $\alpha^{\nu} \, ad_{(\nu)} \mid \left(ad_{(\nu)} \right)^{\sigma}{}_{\tau} = f^{\sigma}{}_{\nu\tau}$ **to mean the association**

$$Ad(b) = \exp \beta = \exp \left(\beta^{\lambda} \, ad_{(\lambda)} \right) \;,\; Ad(a) = \exp \alpha = \exp \left(\alpha^{\lambda} \, ad_{(\lambda)} \right)$$

$$a \;:\; Ad(b) \rightarrow Ad(a) \, Ad(b) \, Ad^{-1}(a)$$

$$\eta = Ad^{T}(a) \, \eta \, Ad(a) \;;\; \forall a \in \mathcal{G}$$

with $\eta_{\lambda\kappa} = - f^{\varrho}{}_{\lambda\sigma} \, f^{\sigma}{}_{\kappa\varrho} \;;\; f^{\varrho}{}_{\lambda\sigma} = \left(ad_{(\lambda)} \right)^{\varrho}{}_{\sigma}$

(171)

$\longrightarrow$

II) continued $\cdots$

Remark concerning definitions of inner- and outer automorphisms

The mathematical realm of automorphisms , inner and outer , is very widespread. While in eq. 171 a well defined group automorphisms is qualified as 'inner' , this may not correspond to strict mathematical usage of the term in other contexts. I refer to refs. [A110-2000] for a general discussion.

It follows thus from eq. 171 that any coordinate transformation of the type sought in this step, which changes the tangent space metric $\left(\right.$ eq. 170 $\left.\right)$

(172) $$\eta_{\,\lambda\,\kappa} \;=\; O_{\,\lambda\nu}\,\eta_{\,(\nu)}\,O_{\,\kappa\,\nu} \;\longrightarrow\; \widetilde{\eta}_{\,\widetilde{\lambda}\,\widetilde{\kappa}} \;=\; \delta_{\,\widetilde{\lambda}\,\widetilde{\kappa}}\,\left(\,e_{\,(\widetilde{\lambda})}\,\right)^{\,2}$$

involves an *outer* automorphism of $Lie_{\,\mathcal{G}}$, which does not appear yet in this step (II) .

Here as a consequence of the transformations in eqs. 157 - 159, 169 - 170 the structure constants are transformed as follows $\left(\right.$ eq. 160 $\left.\right)$, repeated below

(173) $$\widetilde{f}^{\;\widetilde{\varphi}}_{\;\;\widetilde{\sigma}\widetilde{\tau}} \;=\; O_{\,\nu\,\widetilde{\sigma}}\,O_{\,\mu\,\widetilde{\tau}}\,O^{\,\varrho\,\widetilde{\varphi}}\,f^{\,\varrho}_{\;\;\nu\mu} \;\;\longrightarrow\;\; \widetilde{f}^{\;\widetilde{\varphi}}_{\;\;\widetilde{\sigma}\widetilde{\tau}} \;=\; -\,\widetilde{f}^{\;\widetilde{\varphi}}_{\;\;\widetilde{\tau}\widetilde{\sigma}}$$

This is as far as step II carries. But without changing accordingly the matrix elements $^{\sigma}{}_{\tau}$, frozen up to this point, of the matrices as given in eq. 169

(174) $$\left(\,\widetilde{ad}_{\,(\widetilde{\mu})}\,\right)^{\sigma}_{\;\;\;\tau} \;=\; O_{\,\nu\,\widetilde{\mu}}\,\left(\,ad_{\,(\nu)}\,\right)^{\sigma}_{\;\;\tau}$$

$\longrightarrow$

II) end

the automorphism completing relation $\widetilde{ad} \leftrightarrow \widetilde{f}$ does not hold.

(175)
$$\left(\widetilde{ad}_{\,(\widetilde{\mu})} \right)^{\sigma}{}_{\tau} \neq \widetilde{f}^{\,\widetilde{\sigma}}{}_{\widetilde{\mu}\widetilde{\tau}}$$

III) the outer automorphism from $\widetilde{ad}_{\,(\widetilde{\mu})}$ to $\widetilde{\mathbf{ad}}_{\,(\widetilde{\mu})} = S^{-1}\,\widetilde{ad}_{\,(\widetilde{\mu})}\,S\,;\; S = O$

In this step we complete the full , outer- automorphism as given in eq. 163 , generating the associated diagonalized invariant scalar product $($ eqs. 157 and 168 $)$.

Comparing the full 3-tensor transformation of the structure constants $f^{\,\sigma}{}_{\mu\tau}$ in eq. 173 with the *partial* **substitution for the base generators of $Lie_{\,\mathcal{G}}$: $\left(ad_{\,(\mu)} \right)^{\sigma}{}_{\tau}$ in eq. 174 the origin of the external automorphism $($ eqs. 164 - 165 $)$ necessary to complete the covariantly consistent change of variables follows**

(176)
$$\left(\widetilde{\mathbf{ad}}_{\,(\widetilde{\mu})} \right)^{\widetilde{\sigma}}{}_{\widetilde{\tau}} = \left(S^{-1} \right)^{\widetilde{\sigma}}{}_{\sigma'} \left(\widetilde{ad}_{\,(\widetilde{\mu})} \right)^{\sigma'}{}_{\tau'} S^{\tau'}{}_{\widetilde{\tau}}$$
$$S^{\tau'}{}_{\widetilde{\tau}} = O_{\tau'\widetilde{\tau}} \;,\; \left(S^{-1} \right)^{\widetilde{\sigma}}{}_{\sigma'} = O_{\sigma'\widetilde{\sigma}} \longrightarrow$$
$$\left(\widetilde{\mathbf{ad}}_{\,(\widetilde{\mu})} \right)^{\widetilde{\sigma}}{}_{\widetilde{\tau}} = \widetilde{f}^{\,\widetilde{\sigma}}{}_{\widetilde{\mu}\widetilde{\tau}}$$

This ends the résumé of the orthogonalizing and not yet normalizing coordinate transformations on $\mathcal{G}$ and $Lie_{\,\mathcal{G}}$.

Establishing coordinates rendering the metric ortho*normal*

We take up the orthogonalizing coordinate transformations on $\mathcal{G}$ reassigning symbols to the orthogonal form of the metric , also calles Killing-form [A111-1962] , achieved in the previous subsections .

We shall establish the following notation using as suffixes

(177)
$$\begin{aligned} \perp &: \quad \textbf{for orthogonal , not normalized} \\ \vdash &: \quad \textbf{for orthonormal} \end{aligned}$$

In the following the general semisimple compact group $\mathcal{G} = \mathcal{G}_1 \otimes \mathcal{G}_2 \otimes \cdots \otimes \mathcal{G}_l$ shall be restricted to one simple such.

Thus we perform the *orthogonal* substitutions, starting from eq. 176

(178)
$$\begin{aligned} & \left(\widetilde{\textbf{ad}}_{\,(\widetilde{\mu})} \right)^{\widetilde{\sigma}}{}_{\widetilde{\tau}} \quad : \quad \widetilde{\textbf{ad}} \rightarrow \boldsymbol{ad}_{\perp} \quad , \quad \left(\,_{\widetilde{\mu}} \right)^{\widetilde{\sigma}}{}_{\widetilde{\tau}} \rightarrow \left(\,_{\mu} \right)^{\sigma}{}_{\tau} \\ & \widetilde{f}^{\,\widetilde{\sigma}}{}_{\widetilde{\mu}\widetilde{\tau}} \rightarrow \left(\boldsymbol{f}_{\perp} \right)^{\sigma}{}_{\mu\tau} \end{aligned}$$

Going step by step beginning with eq. 176 , the latter takes the form

(179)
$$\left(\boldsymbol{ad}_{\perp\,(\mu)} \right)^{\sigma}{}_{\tau} = \left(\boldsymbol{f}_{\perp} \right)^{\sigma}{}_{\mu\tau}$$

The next equation undergoing the above substitutions →

is eq. 172 , which becomes

(180) $$\widetilde{\eta}_{\widetilde{\lambda}\,\widetilde{\kappa}} \rightarrow \boldsymbol{\eta}_{\perp\,\lambda\,\kappa} = \delta_{\lambda\,\kappa}\left(e_{(\lambda)}\right)^2$$

The next quantity requiring an appropriate substitution is $\widetilde{\underline{I}}_{(\widetilde{\sigma})}$ in eq. 169

(181) $$\widetilde{\underline{I}}_{(\widetilde{\sigma})} \rightarrow \underline{\boldsymbol{I}}_{\perp\,(\sigma)} \;\; ; \;\; \left[\underline{\boldsymbol{I}}_{\perp\,(\sigma)}\,,\,\underline{\boldsymbol{I}}_{\perp\,(\tau)}\right] = \left(\boldsymbol{f}_{\perp}\right)^{\varrho}{}_{\sigma\tau}\,\underline{\boldsymbol{I}}_{\perp\,(\varrho)}$$

This affects the operator and adjoint representation versions in eq. 169

(182) $$\begin{aligned} \underline{\alpha} &= \alpha^{\nu}\,\underline{I}_{(\nu)} \;:\; \underline{\boldsymbol{\alpha}}_{\perp} = \boldsymbol{\alpha}_{\perp}{}^{\nu}\,\underline{\boldsymbol{I}}_{\perp\,(\nu)} \\ \alpha &= \alpha^{\nu}\,ad_{(\nu)} \;:\; \boldsymbol{\alpha}_{\perp} = \boldsymbol{\alpha}_{\perp}{}^{\nu}\,\boldsymbol{ad}_{\perp\,(\nu)} \\ & \left[\boldsymbol{ad}_{\perp\,(\sigma)}\,,\,\boldsymbol{ad}_{\perp\,(\tau)}\right] = \left(\boldsymbol{f}_{\perp}\right)^{\varrho}{}_{\sigma\tau}\,\boldsymbol{ad}_{\perp\,(\varrho)} \end{aligned}$$

A subtlety shall be emphasized here and arises from eqs. 153 and 169 : in order to achieve compatibility with the full transformation $\left(ad_{(\mu')}\right)^{\sigma'}{}_{\tau'} \rightarrow \left(\boldsymbol{ad}_{\perp\,(\mu)}\right)^{\sigma}{}_{\tau}$ a coordinate transformation as implementing the outer automorphism must be performed on $\mathcal{G}$, which is *not* a group similarity transformation in order to bring about the full substitution $\widetilde{I}_{(\widetilde{\nu})} \;:\; \underline{\boldsymbol{I}}_{\perp\,(\nu)}$ in eq. 181 .

Next we turn to eq. 166 →

where the finite adjoint representation matrices $Ad\,(\,a\,(\,\tau\,)\,)$ undergo the outer automorphism associated substitution

(183)
$$\underline{a} \;:\; \underline{\boldsymbol{a}}_{\perp} \;=\; \exp\left(\tau\,\boldsymbol{\alpha}_{\perp}^{\nu}\,\underline{\boldsymbol{I}}_{\perp\,(\nu)}\right)\Big|_{\mathcal{G}} \equiv \exp\left(\tau\,\underline{\boldsymbol{\alpha}}_{\perp}\right)$$
$$a \;:\; \boldsymbol{a}_{\perp} \;=\; \exp\left(\tau\,\boldsymbol{\alpha}_{\perp}^{\nu}\,\boldsymbol{ad}_{\perp\,(\nu)}\right)\Big|_{\mathcal{G}} \equiv \exp\left(\tau\,\boldsymbol{\alpha}_{\perp}\right)$$
$$Ad \;:\; \boldsymbol{Ad}_{\perp} \;\rightarrow\; \boldsymbol{Ad}_{\perp}\left(\boldsymbol{a}_{\perp}\right) = \exp\left(\tau\,\boldsymbol{\alpha}_{\perp}\right)$$

whereupon eq. 166 is transformed into

$$\boldsymbol{Ad}_{\perp}\left(\boldsymbol{a}_{\perp}(\tau)\right)\underline{\boldsymbol{\beta}}_{\perp}\,\boldsymbol{Ad}_{\perp}^{-1}\left(\boldsymbol{a}_{\perp}(\tau)\right) = \left(\boldsymbol{Ad}_{\perp}\left(\boldsymbol{a}_{\perp}(\tau)\right)\right)^{\sigma}{}_{\varrho}\,\boldsymbol{\beta}_{\perp}^{\varrho}\,\boldsymbol{ad}_{\perp\,(\sigma)}$$
$$\rightarrow \boldsymbol{Ad}_{\perp}\left(\boldsymbol{a}_{\perp}(\tau)\right)\boldsymbol{ad}_{\perp\,(\varrho)}\,\boldsymbol{Ad}_{\perp}^{-1}\left(\boldsymbol{a}_{\perp}(\tau)\right) = \left(\boldsymbol{Ad}_{\perp}\left(\boldsymbol{a}_{\perp}(\tau)\right)\right)^{\sigma}{}_{\varrho}\,\boldsymbol{ad}_{\perp\,(\sigma)}$$
(184)

Finally we assign the variable substitution to eq. 157

(185)
$$\left(\;\beta\;,\;\alpha\;\right)_{\eta} : \left(\;\boldsymbol{\beta}_{\perp}\;,\;\boldsymbol{\alpha}_{\perp}\;\right)_{\boldsymbol{\eta}_{\perp}} = \boldsymbol{\beta}_{\perp}{}^{\nu}\,\boldsymbol{\eta}_{\perp\,\nu\,\mu}\,\boldsymbol{\alpha}_{\perp}{}^{\mu}$$
$$\eta \;:\; \boldsymbol{\eta}_{\perp} \;;\; \boldsymbol{\eta}_{\perp\,\nu\,\mu} = \delta_{\nu\,\mu}\left(e_{(\nu)}\right)^{2}$$

with the identifications →

transforming the relations in eq. 168

$$\boldsymbol{\eta}_{\perp\ \nu\ \mu} = -\,tr\ \boldsymbol{ad}_{\perp\ (\nu)}\ \boldsymbol{ad}_{\perp\ (\mu)} = -\ (\ \boldsymbol{f}_{\perp}\ )^{\ \varrho}{}_{\nu\tau}\ (\ \boldsymbol{f}_{\perp}\ )^{\ \tau}{}_{\mu\varrho} \longrightarrow$$

(186) $\boldsymbol{D}\ (\ \boldsymbol{a}\ ) \equiv \boldsymbol{Ad}_{\perp}\ (\ \boldsymbol{a}_{\perp}\ (\ \tau\ )\ )$ as a shorthand

$$\boldsymbol{\eta}_{\perp} = \boldsymbol{D}^{\ T}\ (\ \boldsymbol{a}\ )\ \boldsymbol{\eta}_{\perp}\ \boldsymbol{D}\ (\ \boldsymbol{a}\ )\ \ ;\ \ \forall\ \boldsymbol{a}$$

At this stage we have to go back to restructure eq. 155 having split off the active transformations of the (tangent space-) metric $\eta \longrightarrow \boldsymbol{\eta}_{\perp}$ into an orthoganal outer automorphism and a remainder to be considered below .

The orthogonal substitution as already given in eq. 155 using our present substituded variables becomes

(187) $$\eta = O\ \boldsymbol{\eta}_{\perp}\ O^{\ T}\ \ ;\ \ O\ O^{\ T} = ¶$$

leading to the next *eventually final* step discussed in the next , new subsection .

A1-O $\vdash$ Achieving the ortho*normal* metric

The overall multiple of the unit matrix defining the orthonormal one , denoted $\boldsymbol{\eta}_{\vdash}$ in the following

(188) $$\boldsymbol{\eta}_{\vdash} = C\ ¶\ \ ;\ \ C = k^{\ 2}\ \ ,\ \ k > 0$$

is a matter of convention and shall be denoted by $C > 0$ as in eq. 188 in this subsection . $\longrightarrow$

The last relation in eq. 187 thus transforms into

(189)
$$C\,\boldsymbol{\eta}_{\perp} = S\,\boldsymbol{\eta}_{\vdash}\,S^{T} \; ; \; S = S^{T} = S_{diag}\left(e_{(1)}, \cdots, e_{(G)}\right)$$
$$S_{\mu\,m} = \delta_{\mu\,m}\,e_{(\mu)} \;\rightarrow\; \left(S^{\prime} = S^{-1}\right)_{m\,\mu} = \delta_{m\,\mu}\left(e_{(\mu)}\right)^{-1}$$

In the following we use for all *tensor-* **quantities in the orthonormal basis latin indices , distinguishing them from the orthogonal one , where they are in greek .**

Hence displaying all indices eq. 189 becomes

(190)
$$C\,\boldsymbol{\eta}_{\perp\;\nu\,\mu} = S_{\nu\,n}\,\boldsymbol{\eta}_{\vdash\;n\,m}\,S_{\mu\,m} \;\rightarrow$$
$$\boldsymbol{\eta}_{\vdash\;n\,m} = C\,S^{\prime}_{n\,\nu}\,\boldsymbol{\eta}_{\perp\;\nu\,\mu}\,S^{\prime}_{m\,\mu}$$

The last relation in eq. 186 becomes

(191)
$$S^{2} = \boldsymbol{D}^{T}(\boldsymbol{a})\,S^{2}\,\boldsymbol{D}(\boldsymbol{a}) \;\longrightarrow$$
$$\P \;=\; S^{\prime}\,\boldsymbol{D}^{T}(\boldsymbol{a})\,S\,S\,\boldsymbol{D}(\boldsymbol{a})\,S^{\prime}$$
$$= \left(S\,\boldsymbol{D}(\boldsymbol{a})\,S^{\prime}\right)^{T}\left(S\,\boldsymbol{D}(\boldsymbol{a})\,S^{\prime}\right) = \left(\boldsymbol{D}_{\vdash}(\boldsymbol{a})\right)^{T}\boldsymbol{D}_{\vdash}(\boldsymbol{a})$$
$$\boldsymbol{D}_{\vdash}(\boldsymbol{a}) = S\,\boldsymbol{D}(\boldsymbol{a})\,S^{\prime} = S\,\boldsymbol{D}(\boldsymbol{a})\,S^{-1}$$

$\longrightarrow$

The third relation in eq. 191 rewritten below

(192)
$$\P = (\boldsymbol{D}_{\vdash} (\boldsymbol{a}))^{T} \boldsymbol{D}_{\vdash} (\boldsymbol{a})$$
$$\boldsymbol{D}_{\vdash} (\boldsymbol{a}) = S \, \boldsymbol{D} (\boldsymbol{a}) \, S^{-1}$$

shows that modulo the similarity transformation , as represented by the last relation in eq. 192 , the so defined set formed by the union of $G \times G$ matrices $\{ \bigcup_{\boldsymbol{a}} \boldsymbol{D}_{\vdash} (\boldsymbol{a}) \}$ is equivalent to the adjoint $Ad (\mathcal{G})$ representation of $\mathcal{G}$, which in turn is a *sub-representation* **of the** *vector* **representation of SO_{G} , which I denote $V (SO_{G})$ in the following . The latter is formed by the union of all orthogonal $G \times G$ matrices with determinant 1 .**

(193)
$$Ad (\mathcal{G}) = \{ \bigcup_{\boldsymbol{a}} \boldsymbol{D}_{\vdash} (\boldsymbol{a}) \} \; ; \; V (SO_{G}) = \{ \bigcup O \mid O \, O^{T} = \P , \, det \, O = 1 \}$$
$$\longrightarrow \qquad Ad (\mathcal{G}) \subseteq V (SO_{G})$$

Since $V (SO_{G})$ forms a compact space , so does $Ad (\mathcal{G})$. This proves the consistency of the nonnegative nature of the metric η , defined in eq. 143 , 147 and 168 , with the *required* **restriction that $\mathcal{G}$ be semisimple** *and compact* **.**

It is worth noting here that , always restricting to *simple* **compact groups , the equal or equivalent sign in last relation in eq. 193 holds** $\longrightarrow$

precisely for the smallest simple, compact group

(194) $$Ad \, (\mathcal{G}) \simeq V \, (SO_{G}) \longrightarrow G = 3 \; ; \; \mathcal{G} = SU_{2}$$

We proceed to adapt coordinates and $Ad \, , \, ad$ representations to the similarity transformation as defined in eq. 192 . This requires further adapted notations to which task we turn below.
To this end we refer to the form of the diagonal matrix $S \left(e_{(1)} , \cdots , e_{(G)} \right)$ in eq. 189 and expand eq. 192

(195) $$\begin{array}{l} \P = (\boldsymbol{D}_{\vdash} (\boldsymbol{a}))^{T} \boldsymbol{D}_{\vdash} (\boldsymbol{a}) \\ \boldsymbol{D}_{\vdash} (\boldsymbol{a}) = S \, \boldsymbol{D} (\boldsymbol{a}) \, S^{-1} \; ; \; \boldsymbol{D} (\boldsymbol{a}) \equiv \boldsymbol{Ad}_{\perp} (\boldsymbol{a}_{\perp} (\tau)) \end{array}$$

Next we descend from the general Ad to the logaritmic ad level (eq. 183)

(196) $$\begin{array}{l} \boldsymbol{D}_{\vdash} (\boldsymbol{a}) = \exp (\tau \, \widetilde{\boldsymbol{\alpha}}_{\perp}) \; : \; \boldsymbol{Ad}_{\perp} (\boldsymbol{a}_{\perp}) = \exp (\tau \, \boldsymbol{\alpha}_{\perp}) \\ \widetilde{\boldsymbol{\alpha}}_{\perp} = S \, \boldsymbol{\alpha}_{\perp} \, S^{-1} \longrightarrow \widetilde{\boldsymbol{ad}}_{\perp \, (\nu)} = S \, \boldsymbol{ad}_{\perp \, (\nu)} \, S^{-1} \end{array}$$

We proceed to display *all* components pertaining to eq. 196 introducing the shorthand notation

(197) $$\widetilde{\boldsymbol{ad}}_{\perp \, \widetilde{\sigma} \, (\nu) \, \widetilde{\tau}} := \Phi_{\widetilde{\sigma} \, | \, \nu \, \widetilde{\tau}}$$

$\longrightarrow$

The quantity $\Phi_{\cdot}$ in eq. 197 has the following structure , using eq. 185

$$\Phi_{\widetilde{\sigma}\,|\,\nu\,\widetilde{\tau}} = -\,\Phi_{\widetilde{\tau}\,|\,\nu\,\widetilde{\sigma}}$$

(198) $$\Phi_{\widetilde{\sigma}\,|\,\nu\,\widetilde{\tau}} = S_{\widetilde{\sigma}\,\chi}\,S^{-1}_{\psi\,\widetilde{\tau}}\,\left(\boldsymbol{ad}_{\perp\,(\nu)}\right)^{\chi}{}_{\psi}$$

$$-\,tr\,\widetilde{\boldsymbol{ad}}_{\perp\,(\nu)}\,\widetilde{\boldsymbol{ad}}_{\perp\,(\mu)} = -\,tr\,\boldsymbol{ad}_{\perp\,(\nu)}\,\boldsymbol{ad}_{\perp\,(\mu)} = \boldsymbol{\eta}_{\perp\,\nu\,\mu} = \delta_{\nu\,\mu}\,\left(e_{(\nu)}\right)^{2}$$

The traces in eq. 198 are independent of the similarity transformation defined in eq. 196 . Hence it follows

(199) $$-\,tr\,\widetilde{\boldsymbol{ad}}_{\perp\,(\nu)}\,\widetilde{\boldsymbol{ad}}_{\perp\,(\mu)} = \sum\nolimits_{\widetilde{\sigma}\,\widetilde{\tau}}\left(\Phi_{\widetilde{\sigma}\,|\,\nu\,\widetilde{\tau}}\,\Phi_{\widetilde{\sigma}\,|\,\mu\,\widetilde{\tau}}\right) = \left(S^{2}\right)_{\nu\,\mu}$$

As in the reduction to an orthogonal basis beginning with subsection A1-Cpct has shown, the remaining component , here $_{(\nu)}$ in the quantity $\widetilde{\boldsymbol{ad}}_{\perp\,(\nu)}$ has to be transformed as well . Thus a new quantity arises of the form

(200) $$\widetilde{\boldsymbol{ad}}_{\widetilde{\perp}\,(\widetilde{\varrho})} = S^{-1}_{\nu\,\widetilde{\varrho}}\,\widetilde{\boldsymbol{ad}}_{\perp\,(\nu)} \longrightarrow$$

$$X_{\widetilde{\sigma}\,|\,\widetilde{\varrho}\,\widetilde{\tau}} := S^{-1}_{\nu\,\widetilde{\varrho}}\,\Phi_{\widetilde{\sigma}\,|\,\nu\,\widetilde{\tau}} = S_{\widetilde{\sigma}\,\chi}\,S^{-1}_{\nu\,\widetilde{\varrho}}\,S^{-1}_{\psi\,\widetilde{\tau}}\,\left(\boldsymbol{ad}_{\perp\,(\nu)}\right)^{\chi}{}_{\psi}$$

Combining the antisymmetry relations in eqs. 198 and 179 the *two* such relations follow

(201) $$X_{\widetilde{\sigma}\,|\,\widetilde{\varrho}\,\widetilde{\tau}} = -\,X_{\widetilde{\tau}\,|\,\widetilde{\varrho}\,\widetilde{\sigma}} = -\,X_{\widetilde{\sigma}\,|\,\widetilde{\tau}\,\widetilde{\varrho}}$$

→

and by eq. 201 that the quantity

(202) $X_{\widetilde{\sigma} \,|\, \widetilde{\varrho} \widetilde{\tau}} = K \, (\boldsymbol{f}_{\vdash})_{\widetilde{\sigma} \widetilde{\varrho} \widetilde{\tau}}$; with $K > 0$: a suitable normalization constant

is totally antisymmetric with respect to the *three* indices $\widetilde{\sigma} \widetilde{\varrho} \widetilde{\tau}$.

Next we determine the so induced metric using eq. 200

$$-\, tr \, \widetilde{\boldsymbol{ad}}_{\widetilde{\perp} \, (\widetilde{\varrho})} \, \widetilde{\boldsymbol{ad}}_{\widetilde{\perp} \, (\widetilde{\varrho}\,')} = -\, S^{-1}_{\nu \, \widetilde{\varrho}} \, S^{-1}_{\mu \, \widetilde{\varrho}\,'} \, tr \, \widetilde{\boldsymbol{ad}}_{\perp \, (\nu)} \, \widetilde{\boldsymbol{ad}}_{\perp \, (\mu)}$$

$$= S^{-1}_{\nu \, \widetilde{\varrho}} \, S^{-1}_{\mu \, \widetilde{\varrho}\,'} \, (S^{2})_{\nu \, \mu} \longrightarrow$$

(203)

$$= \delta_{\widetilde{\varrho} \, \widetilde{\varrho}\,'}$$

$$\left(\widetilde{\boldsymbol{ad}}_{\widetilde{\perp} \, (\widetilde{\varrho})} \right)_{\widetilde{\sigma} \, \widetilde{\tau}} = X_{\widetilde{\sigma} \,|\, \widetilde{\varrho} \widetilde{\tau}} := K \, (\boldsymbol{ad}_{\vdash \, (\widetilde{\varrho})})_{\widetilde{\sigma} \, \widetilde{\tau}} \rightarrow$$

$$(\boldsymbol{ad}_{\vdash \, (\widetilde{\varrho})})_{\widetilde{\sigma} \, \widetilde{\tau}} = (\boldsymbol{f}_{\vdash})_{\widetilde{\sigma} \widetilde{\varrho} \widetilde{\tau}} = K^{-1} \, X_{\widetilde{\sigma} \,|\, \widetilde{\varrho} \widetilde{\tau}}$$

The *orthonormal* metric thus becomes

(204)

$$\boldsymbol{\eta}_{\vdash \, \widetilde{\varrho} \, \widetilde{\varrho}\,'} = -\, tr \, \boldsymbol{ad}_{\vdash \, (\widetilde{\varrho})} \, \boldsymbol{ad}_{\vdash \, (\widetilde{\varrho}\,')} = \sum_{\widetilde{\sigma} \, \widetilde{\tau}} \boldsymbol{f}_{\vdash \, \widetilde{\sigma} \, \widetilde{\varrho} \, \widetilde{\tau}} \, \boldsymbol{f}_{\vdash \, \widetilde{\sigma} \, \widetilde{\varrho}\,' \, \widetilde{\tau}}$$

$$= C \, \delta_{\widetilde{\varrho} \, \widetilde{\varrho}\,'} \; ; \; C = k^{2} > 0$$

$\rightarrow$

Combining eqs. 203 and 204 determines the normalization relations

(205) $$K \;=\; k^{\,-1} \;\;\longrightarrow\;\; C \;=\; k^{\,2} \;=\; K^{\,-2}$$

This ends the derivation part of Appendix 1 . It remains to streamline notation to be used in further studies and to add concuding comments in the last subsection below .

A1-A Adapting notation for further use and addenda

Most applications depending on the structure constants of *simple* , *compact* Lie algebras imply as a starting point the orthonormal basis and a definite normalization $(\,C\,)$ as in subsection A1-O .
For this reason we wish to simplify the notation reached in subsection A1-O . It is instructive to compare my derivations here with the associated ones in ref. [A12-1951-1961] the notes of lectures by Giulio Racah . The material covered here in Appendix 1 forms in ref. [A12-1951-1961] 'Lecture 1. General notions on continuous groups' divided into 3 subsections and 15 pages . These contain 27 numbered and 53 in total formulae . In comparison Appendix 1 contains 19 subsections, 42 pages and 115 formulae , up to this point .
Thus we assign the following *new* symbols , first to those in eqs. 203 and 204

$$\left(\boldsymbol{ad}_{\vdash\;(\widetilde{\varrho})}\right)_{\widetilde{\sigma}\,\widetilde{\tau}} = \left(\boldsymbol{f}_{\vdash}\right)_{\widetilde{\sigma}\,\widetilde{\varrho}\,\widetilde{\tau}} \longrightarrow f_{[\,s\;r\;t\,]} = \left(ad_{\,(r)}\right)_{s\;t}$$

(206) $$\boldsymbol{\eta}_{\vdash\;\widetilde{\varrho}\,\widetilde{\varrho}\,'} \longrightarrow \eta_{\,r\;r\,'} = -tr\;ad_{\,(r)}\;ad_{\,(r\,')} = f_{[\,s\;t\;r\,]}\,f_{\left[\,s\;t\;r\,'\,\right]}\;\longrightarrow$$

Next we turn to eqs. 195 and 196 . We define and assign

(207)
$$\boldsymbol{\alpha}_{\vdash} \;=\; (\,\alpha_{\vdash}\,)^{\,\widetilde{\varrho}}\;\boldsymbol{ad}_{\vdash\,(\widetilde{\varrho})} \quad\longrightarrow\quad \alpha \;=\; \alpha^{\,r}\,ad_{\,(r)}$$
$$\boldsymbol{D}_{\vdash}\,(\,\boldsymbol{a}_{\vdash}\,) \;=\; \boldsymbol{Ad}_{\vdash}\,(\,\boldsymbol{a}_{\vdash}\,) \quad\longrightarrow\quad Ad\,(\,a\,) \;=\; \exp\,(\,\tau\,\alpha\,)$$

Next we cast eq. 193 into the form , using the substitutions in eq. 207

(208)
$$Ad\;(\,\mathcal{G}\,) \;=\; \{\,\bigcup_{\,a}\;Ad\,(\,a\,)\,\}\;;\;V\;(\,SO_{\,G}\,) \;=\; \{\,\bigcup\,O\,|\,O\,O^{\,T} \;=\; \P\,,\,det\,O \;=\; 1\,\}$$
$$\longrightarrow \qquad Ad\;(\,\mathcal{G}\,)\;\subseteq\;V\;(\,SO_{\,G}\,)$$

In analagy with $Ad\,(\,\mathcal{G}\,)$ **we define the (adjoint) Lie algebra representation**

(209)
$$ad\,(\,\mathcal{G}\,) \;=\; \{\,\bigcup_{\,\alpha}\;\alpha \;=\; \alpha^{\,r}\,ad_{\,(r)}\,\}$$
$$\left[\,ad_{\,(r)}\,,\,ad_{\,(s)}\,\right] \;=\; f_{\,[\,r\,s\,t\,]}\,ad_{\,(t)}$$

Now we turn to the normalization constant C , defined in eq. 188 . To this end we define first the (second) Casimir imvariant for the adjoint representation and compare with eq. 206

(210)
$$-\,\left(\,\sum_{\,r}\,\left(\,ad_{\,(r)}\,\right)^{\,2}\,\right)_{\,s\,t} \;=\; \sum_{\,r\,u}\,f_{\,[\,s\,r\,u\,]}\,f_{\,[\,t\,r\,u\,]} \;=\; C_{\,2}\,(\,\mathcal{G}\,)\,\delta_{\,s\,t}$$
$$\longrightarrow\quad C \;=\; C_{\,2}\,(\,\mathcal{G}\,)$$

$\longrightarrow$

Eq. 210 – for a simple compact group $\mathcal{G}$ – represents a definition identity ***in the orthonormal basis*** **, by no means an absolute magnitude determination.**

The latter follows just for the smallest $\mathcal{G} \, = \, SU2$ from the conventions

(211)
$$\mathcal{G} \, = \, SU2 \quad : \qquad r, s, t \, = \, 1, 2, 3 \, \longrightarrow \, \left| \, f_{\,[\,r\,s\,t\,]} \, \right| \, = \, 1$$
$$\longrightarrow \quad C \, = \, C_{\,2} \, (\, SU2 \,) \, = \, 2$$

This is in line with the normalization convention for the angular momentum operators with respect to the origin in an $R^{\,3}$ configuration space

(212)
$$J_{\,r} \, = \, \varepsilon_{\,rst} \, x^{\,s} \, \frac{1}{i} \, \partial_{\,x\,;\,t} \quad ; \quad \varepsilon_{\,rst} \, \equiv \, f_{\,[\,r\,s\,t\,]} \quad ; \quad \vec{J} \, = \, (\, J_{\,1} \, , \, J_{\,2} \, , \, J_{\,3} \,)$$
$$\left(\, \vec{J} \, \right)^{\,2} \Big|_{\,ad} \, = \, J \, (\, J \, + \, 1 \,) \, |_{\,J=1} \, = \, 2$$

It is instructive to illustrate here for $\mathcal{G} \, = \, SU2$ and the angular momentum operators defined in eq. 212 , that no absolute normalization for the constant C (eq. 188 derives from the expansion in orthonormal coordinates of $\mathcal{G}$ around the origin , chosen as the unit element .

To this end we express $I_{\,3} \, = \, i \, J_{\,3}$ as derivative operator in cylindrical coordinates

(213)
$$x^{\,1} \, = \, x \, = \, x_{\,\perp} \, \cos \, \varphi \quad , \quad x^{\,2} \, = \, y \, = \, x_{\,\perp} \, \sin \, \varphi \quad , \quad x^{\,3} \, = \, z$$
$$x_{\,\perp} \, = \, \sqrt{\, x^{\,2} \, + \, y^{\,2}} \quad , \quad \varphi \, = \, \arctan \, (\, y \, / \, x \,)$$

$\longrightarrow$

It follows from eq. 213

(214) $$I_3 = \partial / \partial_\varphi$$

We consider a rescaling transformation of the *orthonormal* **structure constants**

(215) $$\lambda \text{ real} \;:\; \begin{cases} f_{[rst]} & \rightarrow \quad f^{\lambda}_{[rst]} = \lambda f_{[rst]} \\ C & \rightarrow \quad C^{\lambda} = \lambda^2 C \\ I_3 & \rightarrow \quad I^{\lambda}_3 = \lambda I_3 \end{cases}$$

Hence the rescaling as defined in eq. 215 is equivalent to a rescaling of the polar angle φ

(216) $$I^{\lambda}_3 = \partial / \partial_{\varphi^{\lambda}} \;;\; \varphi^{\lambda} = \varphi / \lambda$$

Obviously the rescaling $\varphi \rightarrow \varphi / \lambda$ **changes the period of cylindrical functions , but an expansion around a given angle ,** $\varphi_0 = 0$ **say , without all order summation cannot reveal this topological property.**
We shall abide by the conventional normalization of $\mathcal{G} = SU2$ **– i.e.** $\lambda \rightarrow 1$ **– which can be extended consistently to all simple , compact groups , except if arbitrary rescaling factors are explicitly introduced .** $\longrightarrow$

A1-D Remarks on irreducible general representations , roots and weights

Here it is not intended to give an exhaustive treatment of the topics in the subsection title above [A112-2002] , [A113-1986] . Only the alignment of unitary irreducible linear representations with the conventionally normalized orthonormal basis of the adjoint representation is discussed.
To this end we start with the adjoint Lie algebra , in orthonormal convention and reproduce eq. 209 below

(217)
$$ad\,(\,\mathcal{G}\,) \;=\; \{\,\bigcup_{\alpha}\;\alpha \;=\; \alpha^{\,r}\;ad_{\,(r)}\,\}$$
$$\left[\,ad_{\,(r)}\,,\,ad_{\,(s)}\,\right] \;=\; f_{\,[\,r\;s\;t\,]}\;ad_{\,(t)}$$

For an arbitrary , finite dimensional *reducible or irreducible* **and** *unitary* **representation of** $\mathcal{G} \;\rightarrow\; \mathcal{D}\,(\,\mathcal{G}\,)$ **and its 'infinitesimal' antiunitary Lie algebra representing matrices** $d\,(\,\mathcal{G}\,)$ **,** *aligned* **with the orthonormal basis quantities** $Ad\,(\,\mathcal{G}\,)$ **and** $ad\,(\,\mathcal{G}\,)$ **in eqs. 208 and 217 respectively , we use the notation**

(218)
$$d\,(\,\mathcal{G}\,) \;=\; \{\,\bigcup_{\alpha}\;d\,(\,\alpha\,) \;=\; \alpha^{\,r}\;d_{\,(r)}\,\}$$
$$\left[\,d_{\,(r)}\,,\,d_{\,(s)}\,\right] \;=\; f_{\,[\,r\;s\;t\,]}\;d_{\,(t)} \quad ; \quad \left(\,d_{\,(r)}\,\right)^{\,\dagger} \;=\; -\,d_{\,(r)}$$
$$\mathcal{D}\,(\,\mathcal{G}\,) \;=\; \{\,\bigcup_{\alpha}\;\mathcal{D}\,(\,a\,) \;=\; \exp\,(\,\tau\,d\,(\,\alpha\,)\,)\,\} \quad ; \quad \mathcal{D}\,(\,a\,)\,\mathcal{D}^{\,\dagger}\,(\,a\,) \;=\; ¶$$

also : $d_{\,(r)} \;=\; \frac{1}{i}\,\delta_{\,(r)} \quad ; \quad \left(\,\delta_{\,(r)}\,\right)^{\,\dagger} \;=\; \delta_{\,(r)} \;\longrightarrow\; \left[\,\delta_{\,(r)}\,,\,\delta_{\,(s)}\,\right] \;=\; i\,f_{\,[\,r\;s\;t\,]}\;\delta_{\,(t)}$

$\rightarrow$

In parallel with the transformations to the otrhonormal basis in eqs. 206 - 207 – restricted to the adjoint representation , we also reassign upon appropriate *coordinate transformation* **the adapted differential operators bringing those defined in eq. 183 for the orthogonal basis to accord with the orthonormal basis**

$$\underline{a} \; : \; \underline{\boldsymbol{a}}_{\vdash} \; = \; \exp\left(\tau \, \boldsymbol{\alpha}_{\vdash}^{\nu} \, \underline{\boldsymbol{I}}_{\vdash \, (\nu)} \right) \Big|_{\mathcal{G}} \equiv \exp\left(\tau \, \underline{\boldsymbol{\alpha}}_{\vdash} \right) \; \longrightarrow \; \exp\left(\tau \, \underline{\alpha} \right) \equiv \underline{U} \left(\tau \, \underline{\alpha} \right)$$

$$a \; : \; \boldsymbol{a}_{\vdash} \; = \; \exp\left(\tau \, \boldsymbol{\alpha}_{\vdash}^{\nu} \, \boldsymbol{ad}_{\vdash \, (\nu)} \right) \Big|_{\mathcal{G}} \equiv \exp\left(\tau \, \boldsymbol{\alpha}_{\vdash} \right) \; \longrightarrow \; \exp\left(\tau \, \alpha \right)$$

$$Ad \; : \; \boldsymbol{Ad}_{\vdash} \; \rightarrow \; \boldsymbol{Ad}_{\vdash} \left(\boldsymbol{a}_{\vdash} \right) = \exp\left(\tau \, \boldsymbol{\alpha}_{\vdash} \right) \; \longrightarrow \; Ad \left(a \right) \tag{219}$$

In order to sharpen the perspective lets repeat eq. 207 below

$$\boldsymbol{ad}_{\vdash \, (\widetilde{\varrho})} \; = \; \left(\alpha_{\vdash} \right)^{\widetilde{\varrho}} \, \boldsymbol{ad}_{\vdash \, (\widetilde{\varrho})} \; \longrightarrow \; \alpha \; = \; \alpha^{\, r} \, ad_{\, (r)}$$
$$\boldsymbol{D}_{\vdash} \left(\boldsymbol{a}_{\vdash} \right) \; = \; \boldsymbol{Ad}_{\vdash} \left(\boldsymbol{a}_{\vdash} \right) \; \longrightarrow \; Ad \left(a \right) \; = \; \exp\left(\tau \, \alpha \right) \tag{220}$$

Thus the first relation in eq. 219 has to be extended to the differential (Killing-) operators pertinent to the orthonormal coordinates on $\mathcal{G}$

$$\boldsymbol{ad}_{\vdash \, (\widetilde{\varrho})} \; = \; \underline{\boldsymbol{\alpha}}_{\vdash} \, |_{\mathcal{G}} \; = \; \left(\alpha_{\vdash} \right)^{\widetilde{\varrho}} \, \underline{\boldsymbol{I}}_{\vdash \, (\nu)} \Big|_{\mathcal{G}} \; \longrightarrow \; \underline{\alpha} \; = \; \alpha^{\, r} \, \underline{I}_{\, (r)} \Big|_{\mathcal{G}}$$
$$\longrightarrow \; \left[\underline{I}_{\, (r)} \, , \, \underline{I}_{\, (s)} \right] \; = \; f_{\, [\, r \, s \, t \,]} \; \underline{I}_{\, (t)} \Big|_{\mathcal{G}} \tag{221}$$

It is of historical interest, that the so induced unitary finite transformation operators on $\mathcal{G}$ *compact* **, endowed with a group invariant metric and also measure , generate an exhaustive and complete set of finite dimensional unitary representations of** $\mathcal{G}$ **[A114-1933] , [A115-1927] , [A116-1953] .** →

Hermitian versus antihermitian convention

In the following we will suppress the label $|_{\mathcal{G}}$ on the quantities $\underline{U}\,(\tau\,\underline{\alpha})$ and $\underline{I}_{\,(r)}$ defined in eqs. 219 - 221 , turning to these operator quantities next .

$$\underline{U}\,(\tau\,\underline{\alpha}) = \exp\,(\tau\,\underline{\alpha}) \;\;;\;\; \underline{\alpha} = \alpha^{\,r}\,\underline{I}_{\,(r)} = \frac{1}{i}\,\alpha^{\,r}\,\underline{J}_{\,(r)} \;\;;\;\; \underline{I}_{\,(r)} \equiv \frac{1}{i}\,\underline{J}_{\,(r)}$$

(222)
$$\underline{I}^{\,\dagger}_{\,(r)} = -\,\underline{I}_{\,(r)} \quad\leftrightarrow\quad \underline{J}^{\,\dagger}_{\,(r)} = \underline{J}_{\,(r)}$$

$$\left[\,\underline{I}_{\,(r)}\,,\,\underline{I}_{\,(s)}\,\right] = f_{\,[\,r\,s\,t\,]}\,\underline{I}_{\,(t)} \quad\leftrightarrow\quad \left[\,\underline{J}_{\,(r)}\,,\,\underline{J}_{\,(s)}\,\right] = i\,f_{\,[\,r\,s\,t\,]}\,\underline{J}_{\,(t)}$$

The conventions more adapted to applications of quantum mechanics , and thus preferred in physics, are to consistently use self adjoint operators , as generating through the exponential *imaginary* mapping unitary ones, whereas in the mathematical literature the antihermitian convention prevails. While the two are equivalent often misunderstandings result, which should be avoided. A good traceback of the appropriate powers of i can be found in ref. [A117-1963] .
Following – for a while – the selfadjoint operator basis $\underline{J}_{\,(r)}$ we can construct an invariant nonnegative operator (on $\mathcal{G}$) , quadratic with respect to derivatives

(223)
$$\underline{\mathcal{H}}^{\,(2)} = \sum_{s}\,\underline{J}^{\,2}_{\,(s)} \;\;;\;\; \left[\,\underline{J}_{\,(r)}\,,\,\underline{\mathcal{H}}^{\,(2)}\,\right] = 0$$

It is at this point that the Killing-fields at the basis of the operators $\underline{J}_{\,(r)}$ refer here by convention to left-multiplication on $\mathcal{G}$, whereas there exist also the right-multiplication $\longrightarrow$

Killing-fields , giving rise to a second set of *right-multiplication* operators $\underline{J}^{\,R}_{\,(r)}$

(224) $$\left[\,\underline{J}^{\,R}_{\,(r)}\,,\,\underline{J}^{\,R}_{\,(s)}\,\right] = i\,f_{\,[\,r\,s\,t\,]}\,\underline{J}^{\,R}_{\,(t)} \quad ; \quad \left[\,\underline{J}_{\,(r)}\,,\,\underline{J}^{\,R}_{\,(s)}\,\right] = 0 \quad , \quad \forall\; r\,,\,s$$

fully commuting with the left-multiplication associated operators $\underline{J}_{\,(t)}$.

Returning to left-multiplication, we can use $\underline{\mathcal{H}}^{\,(2)}$ in eq. 223 – together with the so called Cartan torus on $\mathcal{G}$ – to split the eigenvalues and projectors on suitable subspaces, which as a consequence of the compact nature of $\mathcal{G}$ reveal a discrete set of eigenvalues , each with a finite multiplicity . The Cartan torus is spanned by a maximal abelean subgroup , a sub-torus of $\mathcal{G}$ of dimension $\overline{r}\; =$ rank of $\mathcal{G}$

(225) $$\begin{array}{lcl} & \left(\,\underline{H}_{\,(1)}\,,\,\cdots\,,\,\underline{H}_{\,(\overline{r})}\,\right) & \leftrightarrow \quad \left(\,\underline{J}_{\,(1)}\,,\,\cdots\,,\,\underline{J}_{\,(\overline{r})}\,\right) \\ \overline{G} \,=\, G \,-\, \overline{r} \;\; \rightarrow & \left(\,\underline{K}_{\,(1)}\,,\,\cdots\,,\,\underline{K}_{\,(\overline{G})}\,\right) & \leftrightarrow \quad \left(\,\underline{J}_{\,(\overline{r}+1)}\,,\,\cdots\,,\,,\,\underline{J}_{\,G}\,\right) \end{array}$$

In fact a permutation on the indices $(1)\,,\,\cdots\,,\,(G)$ should be allowed for, before the segregation in eq. 225 into Cartan torus and its orthogonal (tangent space) complement is performed.

This segregation gives a corresponding separation of the Lie algebra operations

(226) $$\begin{array}{l} \alpha \;=\; \left(\,\alpha^{\,1}\,,\,\cdots\,,\,\alpha^{\,G}\,\right) \leftrightarrow \underline{\alpha} \,=\, \underline{h} \,+\, \underline{\kappa} \\ h \;=\; \left(\,h^{\,1}\,,\,\cdots\,,\,h^{\,\overline{r}}\,\right) \leftrightarrow \underline{h} \,=\, h^{\,r}\,\underline{H}_{\,(r)} \quad ; \quad r \,=\, 1\,,\,\cdots\,,\overline{r} \\ \kappa \;=\; \left(\,\kappa^{\,1}\,,\,\cdots\,,\,\kappa^{\,\overline{G}}\,\right) \leftrightarrow \underline{\kappa} \,=\, \kappa^{\,s}\,\underline{K}_{\,(s)} \quad ; \quad s \,=\, 1\,,\,\cdots\,,\overline{G} \end{array}$$

$\rightarrow$

Returning to the (left-multiplication) commutation rules in eq. 224 we separate indices of the operators $\underline{J}_u$ along h , κ

(227) $$r \, , \, r' \, \cdots = 1, \cdots, \overline{r} \leftrightarrow h \; ; \; s \, , \, t \cdots = 1, \cdots, \overline{G} \leftrightarrow \kappa$$

Adapting to the Cartan basis the structure constants take the form

(228) $$f_{[r_1 r_2 r_3]} = 0 \; , \; f_{[r_1 r_2 s]} = 0$$
$$f \rightarrow \begin{cases} f_{[\; r s_1 s_2]} \\ f_{[s_1 s_2 s_3]} \end{cases} \; ; \; r_{.} \in h \; , \; s_{.} \in \kappa$$

We propose to study *alongside* the adjoint representation *all* irreducible representations of $\mathcal{G}$, adapted to the Cartan torus

representation	action on	action $\longrightarrow$ roots / weights
$\underline{J}_{(r)} \in \underline{h}$	$\underline{\kappa} = \kappa^s \underline{J}_{(s)} \in \underline{\kappa} \rightarrow \left[\underline{J}_{(r)} \, , \underline{\kappa} \right]$	$i f_{[r s t]} \kappa^s \underline{J}_{(t)}$
$ad : \left(i \, ad_{(r)} \right)_{s t} \in h$	$\kappa = \kappa^s \, i \, ad_{(s)} \in \kappa$	$i f_{[r s t]} \kappa^s \, i \, ad_{(t)}$
$\mathcal{D} : \delta_{(r)} \in h$	$z \in L(\mathcal{D})$	$z^\sigma \rightarrow \left(\delta_{(r)} \right)^\sigma{}_\tau \, z^\tau$

(229)

In the row devoted to $\mathcal{D}$ in eq. 229 we have denoted the linear space on which the representation , i.e. the set of *unitary* , *irreducible* matrices forming $\mathcal{D}$ acts , by $L\,(\,\mathcal{D}\,)$ (eq. 218) .

We repeat below the action of the Cartan generators for adjoint and general representations

(230)
$$
\begin{array}{l}
ad\ :\ \underline{J}_{\,(r)} \text{ or } i\,ad_{\,(r)} \quad \longrightarrow \quad \kappa \to i\,ad_{\,(r)}\,\kappa \\
\mathcal{D}\ :\ \delta_{\,(r)} \quad \longrightarrow \quad z \to \delta_{\,(r)}\,z \\
\left(\,i\,ad_{\,(r)}\,\right)_{s\,t} = i\,f_{[\,s\,r\,t\,]} \ ,\ \left(\,\delta_{\,(r)}\,\right)^{\sigma}{}_{\tau} \\
\underline{J}^{\,\dagger}_{\,(r)} = \underline{J}_{\,(r)} \ ,\ \left(\,i\,ad_{\,(r)}\,\right)^{\dagger} = \left(\,i\,ad_{\,(r)}\,\right) \ ,\ \left(\,\delta_{\,(r)}\,\right)^{\dagger} = \left(\,\delta_{\,(r)}\,\right)
\end{array}
$$

Choosing the hermitian (selfadjoint) basis implies that $\underline{J}_{\,(r)}$ are selfadjoint operators with discrete *real* eigenvalues on a compact $\mathcal{G}$, whereas the representation matrices $i\,ad_{\,(r)}\ ,\ \delta_{\,(r)}$ are hermitian finite dimensional .

We note the very different actions pertaining to the Lie algebra generators as well as *one of its* representations – the adjoint one – on one hand and the direct multiplicative action of the Lie algebra *representation-* matrices $\delta_{\,(r)}\ ,\ r = 1, \cdots, \overline{r}$ on $L\,(\,\mathcal{D}\,)$ on the other . The first kind of action is displayed in the first two rows in eq. 229 , the second one in the last row thereof .

Had we chosen to act with the representation matrices of the Cartan generators $\delta_{\,(r)}$ pertaining to $\mathcal{D}$ by commutation with the corresponding representation matrices $\delta_{\,(s)}\ ,\ s = 1, \cdots, \overline{G}$ $\longrightarrow$

$$\kappa^{\mathcal{D}} = \kappa^{s} \delta_{(s)} \; ; \; s = 1, \cdots, \overline{G}$$

(231) $$ad(\mathcal{D}) \leftrightarrow \delta_{(r)} \; : \; \begin{cases} \kappa^{\mathcal{D}} \; \rightarrow \; [\delta_{(r)}, \kappa^{\mathcal{D}}] = i \left(ad_{(r)} \right)_{s\,t} \kappa^{t} \delta_{(s)} \\ \kappa \; \longrightarrow \; ad_{(r)} \kappa \; ; \; \text{for any } \mathcal{D} \end{cases}$$

roots denote the collection of eigenvalues $\{ \varrho = (\varrho^{1}, \cdots, \varrho^{\overline{r}}) \}$ of $ad_{(r)}$, pertaining to the adjoint Lie algebra representation .

weights denote the collection of eigenvalues $\{ w = (w^{1}, \cdots, w^{\overline{r}}) \}$ of $\delta_{(r)}$, pertaining to the Lie algebra representation $\mathcal{D}$.

For the adjoint representation the vanishing root $\varrho \equiv 0$ with $\overline{r}$ fold degenracy , corresponding to the abelean sub-torus (eq. 228) , should be included .

A1-concl Concluding remarks to topical derivations in this appendix

r1) Roots *and* weights displayed as vectors in $\overline{r}$ euclidean dimensions and using cartesian coordinates form a lattice , which can be generated from $\overline{r}$ base points by integer linear multiples. In this connection the minima of absolute eigenvalues relative to $\underline{J}_{(r)} \; ; \; r = 1, \cdots, \overline{r}$ form *also* relevant invariants , in addition to the second and higher Casimir invariants.

I show in the next subsection the roots and weights of $SU3$, the latter with respect to the 3 dimensional (complex) representation $\mathcal{D}3$. Here a derivation of root and weight systems of all simple compact groups is not undertaken . Instead I refer to refs. [A14-1894] , [A113-1986] and [A118-1981] .

→

r2) This appendix shall prepare the ground for a comparison of two kinds of local gauge structures :

a) gauging orientation $\leftrightarrow$ Riemannian metric spaces ([A11-2008])

b) charge-like gauging $\leftrightarrow$ Yang-Mills- or charge-like connections

r3) While such comparison (r2) is well known for classical connection *and* metric or vielbein , it is – at least for the author – much less so for quantized dynamical fields including a metric , acting in dimensions beyond time and space .

A1-SU3 Roots and simplest weights for $\mathcal{G} = SU3$

Adopting the conventions of the eight Gell-Mann matrices [A118-1964] and rescaled by $\frac{1}{2}$ multiplicatively we renumber the generators $\underline{J}_1 , \cdots \underline{J}_8$ to the numbering used here for the Cartan torus , choosing the 3 dimensional representation $\mathcal{D}3$ of $SU3$

conventional	1	2	3	4	5	6	7	8
here	(8)	(3)	(1)	(4)	(5)	(6)	(7)	(2)

(232)

$$\delta_{(1)} = \frac{1}{2} \lambda_3 \leftrightarrow \underline{J}_3 \quad ; \quad \text{eigenvalues } j_3$$

$$\delta_{(2)} = \frac{1}{2} \lambda_8 \leftrightarrow \underline{J}_8 \quad ; \quad \text{eigenvalues } j_8$$

$\longrightarrow$

The chosen basis defines (*by convention* **) the 3 representatiuon of the base-charges of u , d , s , the diagonal elements of** $j_3\,,\; j_8$ **respectively , ordered into three 2 vectors** $(\,j_3\,,\; j_8\,)_{m\,m}$ **with** $m\;=\;1,2,3\;=\;u,d,s$ **, are**

(233)

flavor	$(\,j_3\,,\; j_8\,)$
u	$(\;\frac{1}{2}\,,\;\frac{1}{2\sqrt{3}}\;)$
d	$(\,-\frac{1}{2}\,,\;\frac{1}{2\sqrt{3}}\;)$
s	$(\;0\,,\;-\frac{1}{\sqrt{3}}\;)$

For the octet (adjoint) representation the flavor assignments $\in\;\kappa$ **are**

(234)
$$
\begin{array}{llll}
u\otimes\overline{d} \;\leftrightarrow\; \pi^{+} & & d\otimes\overline{u} \;\leftrightarrow\; \pi^{-} \\
u\otimes\overline{s} \;\leftrightarrow\; K^{+} & & s\otimes\overline{u} \;\leftrightarrow\; K^{-} \\
d\otimes\overline{s} \;\leftrightarrow\; K^{0} & & s\otimes\overline{d} \;\leftrightarrow\; \overline{K^{0}}
\end{array}
$$

and with $(\,j_3\;=0\,,\,j_8\;=\;0\,)\;:$

$$
\begin{cases}
\pi^{0} \;=\; \frac{1}{\sqrt{2}}\left(u\otimes\overline{u}\;-\;d\otimes\overline{d}\right) \\
\eta^{8} \;=\; \frac{1}{\sqrt{6}}\left(\,u\otimes\overline{u}+d\otimes\overline{d}-2\,s\otimes\overline{s}\right)
\end{cases}
$$

(235)

→

The values of (j_3, j_8) for the octet representation are

(236)

flavor	(j_3, j_8)
π^0, η^8	$(0, 0)$
$\pi^\pm$	$\pm(1, 0)$
$K^\pm$	$\pm\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$K^0, \overline{K}^0$	$\pm\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

The weights of the 3 and $\overline{3}$ representations and the roots of SU3 are shown in Fig. A11 below . $\longrightarrow$

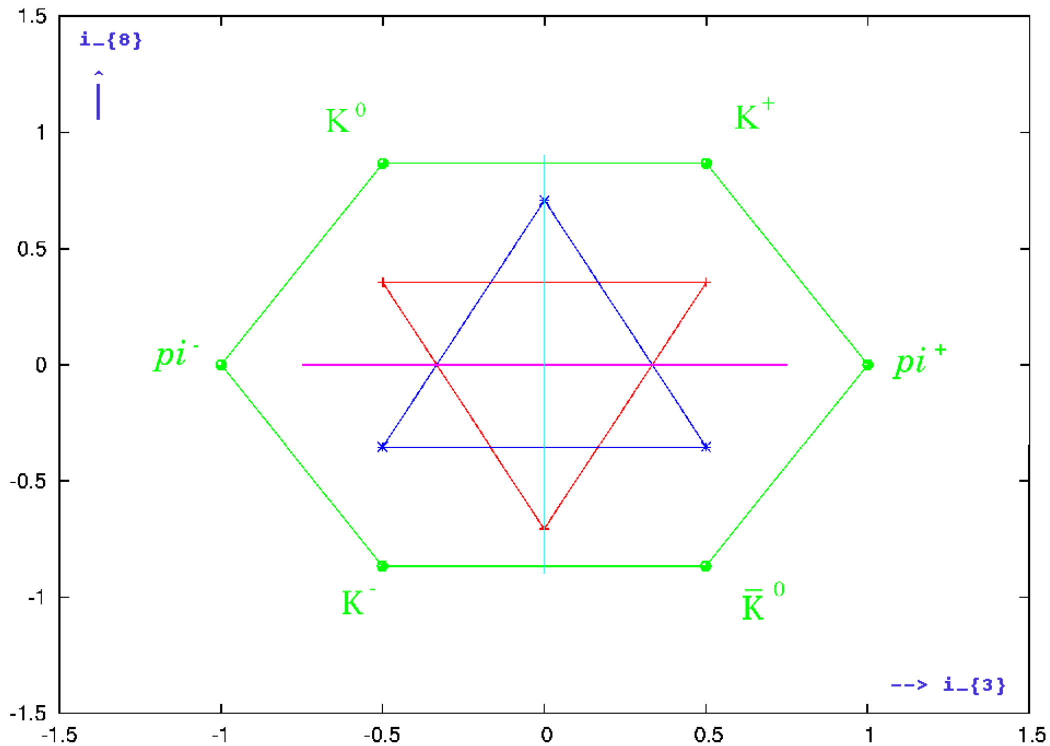

Fig A12 : Weight diagrams of the 3 , $\overline{3}$ representations and the roots of SU3 . $(i_3, i_8) = 0$ states not displayed .

A2 Appendix 2 - Renormalization group equation in QCD

We take the short distance expansion for the current product as defined in eq. 73 repeated below , subject to the renormalization group or rescaling equations, the latter representing *exact, anomalous* **Ward identities for the dilatation current [A21-1976]**

(237) $$\begin{matrix} T \\ (\Pi) \end{matrix} \left\{ J_{\mu} \left(x + \tfrac{1}{2} z \right) J_{\nu} \left(x - \tfrac{1}{2} z \right) \right\} \underset{z \to 0}{\sim} \sum_{\mathcal{O}} C_{\mu\,\nu\,\mathcal{O}}^{\,T\,(\Pi)} (z) \mathcal{O} (x)$$

In the triple association

(238) $$J_{\mu} (x_1) , J_{\nu} (x_2) \rightarrow \mathcal{O} (x_3)$$

we will assume that all three local fields are multiplicatively – *perturbatively* **– renormalizable for simplicity. Mixing effects of** *finite* **groups of operators**

(239) $$\{ \cup \mathcal{O} \mid (\mathcal{O}_1 , \cdots , \mathcal{O}_n) \}$$

do arise and can easily be incorporated [A22-1974] .

We are mainly interested in the $tw = 4 ; \, dim\, 4$ **operators** *later*

(240) $$\{ \mathcal{O} \}_4 = \{ \vartheta_{\mu}^{\mu} , \tfrac{1}{4} g^2 : F_{\mu\nu\, a} F_{a}^{\mu\nu} : , \cdots \}$$

ϑ_{ν}^{μ} : **suitable, conserved energy momentum tensor**

$\longrightarrow$

but discuss the general simply multiplicatively renormalizable case first .

In terms of unrenormalized quantities , generically denoted by the suffix $^{(0)}$ **, and renormalization constants including a Fermi-gauge fixing parameter** η **in order to controle the gauge invariant character of the so defined operators we set**

(241)
$$J_\alpha = (Z_J)^{-1} J_\alpha^{(0)} \quad , \quad \mathcal{O} = (Z_{\mathcal{O}})^{-1} \mathcal{O}^{(0)}$$
$$g = (Z_3)^{3/2} (Z_1)^{-1} g^{(0)} \quad , \quad \eta = (Z_3)^{-1} \eta^{(0)}$$

As renormalization conditions we use a ***finite*** **dummy scale** μ **, as it appears** ***also*** **naturally in dimensional renormalization , with respect to which unrenormalized quantities are insensitive**

(242)
$$d / d\mu \left\{ g^{(0)} , \eta^{(0)} ; J_\alpha^{(0)} , \mathcal{O}^{(0)} , \ldots^{(0)} \right\} = 0$$

For the choice of currents in eq. 237 and $d = 4$ **scalar operators** $\{ \mathcal{O} \}$ **in eq. 240 it follows , always within the (asymptotically) perturbative logic**

(243)
$$C^{T\,(\Pi)}_{\alpha\,\beta\,\mathcal{O}} (z) = \left(g_{\alpha\beta} \Box - \partial_\alpha \partial_\beta \right) C^{T\,(\Pi)}_{J\,\mathcal{O}} (z ; \mu , g , \eta)$$

with $dim\, C^{T\,(\Pi)}_{J\,\mathcal{O}} = 0 \;\; ; \;\; Z_J = 1$

$\longrightarrow$

The short distance distributions $C^{T\,(\Pi)}_{J\,\mathcal{O}}$ then are of the form

(244)
$$C^{T\,(\Pi)}_{J\,\mathcal{O}}\,(z\,;\,\mu\,,\,g\,,\,\eta\,)\;=\;(\,Z_J\,)^2\,(\,Z_{\mathcal{O}}\,)^{-1}\,\widehat{C}^{T\,(\Pi)}_{J\,\mathcal{O}}\,(z\,;\,\mu\,,\,g\,,\,\eta\,)$$
$$g\;=\;(\,Z_3\,)^{3/2}\,(\,Z_1\,)^{-1}\,g^{(0)}\quad,\quad\eta\;=\;(\,Z_3\,)^{-1}\,\eta^{(0)}$$

The μ rescaling equation now follows from eq. 242

$$\left(\begin{array}{l}\mu\,\partial_\mu\,+\,\beta\,(\,g\,)\,\partial_g\,-\,\gamma_{m_\alpha}\,m_\alpha\,\partial_{m_\alpha}\\ -\,2\,\gamma_3\,(\,\eta\,\partial_\eta\,)\,-\,\gamma_{J\,\mathcal{O}}\end{array}\right)\;C^{T\,(\Pi)}_{J\,\mathcal{O}}\,(\,z\,;\,\mu\,,\,g\,,\,m_\beta\,,\,\eta\,)\;=\;0$$

(245)
$$\left\{\begin{array}{l}\beta\,(\,g\,)\;=\;-\,g\,b\,(\,g^2\,)\\ \gamma_{m_\beta}\,(\,g^2\,)\\ \gamma_{J\,\mathcal{O}}\,(\,g^2\,,\,(\,\eta\,)\,)\\ \gamma_3\,(\,g^2\,,\,\eta\,)\end{array}\right\}\;=\;\mu\,d\,/\,d\,\mu\,\left\{\begin{array}{c}\log\left(\,(\,Z_3\,)^{3/2}\,(\,Z_1\,)^{-1}\right)\\ Z_{m_\beta}\\ \log\,(\,Z_{\mathcal{O}}\,/\,Z_J^2\,)\\ \log\,(\,Z_3\,)^{1/2}\end{array}\right\}$$

The brackets in red in eq. 245 shall indicate that upon establishing that $\{\,J\,,\,\mathcal{O}\,\}$ are indeed gauge invariant operators the derivative with respect to the gauge parameter η in the rescaling equation gives zero and also the *combined* anomalous dimensions $\gamma_{J\,\mathcal{O}}$ do not depend on η . For the operators in the group of interest here (eq. 240) the determination of the correct gauge invariant ones has been derived by H. Kluberg-Stern and J. B. Zuber [A24-1975] . $\longrightarrow$

From the *partial* renormalization systematics given given in eqs. 241 - 242 the *redundant* scale μ becomes an essential argument of renormalized quantities

$$\begin{array}{lll} J_{\alpha} & : & \mu\, d / d\mu\, (J_{\alpha})_{\mu} = - \gamma_{J} \, (J_{\alpha})_{\mu} = 0 \\ \mathcal{O} & : & \mu\, d / d\mu\, (\mathcal{O})_{\mu} = - \gamma_{\mathcal{O}} \, \mathcal{O}_{\mu} \\ g & : & \mu\, d / d\mu\, (g)_{\mu} = \beta (g_{\mu}) \\ \eta & : & \mu\, d / d\mu\, (\eta)_{\mu} = - \gamma_{3} \, \eta_{\mu} \end{array} \quad \textbf{with :} \left\{ \begin{array}{l} \gamma_{3} = \gamma_{3} (g^{2} , \eta) \\ \gamma_{J} = 0 \; , \; Z_{J} = 1 \\ \gamma_{\mathcal{O}} = \gamma_{\mathcal{O}} (g^{2} , (\eta)) \end{array} \right.$$

(246)

Three remarks are in order :

1) Perturbative accessibility of renormalization in asymptotically free theories

While the entire renormalization procedure thus (eq. 246) becomes within *perturbative accessibility* – as explained in textbooks [A25] , [A26-1982] – the associated renormalization group equation serves to restore renormalization group invariant properties, in particular such definitions of operators .

2) Infrared instability

is associated with all physical scales *not* accessible to perturbative approximations .

→

3) Quark mass dependence

We neglect in the considerations followed here the quark mass dependence of all Green functions in the deep Euclidean region on quark masses , the latter also *to be renormalized* **and thus** *not* **renormalization group independent [A27-1975] . This is in line with the main short distance contributions , which are sorting out by the twist characteristic leading contributions** *modulo less dominant ones modulo powers of inverse Euclidean distance* **. These dimensional hierarchies** *also* **break down whence the region of perturbative accessibility is transgressed . For small quark masses at a generic scale of** $\sim\ 1\,GeV$ **the quark mass associated mixing of operators with different dimensions sets in in subtle ways governed by approximate chiral symmetry also outside the deep Euclidean region .**

$\longrightarrow$

A2-slsc The sliding scale coupling constant

From eq. 246 the sliding scale coupling constant follows , considering a variation of μ and g_μ

$$t = \log(\overline{\mu}/\mu) \quad ; \quad \partial_t = \overline{\mu}\, d/d\overline{\mu}$$

(247) $$\overline{g} = \overline{g}(t, g_\mu) \quad ; \quad \text{generic variablei is : } g' \to x \quad ; \quad t \to \tau$$

$$\partial_t \overline{g} = \beta(\overline{g}) \quad ; \quad \overline{g}(t = 0, g_\mu) = g_\mu$$

Using the generic variables x , τ the differential equation (eq. 247) becomes

$$(d/d\tau)\, x = \beta(x) \to d\tau = dx/\beta(x) \quad \to$$

$$\tau_2 - \tau_1 = \int_{x_1}^{x_2} dx\, (\beta(x))^{-1} = F(x_2) - F(x_1)$$

(248) $$F(x) = \int_{x_0}^{x} dx'/\beta(x') ; \quad \text{generic : } x_0 \text{ independent of } x_1 , x_2$$

$$x = x(\tau) \to \begin{cases} x_1(\tau_1) \\ x_2(\tau_2) \end{cases}$$

The function $F(x)$ satisfies the (generic) equation

(249) $$\beta(x)\, \partial_x F(x) = 1$$

→

From the generic identities a consequence follows, relevant for the short distance limit of the rescaling equation (eq. 245) , for the solution to the associated homogeneous partial differential equation [a]

(250)
$$(\partial_\tau - \beta(g)\partial_g)\, h(\tau, g) = 0 \longrightarrow$$
$$h = h(\overline{g}(\tau = t, g = g_\mu)) \quad \text{with } \overline{g} \text{ as defined in eq. 247}$$

Leaving aside quark mass dependence for simplicity here , bearing in mind remark 3) above, we turn to the properties of the sliding scale coupling constant , i.e. the function $\overline{g} = \overline{g}(t, g_\mu)$ and the associated differential equation defined in eq. 247 readapted below. The universal independence of the sliding scale coupling constant can be maintained independent of quark masses .

(251)
$$t = \log(\overline{\mu}/\mu) \quad ; \quad \partial_t = \overline{\mu}\, d/d\overline{\mu}$$
$$\partial_t \overline{g} = \beta(\overline{g}) \quad ; \quad \overline{g}(t = 0, g_\mu) = g_\mu$$

Thus we are to determine the function $F(x)$ as defined in eq. 247 such that

(252)
$$t = F(\overline{g}) - F(g) \quad ; \quad g = g_\mu \quad ; \quad F(x) = \int_{x_0}^{x} dy/\beta(y)$$

→

a Note the - sign in the first line of eq. 250 .

We have followed the concise treatment and notation of C. G. Callan [A28-1970] and K. Symanzik [A29-1970] on whose formulation and structural discussion the modern form of the renormalization group equation(s) relies . The two loop renormalizability of nonabelian gauge theories is due to G. t'Hooft [A210-1971(2)] and contain all elements which determine the sliding scale function discussed here. These quotations duely made including [A211-1969] , [A212-1972] , I continue using selected changes of variables, transforming eq. 252

$$s = 2t = \log \left[(\overline{\mu} / \mu)^2 \right]$$

$\kappa = g^2 / \left(16 \pi^2 \right)$ **and** $\kappa \rightarrow \overline{\kappa}$ **generic** $\kappa \rightarrow X , Y$

(253) $$s = F (\overline{\kappa}) - F (\kappa_\mu) \; ; \; \kappa_\mu = g_\mu^2 / \left(16 \pi^2 \right)$$

$$F (X) = \int_X^{X_0} \left(d Y / Y^2 \right) (B (Y))^{-1}$$

$$\beta (y) = - y \, b (y^2) \; ; \; B (Y) = b (y^2) / Y \leftrightarrow Y = y^2 / \left(16 \pi^2 \right)$$

With the substitutions in eq. 253 we have

(254) $$s = F (\overline{\kappa}) - F (\kappa_\mu) = \int_{\overline{\kappa}}^{\kappa_\mu} \left(d Y / Y^2 \right) (B (Y))^{-1}$$

$$B (Y) = b_0 + b_1 Y + b_2 Y^2 + \cdots$$

→

There exist many ways to separate the limiting part $\overline{\kappa} \to 0 \leftrightarrow s \to +\infty$ of the integral in eqs. 253, 254 . We use (two) partial integrations

$$s = (Y B (Y))^{-1} \Big|_{\kappa_\mu}^{\overline{\kappa}} + \int_{\overline{\kappa}}^{\kappa_\mu} (dY / Y) (d / dY) (B (Y))^{-1}$$

(255)

$$\Delta_1 s = s - (Y B (Y))^{-1} \Big|_{\kappa_\mu}^{\overline{\kappa}} \; ; \; \Delta_0 B^{-1} = (B (Y))^{-1}$$

Upon the substitution $s \to \Delta_1 s$ we obtain

$$\Delta_1 s = (\log (Y)) (- d / dY) (B (Y))^{-1} \Big|_{\kappa_\mu}^{\overline{\kappa}} + \Delta_2 F$$

(256)

$$\Delta_2 F = \int_{\overline{\kappa}}^{\kappa_\mu} dY (- \log Y) (- d / dY)^2 (B (Y))^{-1}$$

Going step by step we evaluate first the derivatives as acting on $B^{-1} (Y)$

$$(- d / dY) (B (Y))^{-1} = B' (Y) / B^2 (Y) = \Delta_1 B^{-1} = - \left(B^{-1} \right)'$$

$$(- d / dY)^2 (B (Y))^{-1} = \left(2 \left(B' (Y) \right)^2 - B'' (Y) B (Y) \right) (B (Y))^{-3}$$

$$= \Delta_2 B^{-1} = \left(B^{-1} \right)''$$

$${}' = d / dY , \; {}'' = (d / dY)^2 , \cdots$$

(257)

→

Collecting the definitions in eqs. 255 - 257 we restate

(258)
$$n = 0 \quad \Delta_0 B^{-1} = (B(Y))^{-1}$$
$$n \geq 1 \quad \Delta_n B^{-1} = (-d/dY)^n (B(Y))^{-1}$$

and

(259)
$$s = \left[\left(Y^{-1} \right) \Delta_0 B^{-1}(Y) + (\log Y) \Delta_1 B^{-1}(Y) \right]_{\kappa_\mu}^{\overline{\kappa}} + \Delta_2 F$$
$$\Delta_2 F = \int_{\overline{\kappa}}^{\kappa_\mu} dY \, (-\log Y) \, \Delta_2 B^{-1}(Y)$$

We add two representations valid in the perturbatively accessible region $0 \leq Z \leq \overline{X} = \overline{\kappa}$

---

(260)
$$\left(Y^{-1} \right) \Delta_0 B^{-1}(Y) \big|_{Y = \overline{X}} = \overline{X}^{-1} \left[\begin{array}{c} B^{-1}(0) + \\ + \int_0^{\overline{X}} dZ \, \left(-\Delta_1 B^{-1} \right)(Z) \end{array} \right]$$

$$\overline{X} \to \overline{\kappa_\mu}$$

$\longrightarrow$

Also we anchor $\Delta_2 F$ in eq. 259 at $X \to 0$

$$\Delta_2 F = \int_{\overline{\kappa}}^{\kappa_\mu} dY\,(-\log Y)\,\Delta_2 B^{-1}(Y)$$

(261)

$$= F_2(X_\mu) - F_2(\overline{X})$$

$$F_2(X) = \int_0^X dY\,(-\log Y)\,\Delta_2 B^{-1}(Y)\ ; \quad \begin{cases} X_\mu = \kappa_\mu \\ \overline{X} = \overline{\kappa} \end{cases}$$

We decompose s in eq. 259

(262)

$$s = \Sigma(\overline{X}) - \Sigma(X_\mu)$$

$$\Sigma(X) = \big(X^{-1}\big)\,\Delta_0 B^{-1}(X) + (\log X)\,\Delta_1 B^{-1}(X) - F_2(X)$$

For $X \to X_\mu$ we do not know the form of the functions determining $\Sigma(X)$, *in particular* if we choose the scale μ *outside* the region of perturbative accessibility . But for $X \to \overline{X}$ we can perform an asymptotic expansion for $X \to 0$, assuming a pure power expansion for the functions $\big(B\,,\,B^{-1}\,,\,\cdots(X)\big)$, as they appear in the asymptotic expressions for $\Sigma(\overline{X})$ as defined in eqs. 261 - 262 .

To this end it is enough to determine $\longrightarrow$

the power expansion of $B^{-1}(X)$ **up to second order in X , to which we turn below.**

(263)
$$B(X) = b_0 + b_1 X + b_2 (X)^2 + R_2(X)$$
$$R_2(X) = o(X^2) \text{ for } X \to +0$$

To this end it is convenient to rescale $B(X)$

(264)
$$B(X) = b_0 A(X) \; ; \; b_0 = \frac{1}{3}(33 - 2N_{fl}) > 0$$
$$A(X) = 1 + a_1 X + a_2 X^2 + \widehat{R}_2(X) \; ; \; a_n = b_n / b_0$$
$$\widehat{R}_2 = (b_0)^{-1} R_2 \; ; \; \widehat{R}_2(X) = o(X^2)$$

The rescaling by $(b_0)^{-1}$ **is universal to all three terms on the right hand side of the expression for s in eq. 262 yielding**

(265)
$$s = (b_0)^{-1} \left(\widehat{\Sigma}(\overline{X}) - \widehat{\Sigma}(X_\mu) \right)$$
$$\widehat{\Sigma}(X) = \left(X^{-1} \right) \Delta_0 A^{-1}(X) + (\log X) \Delta_1 A^{-1}(X) - \widehat{F}_2(X)$$
$$\widehat{F}_2(X) = \int_0^X dY \, (-\log Y) \, \Delta_2 A^{-1}(Y)$$

$\longrightarrow$

For A^{-1} we thus have

(266)
$$\Delta_0 A^{-1}(X) = 1 - a_1 X + \left((a_1)^2 - a_2\right) X^2 \quad + R_2^{(\Delta_0 A^{-1})}(X)$$
$$\Delta_1 A^{-1}(X) = a_1 - 2\left((a_1)^2 - a_2\right) X \quad + R_1^{(\Delta_1 A^{-1})}(X)$$
$$\Delta_2 A^{-1}(X) = 2\left((a_1)^2 - a_2\right) \quad + R_0^{(\Delta_2 A^{-1})}(X)$$

$$\Delta_0 A^{-1} = A^{-1} \; ; \; R_n^{(.)}(X) = o(X^n)$$

Here we list the three coefficients of the function $B(\kappa) = -\beta(g)/(g\kappa)$ [A213-1988] (eqs. 253 - 254 , 264) , to which the present asymptotic expansion at short distances is restricted, in the $\overline{MS}$ renormalization scheme

(267)
$$b_0 = \tfrac{1}{3}(33 - 2N_{fl})$$
$$b_1 = \tfrac{2}{3}(9 \times 17 - 19N_{fl})$$
$$b_2 = \tfrac{1}{54}\left(27 \times 2857 - 21 \times 719 N_{fl} + 25 \times 13 N_{fl}^2\right)$$

$$5033 = 7 \times 719 \; , \; 325 = 25 \times 13$$

b_3 has been calculated in ref. [A214-1997] . →

A2-Asy The asymptotic expansion of the sliding scale coupling constant

We go back to the expression for $\widehat{\Sigma}$ in eq. 265

(268)

$$\widehat{\Sigma}(X) = \left(X^{-1}\right)\Delta_0 A^{-1}(X) + (\log X)\Delta_1 A^{-1}(X) - \widehat{F}_2(X)$$

$$\widehat{F}_2(X) = \int_0^X dY\,(-\log Y)\,\Delta_2 A^{-1}(Y)$$

and expand successive terms for $X \to \overline{X} \searrow 0$ using the asymptotic expressions in eq. 266

(269)

$$asy_1 = X^{-1}\Delta_0 A^{-1}(X) = \begin{cases} \left(X^{-1} - a_1\right) + \\ \left((a_1)^2 - a_2\right) X + \\ + R_1^{(Asy_1)}(X) \end{cases}$$

$$asy_2 = (\log X)\Delta_1 A^{-1}(X) = \begin{cases} -a_1\left(\log\left(X^{-1}\right)\right) + \\ 2\left((a_1)^2 - a_2\right)\left(\log\left(X^{-1}\right)\right) X + \\ + \left(\log\left(X^{-1}\right)\right) R_1^{(Asy_2)}(X) \end{cases}$$

$$Asy_1 = asy_1 + a_1 \quad , \quad Asy_2 = asy_2$$

$$R_1^{(Asy_1)} = X^{-1} R_2^{(\Delta_0 A^{-1})} \quad , \quad R_1^{(Asy_2)} = -R_1^{(\Delta_1 A^{-1})}$$

→

The constant $-a_1$ in the expression (in red) for Asy_1 in eq. 269 is not part of the asymptotic short distance expansion . This is so, because of the *two* expressions for the quantity denoted $\widehat{\Sigma}$ in eq. 265 , obtained from twofold partial integrations . This equation is repeated for clarity below

$$s = (b_0)^{-1}\left(\widehat{\Sigma}(\overline{X}) - \widehat{\Sigma}(X_\mu)\right)$$

(270) $$\widehat{\Sigma}(X) = \left(X^{-1}\right)\Delta_0 A^{-1}(X) + (\log X)\,\Delta_1 A^{-1}(X) - \widehat{F}_2(X)$$

$$\widehat{F}_2(X) = \int_0^X dY\,(-\log Y)\,\Delta_2 A^{-1}(Y)$$

This comes from the inherent representation of the function s as shown in eq. 259 and implies a redefinition : $\widehat{\Sigma} \to \Sigma_{asy}$

$$\Sigma_{asy}(X) = \widehat{\Sigma}(X) + a_1 \rightarrow$$

$$s = (b_0)^{-1}\left(\Sigma_{asy}(\overline{X}) - \Sigma_{asy}(X_\mu)\right)$$

$$\Sigma_{asy}(X) = \left(X^{-1}\right)\Delta_0 A^{-1}(X) + a_1 + (\log X)\,\Delta_1 A^{-1}(X) - \widehat{F}_2(X)$$
(271)

$\longrightarrow$

Eq. 269 takes the form

$$Asy_1 = X^{-1}\Delta_0 A^{-1}(X) + a_1 = \begin{cases} X^{-1} + \\ \left((a_1)^2 - a_2\right) X + \\ + R_1^{(Asy_1)}(X) \end{cases}$$

$$Asy_2 = (\log X)\,\Delta_1 A^{-1}(X) \quad = \begin{cases} -a_1\left(\log\left(X^{-1}\right)\right) + \\ 2\left((a_1)^2 - a_2\right)\left(\log\left(X^{-1}\right)\right) X + \\ + \left(\log\left(X^{-1}\right)\right) R_1^{(Asy_2)}(X) \end{cases}$$

$$R_1^{(Asy_1)} = X^{-1} R_2^{(\Delta_0 A^{-1})} \quad , \quad R_1^{(Asy_2)} = -R_1^{(\Delta_1 A^{-1})}$$

(272)

while →

the short distance expansion for $\Sigma_{asy}(X)$ **for** $X \rightarrow \overline{X} \searrow 0$ **becomes**

$$\Sigma_{asy}(X) = Asy_{12}(X) - \widehat{F}_2(X) \ ; \ Asy_{12} = Asy_1 + Asy_2$$

$$Asy_{12} = \begin{cases} X^{-1} - a_1 \left(\log\left(X^{-1}\right)\right) + \\ \left(\left(a_1\right)^2 - a_2\right) X + 2\left(\left(a_1\right)^2 - a_2\right)\left(\log\left(X^{-1}\right)\right) X + \\ + \mathcal{R}_{asy} \end{cases}$$

(273)

$$\mathcal{R}_{asy} = \begin{cases} X^{-1} R_2^{\left(\Delta_0 A^{-1}\right)} - \left(\log\left(X^{-1}\right)\right) R_1^{\left(\Delta_1 A^{-1}\right)} \\ \rightarrow o\left[\left(\log\left(X^{-1}\right)\right) X^2\right] + o\left[X^2\right] \end{cases}$$

$$\widehat{F}_2(X) = \int_0^X dY \left(-\log Y\right) \Delta_2 A^{-1}(Y)$$

We make explicit the entire power series making up the functions shown in eq. 266 →

and the corresponding remainders

$$\Delta_0 A^{-1}(X) \sim \sum_{n=0\,\infty} I_n X^n$$
$$\Delta_1 A^{-1}(X) \sim -\sum_{n=1}^{\infty} n I_n X^{n-1}$$
$$\Delta_2 A^{-1}(X) \sim \sum_{n=2}^{\infty} n(n-1) I_n X^{n-2} \longrightarrow$$
(274)
$$R_2^{(\Delta_0 A^{-1})} \sim \sum_{m=3}^{\infty} I_m X^m$$
$$R_1^{(\Delta_1 A^{-1})} \sim -\sum_{m=2}^{\infty} (m+1) I_{m+1} X^m$$
$$R_0^{(\Delta_2 A^{-1})} \sim \sum_{m=1}^{\infty} (m+1)(m+2) I_{m+2} X^m$$

The combinations of remainders in the expression for $\mathcal{R}_{asy}$ in eq. 273 become

$$X^{-1} R_2^{(\Delta_0 A^{-1})} \sim \sum_{m=2}^{\infty} I_{m+1} X^m$$
(275)
$$-(\log(X^{-1}))\, R_1^{(\Delta_1 A^{-1})} \sim \sum_{m=2}^{\infty} (m+1) I_{m+1} X^m \log(X^{-1})$$

$\longrightarrow$

From eq. 275 it follows

(276) $$\mathcal{R}_{asy} \sim \sum_{m=2}^{\infty} I_{m+1} X^{m-1} \left[(m+1) \log \left(X^{-1} \right) X + X \right]$$

and from eq. 266

$$I_0 = 1 \;,\; I_1 = -a_1 \;,\; I_2 = \Big((a_1)^2 - a_2 \Big) \;\rightarrow$$

(277) $$I_{n+1} X^{n-1} \left[(n+1) \log \left(X^{-1} \right) X + X \right] \Big|_{n=1} = = \begin{cases} \Big((a_1)^2 - a_2 \Big) \times \\ \times \left(2 \log \left(X^{-1} \right) X + X \right) \end{cases}$$

We substitute eq. 277 int the expression for Asy_{12} in eq. 273

(278) $$Asy_{12} \sim \begin{cases} X^{-1} - a_1 \left(\log \left(X^{-1} \right) \right) + \\ + \sum_{m=1}^{\infty} I_{m+1} X^{m-1} \left[(m+1) \log \left(X^{-1} \right) X + X \right] \end{cases}$$

The symbol $\sim$ in eqs. 274 - 278 indicates that the infinite sums are understood as asymptotic expansions .

$\longrightarrow$

Next we expand $\widehat{F}_2$ in eq. 273

$$\widehat{F}_2(X) = \int_0^X dY\,(-\log Y)\,\Delta_2 A^{-1}(Y)$$

(279) $$\sim \sum_{n=0}^{\infty}(n+1)(n+2)\,I_{n+2}\,(J_n(X))$$

$$J_n(X) = \int_0^X dY\,(-\log Y)\,Y^n = \partial_\epsilon \int_0^X dY\,Y^{n-\varepsilon}\Big|_{\varepsilon=0}$$

The integral in the ε extension in eq. 279 converges for small small ε

$$\int_0^X dY\,Y^{n-\varepsilon} = X^{n+1-\varepsilon}/(n+1-\varepsilon) = X^{n+1}X^{-\varepsilon}/(n+1-\varepsilon)$$

(280) $$\partial_\varepsilon \int_0^X dY\,Y^{n-\varepsilon} = \begin{cases} X^{n+1-\varepsilon}/(n+1-\varepsilon)\times \\ \times\left[\left(\log\left(X^{-1}\right)\right) + (n+1-\varepsilon)^{-1}\right] \end{cases} \rightarrow$$

$$J_n(X) = X^n/(n+1)^2\left[(n+1)\log\left(X^{-1}\right)X + X\right]$$

In order to compare the index n of asymptotic expressions in eqs. 279 - 280 with m in eq. 278 we substitute $n = m - 1$, $\rightarrow$

whereupon eq. 280 yields

$$\int_0^X dY\, Y^{m-1-\varepsilon} = X^{m-\varepsilon}/(m-\varepsilon) = X^m X^{-\varepsilon}/(m-\varepsilon)$$

(281) $$\partial_\varepsilon \int_0^X dY\, Y^{m-1-\varepsilon} = \begin{cases} X^{m-\varepsilon}/(m-\varepsilon)\times \\ \times\left[\left(\log\left(X^{-1}\right)\right) + (m-\varepsilon)^{-1}\right] \end{cases} \longrightarrow$$

$$J_{m-1}(X) = X^{m-1}/m^2\left[m\log\left(X^{-1}\right)X + X\right]$$

and the asymptotic expansion for $\widehat{F}_2$ in eq. 279 becomes

$$\begin{aligned} \widehat{F}_2(X) &= \int_0^X dY\,(-\log Y)\,\Delta_2 A^{-1}(Y) \\ &\sim \sum_{m=1}^{\infty}(m)(m+1)\,I_{m+1}\left(J_{m-1}(X)\right) \\ &\sim \sum_{m=1}^{\infty} I_{m+1}\left((m+1)/m\right)X^{m-1}\left[m\log\left(X^{-1}\right)X + X\right] \end{aligned}$$

(282)

$\longrightarrow$

We rename the various parts composing Σ_{asy} in eq. 273

(283)
$$\begin{aligned} \Sigma_{asy}(X) &= Asy_{12}(X) - \widehat{F}_2(X) \\ &= \sigma_{asy}(X) + \mathcal{R}_\sigma(X) \\ \sigma_{asy}(X) &= X^{-1} - a_1 \log\left(X^{-1}\right) \\ \mathcal{R}_\sigma(X) &= -\sigma_{asy}(X) + Asy_{12}(X) - \widehat{F}_2(X) \end{aligned}$$

We collect the expressions for the contributions to Σ_{asy} as given in eq. 283 using eqs. 278 and 282

(284)
$$\begin{aligned} Asy_{12} - \sigma_{asy} &\sim \sum_{m=1}^{\infty} I_{m+1} X^{m-1} \left[(m+1)\log\left(X^{-1}\right) X + X\right] \\ \widehat{F}_2(X) &\sim \sum_{m=1}^{\infty} I_{m+1} X^{m-1} \left\{ \begin{array}{l} ((m+1)/m) \times \\ \times \left[m \log\left(X^{-1}\right) X + X\right] \end{array} \right\} \end{aligned}$$

$$\longrightarrow \quad \mathcal{R}_\sigma(X) \sim -\sum_{m=1}^{\infty} m^{-1} I_{m+1} X^m$$

It becomes clear that the cancellation of all terms $\propto X^m \log\left(X^{-1}\right)$ in the expansion of $\mathcal{R}_\sigma$ is not accidental . $\longrightarrow$

Collecting results concerning the asymptotic expansion of the sliding scale coupling constant

We thus arrive at the resulting structure of the running (inverse) coupling constant, setting

$$Z = b_0 \log \left(\overline{\mu}^2 / \Lambda^2 \right) \equiv b_0 s \;\; ; \;\; X = \overline{\kappa}_{\overline{\mu}}$$

$$\Sigma_{asy}(X) = Z \;\; \leftrightarrow \;\; X = \overline{X}(Z)$$

$$\Sigma_{asy}(X) = \sigma_{asy}(X) + \mathcal{R}_\sigma(X)$$

(285) $$\sigma_{asy} = X^{-1} - a_1 \log(X^{-1}) \;\; ; \;\; \mathcal{R}_\sigma(X) \sim - \sum_{m=1}^{\infty} m^{-1} I_{m+1} X^m$$

for $\overline{\mu}\,,\, Z,\, ,\, X^{-1} \rightarrow \infty$ **and** Λ **fixed**

$$I_0 = 1 \;\; , \;\; I_1 = -a_1 \;\; , \;\; I_2 = \Big((a_1)^2 - a_2 \Big)$$

$$a_m = b_m / b_0 \;\; , \;\; m = 1, 2, \cdots$$

In eq. 285 we substituted material contained in eqs. 270 - 284 .

The definitions of the functions $\beta\,,\, B = b_0 A\,,\, A\,,\, A^{-1}$ **are collected below .**

We repeat and complete eq. 253 using eq. 264 $\longrightarrow$

$$s = 2t = \log\left[(\overline{\mu}/\mu)^2\right]$$

$\kappa = g^2/\left(16\pi^2\right)$ **and** $\kappa \to \overline{\kappa}$ **generic** $\kappa \to X, Y$

$$s = F(\overline{\kappa}) - F(\kappa_\mu) \; ; \; \kappa_\mu = g_\mu^2/\left(16\pi^2\right)$$

$$F(X) = \int_X^{X_0} \left(dY/Y^2\right) (B(Y))^{-1}$$

(286) $$\beta(y) = -y\,b(y^2) \; ; \; B(Y) = b(y^2)/Y \;\leftrightarrow\; Y = y^2/\left(16\pi^2\right)$$

$$\longrightarrow \begin{cases} B(X) = b_0 A(X) \; ; \; b_0 = \frac{1}{3}(33 - 2N_{fl}) > 0 \\ B(X) \sim \sum_{m=0}^{\infty} b_m X^m \\ A(X) \sim \sum_{m=0}^{\infty} a_m X^m \; ; \; a_m = b_m/b_0 \\ A^{-1}(X) \sim \sum_{m=0}^{\infty} I_m X^m \\ a_0 = I_0 = 1 \; , \; I_1 = -a_1 \; , \; \cdots \end{cases}$$

A recent 4-loop approximated evaluation has been performed by S. Bethke [A215-2009] $\longrightarrow$

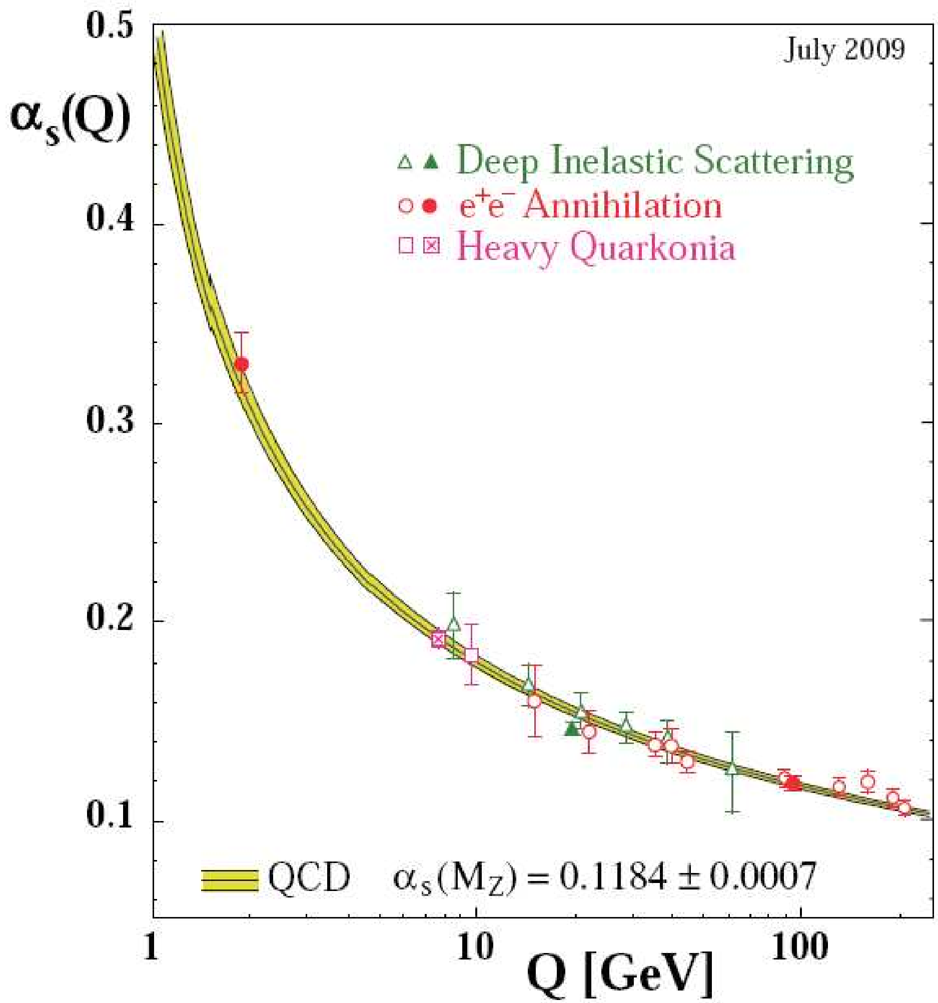

Fig A21 : $\alpha_s(Q) = 4\pi\,\kappa_{\overline{\mu}=Q}$ **from ref. [A215-2009] .**

Back to the asymptotic expansion of the sliding scale coupling constant

We take up eq. 285 and invert the functional relation

$$\Sigma_{asy}(X) = Z \quad \leftrightarrow \quad X = \overline{X}(Z)$$

(287)

$$Z = b_0 \log\left(\overline{\mu}^2/\Lambda^2\right) \equiv b_0\, s$$

$$\Sigma_{asy}(X) = \sigma_{asy}(X) + \mathcal{R}_\sigma(X)$$

$$\sigma_{asy} = X^{-1} - a_1 \log(X^{-1}) \;\; ; \;\; \mathcal{R}_\sigma(X) \sim -\sum_{m=1}^{\infty} m^{-1} I_{m+1} X^m$$

in the form suitable for successive approximations

(288)

$$X^{-1} = Z + a_1 \log(X^{-1}) - \mathcal{R}_\sigma(X) = Z + f(X) \longrightarrow$$

$$f(X) \sim a_1 \log(X^{-1}) + \sum_{m=1}^{\infty} m^{-1} I_{m+1} X^m$$

$$1/X_{\nu+1}(Z) = Z + f\,(X_\nu(Z))$$

starting with the substitution for $\nu = 0 \;\; : \;\; f\,(X_{\nu=0}(Z)) = 0$

In the successive approximation procedure furthermore the function $f\,(X \ldots)$ can be evaluated in various suitable approximations . $\longrightarrow$

We first give the two diverging terms for $Z \longrightarrow \infty$ keeping only the first term in $f \approx f_{(1)} = a_1 \log(X^{-1})$

$$
\begin{aligned}
1 / X^{(0)}_{\nu=1}(Z) &= Z \\
1 / X^{(1)}_{\nu=2} &= Z + a_1 \log[Z] = b_0 s + a_1 \log[b_0 s] \\
\ldots & \quad a_1 = b_1 / b_0
\end{aligned} \tag{289}
$$

$$
f \longrightarrow f_{(\varrho)}(X) \quad \text{with} \quad f_{(1)}(X) = a_1 \log(X^{-1})
$$

The above diverging terms for $Z \longrightarrow \infty$ entail the universal character of the first *two* coefficients – b_0, b_1 – of the $\beta -$ function in any renormalization scheme .

While the above path of successive approximations may not be optimally converging whence extended to terms vanishing for $Z \longrightarrow \infty$, these emerge as a double sequence

$$
1 / X^{(\varrho)}_{\nu+1}(Z) = Z + f_{(\varrho)}\left(X^{(\leq\varrho)}_{\nu}(Z)\right) \tag{290}
$$

We are here not interested in a high level of precision of the approximations, only illustrating within the perturbatively accessible region of QCD the structure of asymptotic expansions. For the evaluations involving the first four orders (in X) of the beta-function I refer to ref. [A215-2009] . $\longrightarrow$

For the purpose of illustration we give the next approximation corresponding to $\nu = 3 \, , \, (\varrho) = 1$

$$\begin{aligned} 1 \, / \, X^{\,(\varrho)=1}_{\,\nu=3} \, (\, Z \,) \quad & = \quad Z \, + \, a_{\,1} \, \log \, [\, Z \, + \, a_{\,1} \, \log \, [\, Z \,] \,] \\ & = \quad Z \, + \, a_{\,1} \, \log \, [\, Z \,] \, + \, a_{\,1} \, \log \, \left[1 \, + \, a_{\,1} \, Z^{\,-1} \, \log \, [\, Z \,] \, \right] \\ & \sim \quad Z \, + \, a_{\,1} \, \log \, [\, Z \,] \, + \, a^{\,2}_{\,1} \, Z^{\,-1} \, \log \, [\, Z \,] \, + \, o \, \left(\, Z^{\,-1} \, \log \, Z \, \right) \end{aligned}$$

(291)

The third term in the third line of eq. 291 is the first of its kind vanishing for $Z \, \rightarrow \, + \infty$ **.**

A2-mass **Quark mass renormalization transposed to quark bilinear operator insertion** $: \, \overline{q}^{\,c} \, q^{\,c} \, : |_{\,0}$

and renormalization **using QCD with exactly vanishing quark mass(es)**

A well known problem of electron mass - and analogously quark mass induced effects goes back to the general operator product expansion discussed by K. Wilson [A216-1969] in the light of QED and the renormalization group equation as specifically formulated by M. Gell-Mann and F. Low [A217-1954] ***and*** **extended to QCD .** $\longrightarrow$

We rescale the renormalization group equation relative to its conventional form in eq. 245

$$\begin{pmatrix} \mu^2 \partial_\mu^2 + (-\kappa^2 B)\,\partial_\kappa \\ -\kappa\,\Gamma_{m_\alpha} m_\alpha \partial_{m_\alpha} \\ -\gamma_3 (\eta\,\partial_\eta) - \Gamma_{J\mathcal{O}} \end{pmatrix} C^{T\,(\Pi)}_{J\,\mathcal{O}} \left(z\,;\,\mu^2, \kappa, m_\beta, \eta \right) = 0$$

$$\left\{ \begin{array}{l} \kappa\, B(\kappa) = -\beta(g)/g \\ \Gamma_{m_\beta}(\kappa) = \frac{1}{2}\gamma_{m_\beta} \\ \Gamma_{J\mathcal{O}}(\kappa, (\eta)) = \frac{1}{2}\gamma_{J\mathcal{O}} \\ \gamma_3(\kappa, \eta) \equiv \gamma_3(g^2, \eta) \end{array} \right\} = \mu^2 \partial_{\mu^2} \left\{ \begin{array}{c} 2\,\log\left((Z_3)^{3/2} (Z_1)^{-1} \right) \\ Z_{m_\beta} \\ \log\left(Z_{\mathcal{O}} / Z_J^2 \right) \\ 2\,\log\,(Z_3)^{1/2} \end{array} \right\}$$

$$\kappa = \alpha_s/(4\pi) = g^2/(16\pi^2)$$

(292)

In the $\overline{\text{MS}}$ scheme – and ignoring the precise form of normal orderings – or rather following the most thoughtful suggestion of S. Weinberg [A218-1973] $\longrightarrow$

the mass rescaling functions $\Gamma_{\beta} \longrightarrow \Gamma_{m}$ become independent of quark flavor β and of the quark masses [A219-1982] , [A220-1994]

$$\Gamma_{m\,\beta} \longrightarrow \Gamma_{m} \equiv \kappa\, G_{m}$$

$$G_{m} = g_{0} + g_{1}\,\kappa + \cdots \sim \sum_{n=0}^{\infty} g_{n}\,\kappa^{n}$$

(293)

$$g_{0} = 4 \;,\; g_{1} = \tfrac{2}{9}\left(101 - 10\,N_{fl}\right)$$

$$g_{2} = 1249 - \left[\frac{2216}{27} + \frac{160}{3}\zeta(3)\right] N_{fl} - \frac{140}{81} N_{fl}^{2}$$

$$\ldots$$

The pertinent rearrangement of normal orderings and of renormalizatiun group invariant quantities to five loop order has been carried out in ref. [A221-2006] .

The obstacles thus outlined and surpassed the *sliding scale* quark mass function(s) inherit universality and perturbative accessibility equal to the *sliding scale* coupling constant .

We are led to consider the pair of rescaling equations $\longrightarrow$

using the same notations as defined in eq. 253

(294)
$$s = 2t = \log\left[(\overline{\mu}/\mu)^2\right]$$
$$\kappa = g^2/\left(16\pi^2\right) \quad \textbf{generic} \left\{\begin{array}{c}\kappa \\ \overline{\kappa}\end{array}\right\} \rightarrow \left\{\begin{array}{c}X, Y \\ \overline{X}, \overline{Y}\end{array}\right\}$$
$$G_m \rightarrow G$$
$$m_q \rightarrow m \quad \textbf{generic} \left\{\begin{array}{c}m \\ \overline{m}\end{array}\right\} \rightarrow \left\{\begin{array}{l}m(\mu) \rightarrow m_0 \\ \overline{m}(s; m_0)\end{array}\right\}$$

$$\longrightarrow \frac{d}{ds}\left\{\begin{array}{c}\overline{X} \\ \overline{m}\end{array}\right\} = -\left\{\begin{array}{l}\overline{X}^2 B(\overline{X}) \\ \overline{X} G(\overline{X})\,\overline{m}\end{array}\right\}$$

It is *obvious* that the universal sliding scale mass function cannot be calculated also in the perturbatively accessible region using any version of a quark mass *dependent* propagator . This makes comparison with data , where quark mass dependent thresholds of *hadrons* appear, which depend even nonperturbatively on quark masses, a step more remote – yet not impossible – . This said we proceed $\longrightarrow$

to solve the differential equations as defined in eq. 294 but using the results already established for the sliding scale coupling constant in the last subsection. Thus we introduce the dimensionless quark mass function

(295) $$f(Y) = \log\left[\frac{\overline{m}}{m_0}\right] \quad ; \quad Y \rightarrow \overline{X}$$

f **still depends through the initial conditions on the** *a priori* **arbitrary scale** μ **, which is however replaced by – an appropriate multiple of – the renormalization group invariant scale** Λ **in a way** *related* **to the asymptotic expansion of the running coupling constant .**
The function f **defined in eq. 295 satisfies the differential equation**

(296) $$\frac{d}{dY} f(Y) = \mathcal{Q}(Y) \quad ; \quad \mathcal{Q} = Y^{-1} \frac{G(Y)}{B(Y)}$$

$$G = g_0 H \quad , \quad B = b_0 A$$

which can be integrated $\longrightarrow$

yielding the initial value dependent relation

(297)

$$f\left(\overline{X}\right) - f\left(X_0\right) = \frac{g_0}{b_0} \int_{X_0}^{\overline{X}} \frac{dY}{Y} \mathcal{Q}_{red}(Y)$$

$$\mathcal{Q}_{red} = H(Y) / A(Y) \sim \sum_{n=0}^{\infty} q_n Y^n \; ; \; q_0 = 1$$

We proceed the same way as in the asymptotic expansion of the coupling constant as collected in eq. 286 , integrating the first term in the expansion of the reduced function $\mathcal{Q}_{red}$ in eq. 297

(298)

$$\frac{1}{Y} \mathcal{Q}_{red} = \frac{1}{Y} + \mathcal{R}_1\left(Y^{-1} \mathcal{Q}_{red}\right)$$

$$\mathcal{R}_1 \rightarrow \mathcal{R}_Q \; ; \; \mathcal{R}_Q \sim \sum_{n=1}^{\infty} q_n Y^{n-1}$$

$$q_1 = \frac{g_1}{g_0} - \frac{b_1}{b_0} \; , \; q_2 \cdots$$

Thus eq. 297 $\longrightarrow$

becomes, anchoring the integrals of $\mathcal{R}_Q$ **at** $X = 0$

$$\int_0^X \mathcal{R}_Q(Y)\,dY \equiv M(X)$$

$$f\left(\overline{X}\right) - f(X_0) = \frac{g_0}{b_0}\left[\log\left(\overline{X}\right) + M\left(\overline{X}\right) - \log(X_0) - M(X_0)\right]$$

(299)

$$M\left(\overline{X}\right) \sim \sum_{n=1}^{\infty} \frac{1}{n}\; q_n \left(\overline{X}\right)^n$$

remembering $f(Y) = \log\left[\frac{\overline{m}}{m_0}\right]$; $Y \to \overline{X}, X_0$

We note the values of the critical mass rescaling exponent – denoted c_{mar} **– for** $N_{fl} = 3$ **to** 6

(300)

$$c_{mar}(N_{fl}) = \frac{g_0}{b_0} = \frac{4 \times 3}{33 - 2N_{fl}}$$

→

N_{fL}	3	4	5	6
c_{mar}	$4/9 \sim 0.44$	$12/25 = 0.48$	$12/23 \sim 0.52$	$4/7 \sim 0.57$

(301)

$\longrightarrow \quad 0.4 < c_{mar} < 0.6$ **for** $3 \leq N_{fl} \leq 6$

We rewrite eq. 299 *separating variables*

(302)

$$\log\left[\frac{\overline{m}}{m^*}\right] - c_{mar}\left[\log\left(\overline{X}\right) + M\left(\overline{X}\right)\right] =$$

$$= \quad \log\left[\frac{m_0}{m^*}\right] - c_{mar}\left[\log\left(X_0\right) + M\left(X_0\right)\right]$$

The reference mass denoted m^* **in eq. 302 is completely arbitrary , yet we** *restrict* **it to be renormalization group invariant .**

Next we exponentiate both sides of eq. 302 $\longrightarrow$

$$\frac{\overline{m}}{m^*}\left[(\overline{X})^{-c_{mar}}\right] EM(\overline{X}) =$$

$$= \frac{m_0}{m^*}\left[(X_0)^{-c_{mar}}\right] EM(X_0)$$

$$EM(X) = \exp[-c_{mar} M(X)]$$

(303) $M(X):$ $\begin{cases} \textbf{universal , quark mass independent function} \\ \sim \sum_{n=1}^{\infty} \frac{1}{n} q_n (X)^n \quad \textbf{in perturbatively accessible region} \end{cases}$

↘

$EM(X) = 1 + \mathcal{R}_{EM}(X):$ $\begin{cases} \textbf{universal , quark mass independent function} \\ \mathcal{R}_{EM} \sim \sum_{n=1}^{\infty} g_{EM\,n} (X)^n \\ \textbf{in perturbatively accessible region} \end{cases}$

$$g_{EM\,1} = -c_{mar}\, q_1 , \cdots$$

→

We collect the expressions composing $g_{EM\,1}$ in the last relation of eq. 303 below, using eqs. 267 (b_0 , b_1) , 293 (g_0 , g_1) , 298 (q_1) , 300 and 301 (c_{mar})

$$g_{EM\,1} = -c_{mar}\, q_1 \quad ; \quad c_{mar}(N_{fl}) == \frac{g_0}{b_0} = \frac{4 \times 3}{33 - 2N_{fl}}$$

(304) $$q_1 = \frac{g_1}{g_0} - \frac{b_1}{b_0} \; ;$$

$$\begin{array}{lllclll} b_0 & = & \frac{1}{3}\left(33 - 2N_{fl}\right) & & g_0 & = & 4 \\ b_1 & = & \frac{2}{3}\left(9 \times 17 - 19N_{fl}\right) & , & g_1 & = & \frac{2}{9}\left(101 - 10N_{fl}\right) \end{array}$$

We evaluate the ratios forming the expresion for $g_{EM\,1}$ in eq. 304 only for $N_{fl} = 3$ and 5 and also neglect all $g_{EM\,n>1}$ for simplicity and to show the structural effects in a coherent way, leaving *subsequent* **systematic approximations aside . The latter include heavy flavor matching if we go deep enough inside the region of perturbative accessibility .** $\longrightarrow$

$$N_{fl} = 3,5 \; : \; c_{mar} = \frac{4}{9} \, , \, \frac{12}{23}$$

(305)
$$\left\{ \begin{array}{lllll|lllll} b_0 & = & 9 & , & \frac{23}{3} & g_0 & = & 4 & , & 4 \\ b_1 & = & 64 & , & \frac{116}{3} & g_1 & = & \frac{142}{9} & , & \frac{34}{3} \end{array} \right.$$

It follows

(306)
$$c_{mar} = \frac{4}{9} \, , \, \frac{12}{23} \quad , \quad \frac{g_1}{g_0} = \frac{71}{18} \, , \, \frac{17}{6} \quad , \quad \frac{b_1}{b_0} = \frac{64}{9} \, , \, \frac{116}{23}$$

$$q_1 = - \frac{19}{6} \, , \, - \frac{305}{138}$$

$$g_{EM\,1} = - c_{mar} \, q_1 = \frac{38}{27} \sim 1.41 \, , \, \frac{610}{23*23} \sim 1.15$$

Universal quark mass rescaling – strengths and limits

With the criteria layed out in the last subsection we cast eq. 303 into the form

(307)
$$\overline{m} = m_0 \; \frac{EM \; (X_0)}{(X_0)^{\, c_{mar}}} \; \frac{\left(\overline{X} \right)^{\, c_{mar}}}{EM \; \left(\overline{X} \right)}$$

$\longrightarrow$

It is here important to maintain clarity of notions and I repeat the generic form of the function $EM(X)$ in eq. 303 below , using the values og the first two sets of renormalization coefficients for $N_{fl} = 3$ in eq. 306

$$EM(X) = \exp[-c_{mar} M(X)]$$

$$M(X) : \begin{cases} \textbf{universal , quark mass independent function} \\ \sim \sum_{n=1}^{\infty} \frac{1}{n} \; q_n (X)^n \quad \textbf{in perturbatively accessible region} \end{cases}$$

$\searrow$

(308)

$$EM(X) = 1 + \mathcal{R}_{EM}(X) : \begin{cases} \textbf{universal , quark mass independent function} \\ \mathcal{R}_{EM} \sim \sum_{n=1}^{\infty} g_{EM\,n} (X)^n \\ \textbf{in perturbatively accessible region} \end{cases}$$

$$g_{EM\,1} = -c_{mar} q_1 , \cdots$$

$$N_{fl} = 3 : c_{mar} = \frac{4}{9} \; ; \quad g_{EM\,1} = -c_{mar} q_1 = \frac{38}{27} \; , \quad q_1 = -\frac{19}{6}$$

$$N_{fl} = 5 : c_{mar} = \frac{12}{23} \; ; \quad g_{EM\,1} = -c_{mar} q_1 = \frac{610}{529} \; , \quad q_1 = -\frac{305}{138}$$

$\longrightarrow$

After an apparent detour we separate the universal repsonse function in the 'deep euclidean $\leftrightarrow$ perturbatively accessible' rescaling function of mass versus coupling constant and thus versus scale in the following way

$$\frac{\overline{m}}{m^{*}} = \exp\left(- c_{mar} \left[\log \frac{1}{\overline{X}} - M(\overline{X}) \right] \right)$$

(309)

$$M(X) \sim \sum_{n=1}^{\infty} \frac{1}{n} q_{n} (X)^{n}$$

The sliding scale is related to the coupling constant in eq. 285 reproduced below

$$Z = b_0 \log\left(\overline{\mu}^{2} / \Lambda^{2}\right) \equiv b_0 s \; ; \; X = \overline{\kappa}_{\overline{\mu}}$$

$$\Sigma_{asy}(X) = Z \quad \leftrightarrow \quad X = \overline{X}(Z)$$

(310)

$$\Sigma_{asy}(X) = \sigma_{asy}(X) + \mathcal{R}_{\sigma}(X)$$

$$\sigma_{asy} = X^{-1} - a_1 \log(X^{-1}) \; ; \; \mathcal{R}_{\sigma}(X) \sim - \sum_{m=1}^{\infty} m^{-1} I_{m+1} X^{m}$$

$\longrightarrow$

The adopted two loop or second order approximation eqs. 309 and 310 amounts (for $N_{fl} = 3$) to the substitutions

(311)
$$M(X) \sim q_1 X = -\frac{19}{6} X$$
$$Z \sim X^{-1} - a_1 \log(X^{-1}) \longrightarrow$$
$$X^{-1} \sim Z + a_1 \log Z$$
$$q_1 = -\frac{19}{6}, -\frac{305}{138}, \quad a_1 = \frac{64}{9}, \frac{116}{23}$$

Eqs. 309 , 310 become

(312)
$$\frac{\overline{m}}{m^*} \sim \exp\left(-c_{mar}\left[\log\frac{1}{\overline{X}} - q_1 \overline{X}\right]\right)$$
$$Z = b_0 \log\left(\overline{\mu}^2 / \Lambda^2\right) \sim \overline{X}^{-1} - a_1 \log\left(\overline{X}^{-1}\right)$$

$$N_{fl} = 3 : b_0 = 9, \quad c_{mar} = \frac{4}{9}, \quad q_1 = -\frac{19}{6}, \quad a_1 = \frac{64}{9}$$
$$N_{fl} = 5 : b_0 = \frac{23}{3}, \quad c_{mar} = \frac{12}{23}, \quad q_1 = -\frac{305}{138}, \quad a_1 = \frac{116}{23}$$

We proceed to transform the second relation in eq. 312 →

anchoring the mass scale at $\mu^{*} = 1$ GeV and normalizing the running strong coupling constant (square) relative to $\overline{\alpha}_{s} = 4\pi\overline{X}$

$$Z = 2 b_{0} \left[\log \frac{\overline{\mu}}{\mu^{*}} + A \right] \quad ; \quad EA \equiv e^{A} = \frac{\mu^{*}}{\Lambda} \quad ; \quad \mu^{*} = 1 \text{ GeV}$$

(313) $$X = \frac{\alpha_{s}}{4\pi} \text{ and generic } X \rightarrow \overline{X} \ , \ \alpha_{s} \rightarrow \overline{\alpha}_{s}$$

$$\log \left[\frac{\overline{\mu}}{\mu^{*}} + A \right] \sim \left(\frac{2\pi}{b_{0}} \right) (\overline{\alpha}_{s})^{-1} - \frac{a_{1}}{2 b_{0}} \log \left(\frac{4\pi}{\overline{\alpha}_{s}} \right)$$

We first show three figures : (1) repeating Fig A21 , (2) Fig A22 : comparing with the two loop approximate rescaling with the four loop based $\alpha_{s}(Q)$ on Fig A21 from ref. [A215-2009] , (3) Fig A23 : universally rescaled running quark masses with unspecified reference scale m^{*} and fixed ratios $m_{d} : \frac{1}{2}(m_{d} + m_{u}) : m_{u} = 5 : 4 : 3$.

More detailed description of these three figures is given subsequently . →

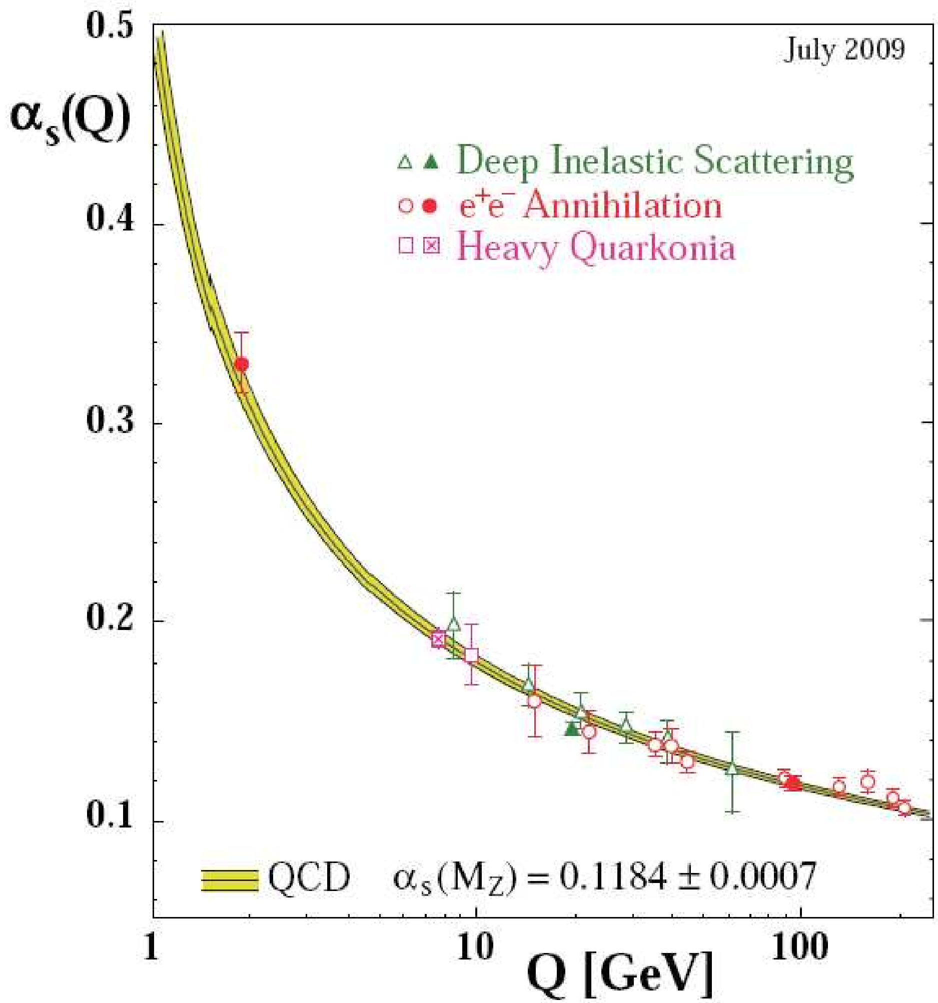

Fig A21 : $\alpha_s(Q) = 4\pi\,\kappa_{\overline{\mu}=Q}$ **from ref. [A215-2009] .**

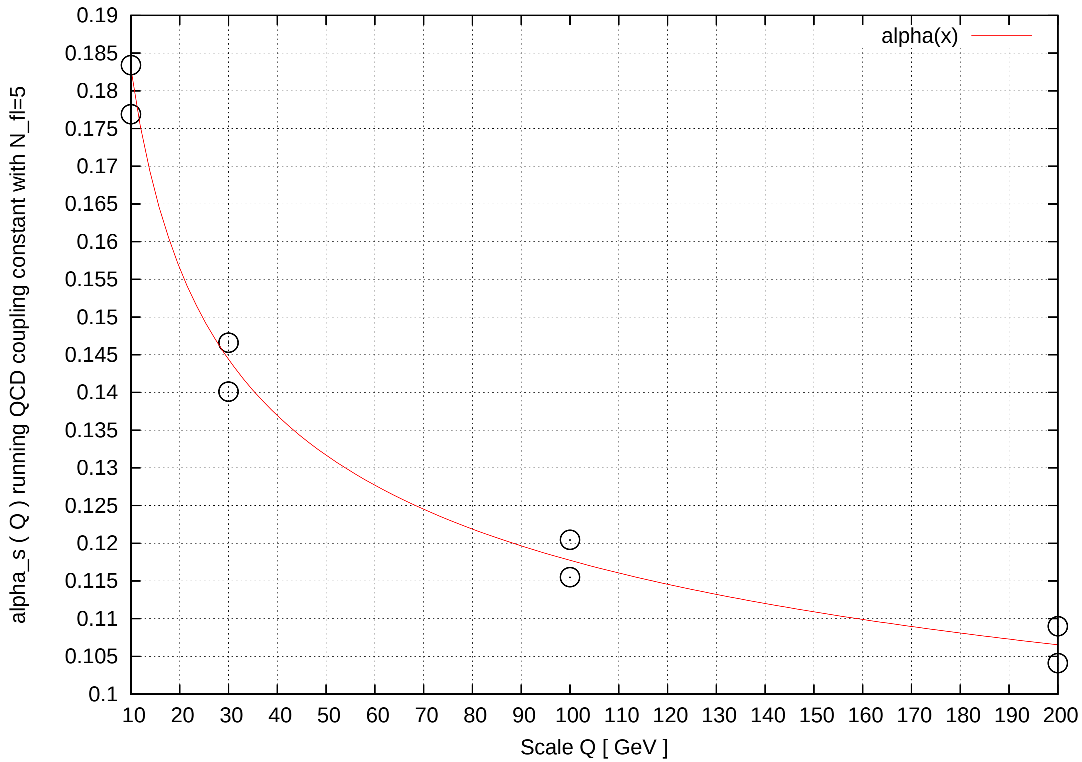

Fig A22 : $\alpha_s(Q) = 4\pi\,\kappa_{\overline{\mu}=Q}$ **from eq. 311 compared with Fig. A21 .**

→

A2-44c

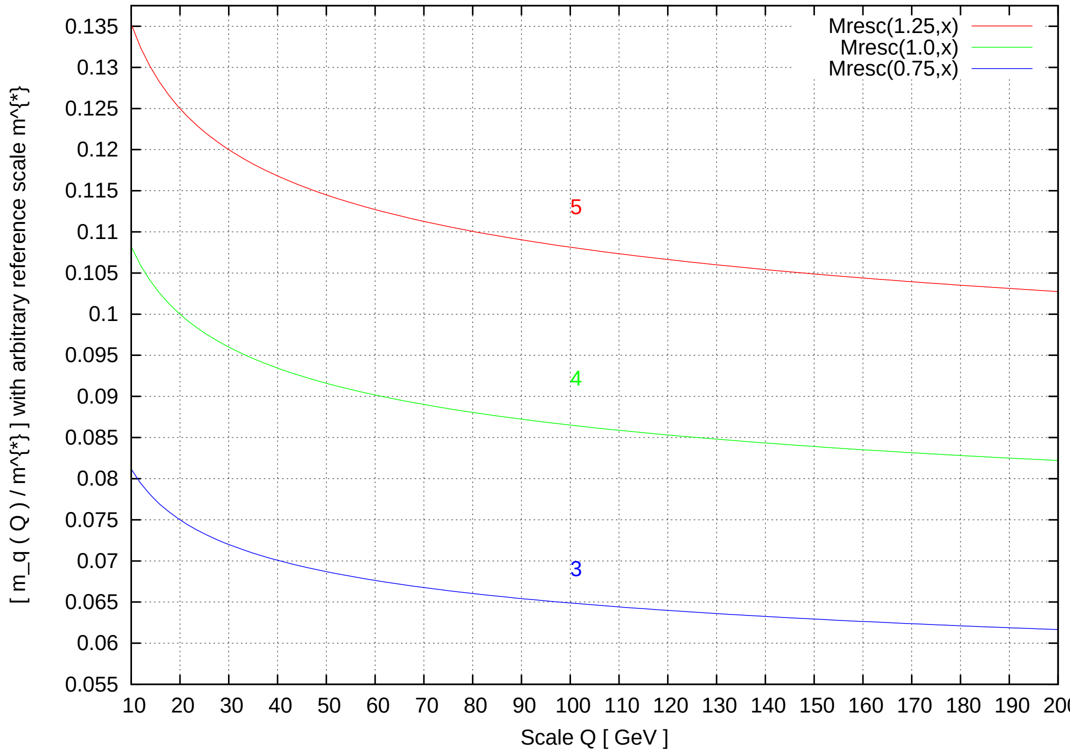

Fig A23 : $m_q(Q)/m^*$ **with fixed ratio of rescaled quark masses**

$m_u : \frac{1}{2}(m_d + m_u) : m_d = 3 : 4 : 5$ **from eq. 309 .**

→

To Fig A21 : In the four loop evaluation of the running coupling constant in ref. [A215-2009] the renormalization group invariant quantity is obtained in the $\overline{MS}$ renormalization scheme

$$\Lambda_5^{(4)} = 213 \pm 9 \text{ MeV} \longleftrightarrow \alpha_s(m_Z) = 0.1184 \pm 0.0007$$

(314)
$$\lambda_5^{(4)} = 213 \text{ MeV} \rightarrow \begin{cases} \Lambda_4^{(4)} = 296 \text{ MeV} \\ \Lambda_3^{(4)} = 338 \text{ MeV} \end{cases}$$

The matching between $N_{fl} = 5 \rightarrow 4 \rightarrow 3$ in ref. [A215-2009] involves the modeling of the b- and c-flavor associated thresholds through the perturbatively assigned b- and c-quark *pole*-masses $m_b = 4.7$ GeV , $m_c = 1.5$ GeV . This is a nonuniversal way to rescale quark masses , and thus does not follow the strict quark mass rescaling at zero quark mass , used here . As a comparison in determining up and down quark masses at an $\overline{MS}$ scale of 2 GeV , Dominguez, Nasrallah, Röntsch and Schilcher [A223-2008] use

(315)
$$\Lambda_3^{(4)} = 381 \pm 16 \text{ MeV} \leftrightarrow \alpha_s(m_\tau) = 0.344 \pm 0.009$$

and adopting the scheme of quark mass rescaling at zero mass obtain for the u,d,s quark mass ratios

(316)
$$\begin{array}{ccccccc} m_u & : & \frac{1}{2}(m_d + m_u) & : & m_d & : & m_s \\ 2.9 \pm 0.2 & : & 4.1 \pm 0.2 & : & 5.3 \pm 0.5 & : & 102 \pm 8 \end{array}$$

→

To Fig A22 : The following value was used : $\alpha_s(M_Z) = 0.184$ **corresponding – for the two loop running as defined in eq. 311 – to** $\Lambda_5^{(2)} \sim 408$ **MeV . The so determined running coupling constant is compared with the 1 -** σ **limits of the same quantity as determined in 4 loop order in ref. [A215-2009] confirming the validity of the two loop approximation in the range** 10 **GeV** $\leq Q \leq 200$ **GeV within the accuracy claimed in ref. [A215-2009] .**

To Fig A23 : Here the strength and weakness of the mass rescaling at zro mass within the perturbatively accessible region is illustrated using as a guide ***only*** **the ratio of u,d quark masses**

(317)
$$\begin{array}{ccccc} m_u & : & \frac{1}{2}(m_d + m_u) & : & m_d \\ 3 & : & 4 & : & 5 \end{array}$$

It seems appropriate to me to refer to the ***in principle*** **approach of rescaling in a universal way the coupling constant and quark masses** ***initially*** **restricting all analysis to the perturbatively accessible region , citing (adapting) the pertinent comment by Murray Gell-Mann :**
'Rising when last (first) seen .'
On the other hand the progress achieved in transgressing the perturbatively accessible region , using universal mass rescaling , in refs. [A223-2008] , [A221-2006] and references cited therein, is significant, based on improved treatment of finite energy sum rules pioneered by Shifman , Vainshtain and Zakharov [A224-1979] . →

To Fig A23 *continued* **: It is worth noting that the value of the gauge boson condensate, found in ref. [A223-2008] , approximated as**

(318) $$\langle \Omega \,|\, \frac{\alpha_s}{\pi} \; : F^{A}_{\mu\nu} F^{\mu\nu\, A} : |\, \Omega \rangle \;\rightarrow\; 0.06 \text{ GeV}^4$$

is 5 times larger , than its original estimate in ref. [A224-1979] .

The basics of chiral expansions in assessing ratios of the u,d,s quark masses continue to provide additional benchmarks at low hadron energies [A225-2001] and references cited therein, while fine details of these ratios can be subject to improvement . Finally the validity of chiral expansions as guidelines for lattice calculations present another *strategy in principle* **[A226-2008] .**

We add here a few representative determination of $\alpha_s(m_Z)$

	$\alpha_s(m_Z)$	processes	source	authors
(319)	0.1176 ± 0.0020	average	[A227-2008]	PDG
	0.1172 ± 0.0022	thrust distributions at LEP	[A228-2008]	Becher , Schwartz
	0.1184 ± 0.0007	average	[A215-2009]	Bethke

References

[1-2008] N. Yamamoto, 'Instanton-induced crossover in dense QCD', JHEP 0812 (2008) 060 , arXiv:0810.2293 [hep-ph] , and references quoted therein.

[2-2006] U. M. Heller, Plenary talk at XXIVth International Symposium on Lattice Field Theory (Lattice 2006). To appear as PoS (LAT2006) 011, 'Recent progress in finite temperature lattice QCD', arXiv:hep-lat/0610114v1 .

[3-1981] P. Minkowski, 'On the ground-state expectation value of the field strength bilinear in gauge theories and constant classical fields', Nucl.Phys.B177 (1981) 203 .

[4-2001] S. Kabana and P. Minkowski, ' Mapping out the QCD phase transition in multiparticle production', New J.Phys.3 (2001) 4, hep-ph/0010247 .

[5-2007] C. Bernard et al., 'Baryon masses with improved staggered quarks', by MILC Collaboration, To appear in the proceedings of 25th International Symposium on Lattice Field Theory, Regensburg, Germany, 30 Jul - 4 Aug 2007, PoS LAT2007 (2007) 137, arXiv:0711.0021 [hep-lat] .

[6-1973] J.M. Kosterlitz and D.J. Thouless, 'Ordering, metastability and phase transitions in two-dimensional systems', J.Phys.C6 (1973) 1181-1203 .

References

[6a-2009] S. Kabana and P. Minkowski, ’On the thermal phase structure of QCD at vanishing chemical potentials’, preprint in preparation.

References

[7-1988] P. Minkowski, ’Color superconducvtivity and the associated phase transition in QCD’, in ’Hadron Interactions’, Proceedings of the International Symposium on ”Hadron Interactions – Theory and Phenomenology”, Bechyne, Czechoslovakia, 26. June - 1. July, 1988, J. Fischer, P. Kolář, V. Kundrát eds., Prague 1988 .

[8-1993] W. H. Zurek, ’Cosmic strings in laboratory superfluids and remnants of other phase transitions’, Acta Physica Polonica B7 (1993) 1301 ,
URL $\longrightarrow$ http://th-www.if.uj.edu.pl/acta/vol24/pdf/v24p1301.pdf .

[9-1937] P. L. Kapitza, Nature 141 (1937) 74 , JETP 11 (1941) , 1 and 581 .

[10-1941] L. D. Landau, JETP 11 (1941) 592 and L. D. Landau and E. M. Lifshitz, ’Statistical Mechanics’, Pergamon Press, New York 1958 .

[11-1998] Complete list works of L D Landau, Physics-Uspekhi 41 (6), June, 1998.

[12-1961] L. P. Pitaevskii, JETP 13 (1961) 451 .

[13-195(6)7] L. D. Landau, ’The theory of a Fermi liquid’, JETP 3 (1957) 920 (english translation) ,
original : JETP 30 (1956) 1058 ;
’Oscillations in a Fermi liquid’, JETP 5 (1957) 101 (english translation) .

References

[14-195(6)7] L. N. Cooper, 'Bound Electron Pairs in a Degenerate Fermi Gas', Phys. Rev 104 (1956) 1189 - 1190 ,

J . Bardeen, L. N. Cooper, and J. R. Schrieffer, 'Microscopic Theory of Superconductivity', Phys. Rev. 106 (1957) 162 - 164 ,

J. Bardeen, L. N. Cooper, and J. R. Schrieffer, 'Theory of Superconductivity', Phys. Rev. 108 (1957) 1175 .

[15-1947] N. N. Bogoliubov, J. Phys. (USSR) 11 (1947) 23 .

[16-2002] J.M. Vogels,K. Xu,C. Raman,J.R. Abo-Shaeer and W. Ketterle, 'Experimental observation of the Bogolyubov transformation for a Bose-Einstein condensed gas', Phys.Rev.Lett.88 (2002) 060402, cond-mat/0109205 .

[17-197(3)4] D.J. Gross and F. Wilczek,

'Asymptotically free gauge theories. 1', Phys.Rev.D8 (1973) 3633-3652 ,

'Asymptotically free gauge theories. 2', Phys.Rev.D9 (1974) 980-993 .

[18-197(3)4] D. Politzer, ' Reliable Perturbative Results for Strong Interactions?', Phys. Rev. Lett. 30 (1973) 1346 ,

H. Georgi and D. Politzer, 'Electroproduction scaling in an asymptotically free theory of strong interactions', Phys. Rev. D9 (1974) 416 .

References

[19-1975] H. Kluberg-Stern and J. B. Zuber, ’Ward Identities and Some Clues to the Renormalization of Gauge Invariant Operators’, Phys.Rev. D12 (1975) 467 ,
’Renormalization of Nonabelian Gauge Theories in a Background Field Gauge. 1. Green Functions’, Phys.Rev. D12 (1975) 482 and
’Renormalization of Nonabelian Gauge Theories in a Background Field Gauge. 2. Gauge Invariant Operators’, Phys.Rev.D12 (1975) 3159-3180 .

References

[z11-2000] K. Holland, (Bern U.), U.-J. Wiese, (MIT, LNS), ’The Center symmetry and its spontaneous breakdown at high temperatures’, BUTP-2000-34, MIT-CTP-3045, Nov 2000. 35pp., published in the Festschrift in honor of B.L. Ioffe, ’At the Frontier of Particle Physics / Handbook of QCD’, edited by M. Shifman, vol. 3, 1909-1944, hep-ph/0011193 .

References

[A11-2008] P. Minkowski, ’On Heraklitean space-time’, contribution to the ”Third International Conference on the Nature and Ontology of Spacetime”, Concordia University, Montreal, Quebec, Canada, 13. June 2008 .

[A12-1951-1961] G. Racah, ’Group theory and spectroscopy’, CERN Yellow report 61-8, 1961, URL : http://doc.cern.ch/cernrep/1961/1961-008/1961-008.html , reprinted from lectures dlivered (by the author) at the Institute for Advanced Study in Princeton, 1951 ; and references therein quoted below.

[A13-1893] S. Lie and G. Scheffers, ’Vorlesungen über continuierliche Gruppen, mit geometrischen und anderen Anwendungen’, Leipzig 1893 .

[A14-1894] É. Cartan, ’Sur la structure des groupes de transformations finis et continues’, Thèse, Paris 1884 , IIème édition 1933 .

[A15-1903] L. Bianchi, ’Lezioni sulla teoria dei gruppi continui finiti di trasformazioni’, Pisa 1903 , IIa edizione 1918 .

$\longrightarrow$

References

[A16-1933] L.P. Eisenhart, ’Continuous groups of tranformations’, Princeton 1933.

[A17-1933] H. Weyl, ’The structure and representation of continous groups’, I and II, Princeton 1933 and 1934-35.

[A18-1890] W. Killing, ’Die Zusammensetzung der stetigen endlichen Transformationsgruppen’, Mathematische Annalen 31 (1888) 252-290; 33 (1888) 1-48; 34 (1889) 57-122; 36 (1890) 161-189 .

[A19-1897] F. Klein and A. Sommerfeld, ’Über die Theorie des Kreisels’, 4 volumes, Teubner Verlag 1897 .

[A110-2000] T. Breuer, ’Characters and Automorphism Groups of Compact Riemann Surfaces’, London Mathematical Society Lecture Note Series (No. 280) , September 2000 .

[A111-1962] N. Jacobson, ’Lie Algebras’ , Wiley Interscience, 1962 .

[A112-2002] A. W. Knapp, ’Lie Groups Beyond an Introduction’, Second Edition , Progress in Mathematics, vol. 140, Birkhuser, Boston, 2002. ISBN: 0-8176-4259-5; 812+xviii pages .

[A113-1986] L. S. Pontryagin, ’Topological Groups’, Gordon and Breach Sci. Publ., New York/London/Paris (1986).

[A114-1933] A. Haar, ’Der Massbegriff in der Theorie der kontinuierlichen Gruppen’, Ann. Math. (2) 34 (1933) 147-169 .

⟶

References

[A115-1927] F. Peter und H. Weyl, ’Die Vollständigkeit der primitiven Darstellungen einer geschlossenen kontinuierlichen Gruppe’ , Ann. Math. 97 (1927) 737-755 .

[A116-1953] G. W. Mackey, ’Induced Representations of Locally Compact Groups II. The Frobenius Reciprocity Theorem’, The Annals of Mathematics, Second Series, Vol. 58, No. 2 (Sep., 1953) 193-221 .

[A117-1963] S. Kobayashi and K. Nomizu, ’Foundations of Differential Geometry’, 2 volumes, John Wiley and Sons, Inc., Hoboken, NJ 1963, Wiley Classics Library 2009 .

[A118-1981] H. Georgi, ’Lie algebras in particle physics’, Benjamin/Cummings, Reading, MA 1981 .

[A118-1964] M. Gell-Mann and Y. Ne’eman, ’The eightfold way’, W. A. Benjamin, Imc. , New York 1964 .

References

[A21-1976] P. Minkowski, ’On the anomalous divergence of the dilatation current in gauge theories’, Bern preprint 1976 , URL : http://www.mink.itp.unibe.ch .

[A22-1974] D.J. Gross and F. Wilczek, ’Asymptotically free gauge theories. 2’, Phys.Rev.D9 (1974) 980-993 and ref. [17-197(3)4] .

[A23-1974] H. Georgi and D. Politzer, ’Electroproduction scaling in an asymptotically free theory of strong interactions’, Phys. Rev. D9 (1974) 416 and ref. [18-197(3)4] .

[A24-1975] H. Kluberg-Stern and J. B. Zuber, ’Ward Identities and Some Clues to the Renormalization of Gauge Invariant Operators’, Phys.Rev. D12 (1975) 467 ,
’Renormalization of Nonabelian Gauge Theories in a Background Field Gauge. 1. Green Functions’, Phys.Rev. D12 (1975) 482 and
’Renormalization of Nonabelian Gauge Theories in a Background Field Gauge. 2. Gauge Invariant Operators’, Phys.Rev.D12 (1975) 3159-3180 .

[A25] C. Itzykson and J. B. Zuber, ’Quantum field theory’, McGraw-Hill, New York 1980.
M. E. Peskin, D. V. Schroeder, ’An Introduction to quantum field theory’, Addison-Wesley, Reading, MA 1995 .
J. Zinn-Justin, ’Quantum field theory and critical phenomena’, The International Series of Monographs on Physics, Vol. 113 , Oxford University Press, Oxford 2002 .

→

References

[A26-1982] G. Altarelli, 'Partons in quantum chromodynamics', NFN-ROME-783, Jul 1981. 211pp., Phys.Rept.81 (1982) .

[A27-1975] H. Fritzsch and P. Minkowski, 'Psi Resonances, Gluons and the Zweig Rule', Caltech preprint CALT-68-492, Mar 1975. 61pp., Nuovo Cim.A30 (1975) 393 .

[A28-1970] C. G. Callan, 'Broken scale invariance in scalar field theory', Phys.Rev.D2 (1970) 1541-1547 .

[A29-1970] K. Symanzik, 'Small distance behavior in field theory and power counting', Commun.Math.Phys.18 (1970) 227-246 .

[A210-1971(2)] G. 't Hooft, 'Renormalizable Lagrangians for Massive Yang-Mills Fields', Nucl.Phys.B35 (1971) 167-188 ,
G. 't Hooft and M.J.G. Veltman, 'Regularization and Renormalization of Gauge Fields', Nucl.Phys.B44 (1972) 189-213 .

[A211-1969] I. B. Khriplovich, 'Greens functions in theories with a non-abelian gauge group' , Yad. Fiz. 10 (1969) 409 , engl. transl. Sov. J. Nucl. Phys. 10 (1970) 235 .

[A212-1972] G. 't Hooft, Remarks on the ultraviolet stable sign of the beta function at the Colloquium on renormalization of Yang-Mills fields and applications to particle physics, Marseille 1972 .

References

[A213-1988] S. G. Gorishnii , A. L. Kataev and S. A. Larin, 'Next-To-Leading $O(\alpha_s^3)$ QCD Correction to $\Sigma_t\ (e^+ e^- \to$ Hadrons $)$: Analytical Calculation and Estimation of the Parameter Lambda (MS)', JINR-E2-88-254, Apr 1988. 18pp., Phys.Lett.B212 (1988) 238-244 , also in the same proceedings as ref. [7-1988] .

[A214-1997] T. van Ritbergen, J.A.M. Vermaseren and S.A. Larin, 'The Four loop beta function in quantum chromodynamics', UM-TH-97-01, NIKHEF-97-001, Jan 1997. 9pp., Phys.Lett.B400 (1997) 379 , hep-ph/9701390 .

[A215-2009] S. Bethke, 'The 2009 Wolrd Average of $\alpha_s(M_Z)$ ', arXiv:0908.1135 [hep-ph] .

[A216-1969] K. Wilson, ' Nonlagrangian models of current algebra', Phys. Rev. 179 (1969) 1499 .

[A217-1954] M. Gell-Mann and F. Low, 'Quantum electrodynamics at small distances', Phys. Rev. 95 (1954) 1300 .

[A218-1973] S. Weinberg, 'New approach to the renormalization group', Phys. Rev. D8 (1973) 3497 .

[A219-1982] O. V. Tarasov, Dubna report JINR P2-82-900 (1982) , unpublished .

[A220-1994] K. G. Chetyrkin, C. A. Dominguez, D. Pirjol and K. Schilcher, 'Mass singularities in light quark correlators : the strange quark case', Phys.Rev.D51 (1995) 5090-5100 , hep-ph/9409371 .

References

[A221-2006] P. A. Baikov, K. G. Chetyrkin and J. H. Kühn, Phys. Rev. Lett. 96 (2006) 012003 .

P. A. Baikov, K. G. Chetyrkin and J. H. Kühn, 'Massless propagators: Applications in QCD and QED', SFB-CPP-08-06, TTP08-03, Oct 2008. 7pp., Contributed to 8th International Symposium on Radiative Corrections (RADCOR 2007): Application of Quantum Field Theory to Phenomenology, Florence, Italy, 1-6 Oct 2007, PoS RADCOR2007 (2007) 023 , arXiv:0810.4048 [hep-ph] .

[A222-2008] P. A. Baikov, K. G. Chetyrkin and J. H. Kühn, 'Hadronic Z- and tau-decays in order α_s^4 ', Phys.Rev.Lett.101 (2008) 012002 , arXiv:0801.1821v2 [hep-ph] .

[A223-2008] C.A. Dominguez, N.F. Nasrallah, R.H. Röntsch and K. Schilcher, 'Up and down quark masses from Finite Energy QCD sum rules to five loops', UCT-TP-271-08, Jun 2008. 4pp. , Phys.Rev.D79 (2009) 014009 , arXiv:0806.0467 [hep-ph] .

[A224-1979] M. A. Shifman , A. I. Vainshtein and V. I. Zakharovi , 'QCD and resonance physics. Theoretical Foundations.' , Nucl. Phys. B147 (1979) 385 , 448 .

[A225-2001] J. Gasser and H. Leutwyler , Annals Phys. 158 (1984) 142 ,

H. Leutwyler, 'Nonlattice determinations of the light quark masses', Nucl.Phys.Proc.Suppl.94 (2001) 108 , hep-ph/0011049 .

References

[A226-2008] B. Blossier et al., European Twisted Mass Collaboration, 'Light quark masses and pseudoscalar decay constants from N(f)=2 Lattice QCD with twisted mass fermions', DESY-07-148, FTUV-07-2709, IFIC-07-57, MS-TP-07-23, RM3-TH-07-11, ROM2F-2007-16, SFB-CPP-07-58, TUM-HEP-676-07, Sep 2007. 20pp., JHEP 0804 (2008) 020 , arXiv:0709.4574 [hep-lat] .

[A227-2008] C. Amsler et al. (Particle Data Group), Physics Letters B667 (2008) 1 .

[A228-2008] T. Becher and M. D. Schwartz, 'A Precise determination of alpha(s) from LEP thrust data using effective field theory', FERMILAB-PUB-08-048-T, Mar 2008, 34pp., JHEP 0807 (2008) 034 , arXiv:0803.0342 [hep-ph] .